

Magnetohydrodynamic Free Convection in a Vertical Slot

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ABSTRACT

The steady magnetohydrodynamic free convection in an asymmetrically heated vertical slot in the presence of a uniform transverse magnetic field has been studied. An exact solution of the governing equation has been obtained. The numerical results for the velocity field and the temperature distribution are presented graphically for various values of the Hartmann number and Grashof number. It is found that the magnitude of the velocity field decreases with increase in Hartmann number. It is also observed that the temperature decreases with increase in either Hartmann number or Grashof number.

Keywords: Magnetohydrodynamics, free convection, Hartmann number, Grashof number and asymmetric.

1. INTRODUCTION

Many analyses of laminar fully developed free convection flow in a vertical parallel-plate channel with prescribed temperatures at the boundary walls are available in the literature. The study of free convective magnetohydrodynamic flow through vertical channels have received considerable attention because of its wide range of applications in the design of MHD generators, cross-field accelerators, shock tubes, pumps, flow meters, cooling of electronic devices and solar energy collectors etc.. In many cases the flow in these devices will be accompanied by heat either that dissipated internally through viscous or Joule heating or that produced by electric currents in the walls. Recently, Aung and Worku [1], Cheng et al. [2] and Hamadah and Wirtz [3] have studied the mixed convection in a vertical channel with symmetric and asymmetric heating of the walls. These authors pointed out that the buoyancy force can cause flow reversal for both upward flow and downward flow. Poorts [4] made an exhaustive analysis of the electrically conducting fluid flow between two heated plates including the Joule heating, viscous dissipation and internal heat sources in the energy equation. Alireza and Sahai [5] studied the effect of temperatures-dependent transport properties on the developing magnetohydrodynamic flow and heat transfer in a parallel plate channel whose walls are held at constant and equal temperatures. The study of Aung and Worku [6] on mixed convection flow through the vertical channel with asymmetric heating of the wall is a great interest of subject based on similarity solutions. Bühler [7] has been discussed free convection flow in a vertical gap. Weidman [8] studied the convective regime flow in a vertical slot and continuum of solutions from capped to open ends.

The aim of the present paper is to study the laminar fully developed free convective flow in a vertical slot with asymmetric heating of the walls in the presence of a uniform transverse magnetic field. We discussed the velocity field and temperature distribution for magnetic field intensity. It is found that the velocity decreases while the temperature increases with increase in Hartmann number M . It is also found that both the velocity

and the temperature decrease with increase in wall temperature parameter $\theta_0(0)$. It is found that the critical wall temperature $(\theta_0)_{crit.}$ at the cold wall ($\eta = -1/2$) decreases with increase in either Hartman number M or Grashof number Gr .

2. FORMULATION OF THE PROBLEM AND ITS SOLUTIONS

We consider a two-dimensional free convective hydromagnetic fully developed flow of a viscous incompressible electrically conducting fluid confined to a vertical channel. The distance between the walls is d . A uniform magnetic field of strength B_0 is imposed perpendicular to the walls of the vertical channel. Choose a Cartesian coordinate system with x -axis vertically upwards along the direction of flow and y -axis perpendicular to it. The origin of the axes is such that the channel walls are at positions $y = -d/2$ and $y = d/2$ [see Figure 1]. The velocity components are (u, v) relative to the Cartesian frame of reference. We do not model the pressure drop across the end caps and only consider the fully-developed flow far from the end caps.

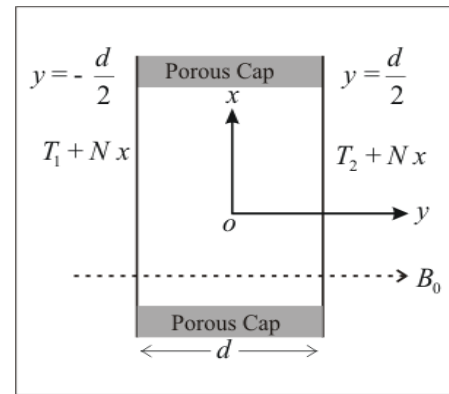


Figure 1: Geometry of the problem

The Boussinesq approximation is assumed to hold and for the evaluation of the gravitational body force, the density is assumed to be dependent on the temperature according to the equation of state

$$\rho = \rho_0[1 - \beta(T - T_0)], \quad (1)$$

where T , ρ , β , T_0 and ρ_0 are respectively, the fluid temperature, the fluid density, thermal expansion coefficient, the temperature and the density in the reference state.

Flow away from the top and bottom ends of the cavity is rectilinear so that $u = u(y)$, $v = 0$. In this case the equation of

continuity is satisfied identically. The y -component momentum equation can be written as $\frac{1}{\rho} \frac{\partial p}{\partial y} = 0$ which implies that

$p = p(x)$. Let us assume that the induced magnetic field produced by the motion of the conducting fluid is negligible so that $\vec{B} = (0, B_0, 0)$. This assumption is valid for a small magnetic Reynolds number. The solenoidal equation $\nabla \cdot \vec{B} = 0$ gives $B_y = \text{constant} = B_0$ everywhere in the flow. On using Boussinesq approximation (1), the momentum and energy equations are simplified to the following form

$$-\frac{\partial p}{\partial x} - \rho_0 g + \rho_0 g \beta (T - T_0) + \mu \frac{d^2 u}{dy^2} - B_0 j_z = 0, \quad (2)$$

$$u \frac{\partial T}{\partial x} = k \frac{d^2 T}{dy^2}, \quad (3)$$

where μ is the coefficient of viscosity and k the thermal diffusivity.

Since the induced magnetic field is neglected and flow is steady, the Maxwell's equation $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ yields $\nabla \times \vec{E} = 0$ which gives $\frac{\partial E_x}{\partial y} = 0$ and $\frac{\partial E_z}{\partial y} = 0$. This implies that $E_x = \text{constant}$ and $E_z = \text{constant}$ everywhere in the flow. We choose these constants equal to zero as there is no external electric field. Then, the Ohm's law

$$\vec{J} = \sigma(\vec{E} + \vec{q} \times \vec{B}) \quad (4)$$

gives

$$j_x = 0 \text{ and } j_z = \sigma B_0 u. \quad (5)$$

The temperature field in the cavity may be written as

$$T - T_0 = Nx + (T_2 - T_1)\theta, \quad (6)$$

where N is the temperature gradient.

On the use of (5) and (6), equation (2) becomes

$$\frac{dp^*}{dx} = g\beta(T - T_0) + \nu \frac{d^2 u}{dy^2} - \frac{\sigma B_0^2}{\rho_0} u, \quad (7)$$

where

$$p^* = \frac{p}{\rho_0} + g x - \frac{1}{2} g \beta N x^2. \quad (8)$$

The velocity and temperature boundary conditions are

$$\begin{aligned} u &= 0 \text{ at } y = \pm \frac{d}{2}, \\ T &= T_1 + Nx \text{ at } y = -\frac{d}{2} \\ T &= T_2 + Nx \text{ at } y = \frac{d}{2} \end{aligned} \quad (9)$$

Introducing the non-dimensional variables

$$\eta = \frac{y}{d}, \quad u_1 = \frac{u d}{\nu}, \quad (10)$$

equations (3) and (7) become

$$\frac{d^2 u_1}{d\eta^2} - M^2 u_1 + Gr \theta = -\alpha, \quad (11)$$

$$\frac{d^2 \theta}{d\eta^2} = Ec u_1, \quad (12)$$

where $M = B_0 d (\sigma / \rho_0 \nu)^{\frac{1}{2}}$ is the Hartmann number,

$Gr = \frac{g \beta (T_2 - T_1) d^3}{\nu^2}$ the Grashof number and $Ec = \frac{N \nu d}{k(T_2 - T_1)}$

the Eckert number and $\alpha = \frac{d^3}{\nu^2} \left(-\frac{\partial p^*}{\partial x} \right)$ the non-dimensional pressure gradient.

The velocity and the temperature boundary conditions (9) become

$$\begin{aligned} u_1 &= 0 \text{ at } \eta = \pm \frac{1}{2}, \\ \theta &= -\theta_0 \text{ at } \eta = -\frac{1}{2} \text{ and } \theta = 1 - \theta_0 \text{ at } \eta = \frac{1}{2}, \end{aligned} \quad (13)$$

where the parameter θ_0 measures the continuous cross-channel variation of the reference temperature T_0 . Assuming no external pressure gradient ($\alpha = 0$), the solutions of (11) and (12) subject to the boundary conditions (13) are

$$\begin{aligned} u_1(\eta) &= \frac{Gr}{m_1^2 - m_2^2} \left[\left(\frac{\cosh m_2 \eta}{\cosh \frac{m_2}{2}} - \frac{\cosh m_1 \eta}{\cosh \frac{m_1}{2}} \right) \left(\frac{1}{2} - \theta_0 \right) \right. \\ &\quad \left. + \frac{1}{2} \left(\frac{\sinh m_2 \eta}{\sinh \frac{m_2}{2}} - \frac{\sinh m_1 \eta}{\sinh \frac{m_1}{2}} \right) \right], \end{aligned} \quad (14)$$

$$\begin{aligned} \theta(\eta) &= \frac{1}{m_1^2 - m_2^2} \left[\left(\frac{m_1^2 \cosh m_2 \eta}{\cosh \frac{m_2}{2}} - \frac{m_2^2 \cosh m_1 \eta}{\cosh \frac{m_1}{2}} \right) \left(\frac{1}{2} - \theta_0 \right) \right. \\ &\quad \left. + \frac{1}{2} \left(\frac{m_1^2 \sinh m_2 \eta}{\sinh \frac{m_2}{2}} - \frac{m_2^2 \sinh m_1 \eta}{\sinh \frac{m_1}{2}} \right) \right], \end{aligned} \quad (15)$$

where

$$\begin{aligned} m_1 &= \alpha + i\beta, \quad m_2 = \alpha - i\beta, \\ \alpha &= \frac{1}{\sqrt{2}} \left[M^2 + \left(M^4 - 4EcGr \right)^{\frac{1}{2}} \right]^{\frac{1}{2}} \\ \beta &= \frac{1}{\sqrt{2}} \left[M^2 - \left(M^4 - 4EcGr \right)^{\frac{1}{2}} \right]^{\frac{1}{2}} \end{aligned} \quad \left. \vphantom{\begin{aligned} m_1 \\ m_2 \end{aligned}} \right\} \text{for } M^2 > \sqrt{4EcGr} \quad (16)$$

$$\begin{aligned} \alpha &= \frac{1}{2} \left[\sqrt{4EcGr} + M^2 \right]^{\frac{1}{2}} \\ \beta &= \frac{1}{2} \left[\sqrt{4EcGr} - M^2 \right]^{\frac{1}{2}} \end{aligned} \quad \left. \vphantom{\begin{aligned} \alpha \\ \beta \end{aligned}} \right\} \text{for } M^2 < \sqrt{4EcGr}. \quad (17)$$

Equation (14) shows that the velocity distribution is a combination of symmetric and antisymmetric solutions. It is

observed from the above equations (14) and (15) that both the velocity field and the temperature distribution depend on the Grashof number Gr . The solutions of the equations (11) and (12) given by (14) and (15) are valid for either $M^2 > \sqrt{4EcGr}$ or $M^2 < \sqrt{4EcGr}$.

3 RESULTS AND DISCUSSION

In order to study the effects of magnetic field, Grashof number and temperature on the velocity field u_1 and temperature distribution θ , we have presented the non-dimensional velocity u_1 and the temperature θ against η for various values of Hartmann number M , Grashof number Gr and the temperature parameter θ_0 in Figures 2 -7 for $Ec=1$. It is observed from Fig.2, that the velocity u_1 decreases with increase in Hartmann number M . This is expected, as the magnetic field exerts a retarding influence on the flow field. Figure 3 shows that the velocity field at any point increases with increase in Grashof number Gr . It is seen from Figure 4 that the velocity decreases with increase in θ_0 . It is observed from Figures 5 and 6 that at any point the temperature increases with increase in M^2 while decreases with increase in θ_0 . Figure 7 shows that temperature decreases with increase in Gr . Figures 2-4 show that there are no flow reversal near the cold wall, as it is seen in the case of without magnetic field [see Weideman [8]], this due to the fact that we have taken the values of θ_0 less than the critical values of θ_0 at the cold wall (critical values of θ_0 are given in Table 2). The incipient flow reversal will occur only for those values of θ_0 which are greater than the critical values of θ_0 at the cold plate.

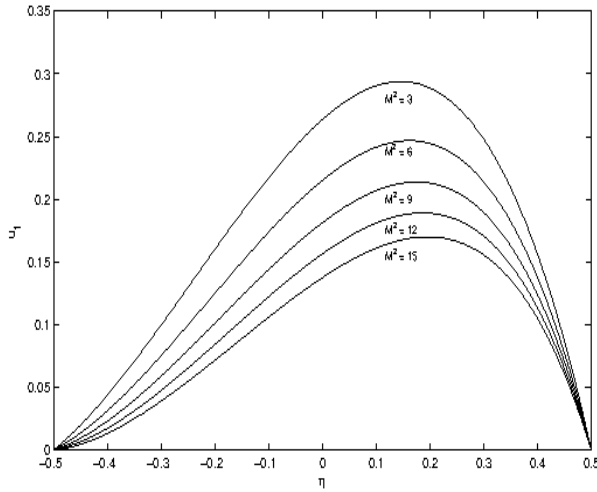


Figure 2: Variation of u_1 for $\theta_0 = 0.2$ and $Gr = 10$

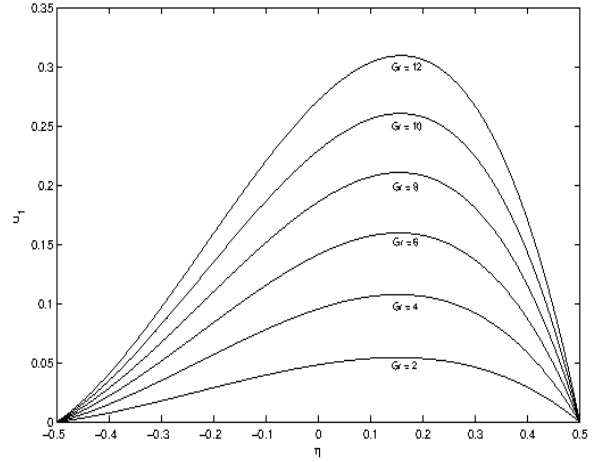


Figure 3: Variation of u_1 for $M^2 = 5$ and $\theta_0 = 0.2$

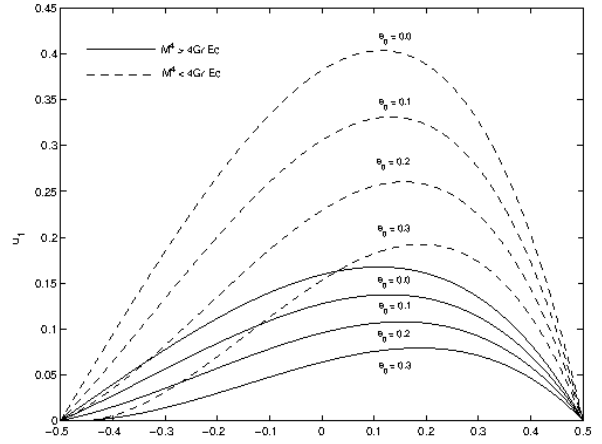


Figure 4: Variation of u_1 for the case $M^4 >$ and $< 4GrEc$.

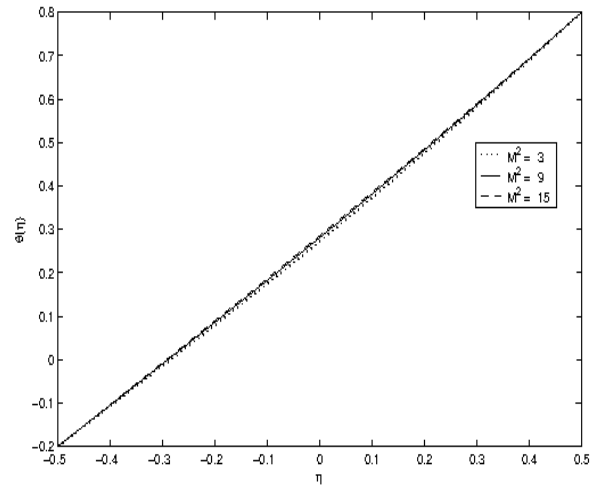


Figure 5: Variation of θ for $\theta_0 = 0.2$ and $Gr = 10$.

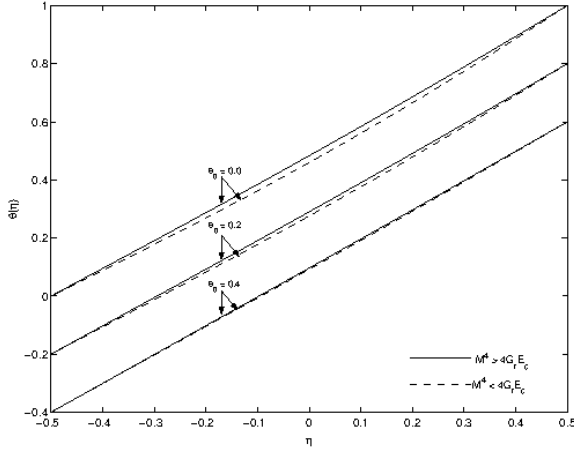


Figure 6: Variation of θ for the case $M^4 >$ and $< 4GrEc$.

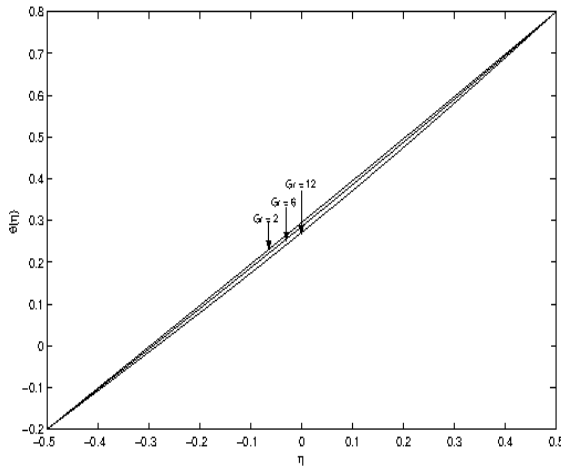


Figure 7: Variation of θ for $M^2 = 5$ and $\theta_0 = 0.2$.

The non-dimensional shear stress at the cold wall ($\eta = -1/2$) and hot wall ($\eta = 1/2$) are given by $\tau_{x_1} = \left(\frac{du_1}{d\eta} \right)_{\eta=-\frac{1}{2}}$ and

$$\tau_{x_2} = \left(\frac{du_1}{d\eta} \right)_{\eta=\frac{1}{2}}, \text{ where}$$

$$\left(\frac{du_1}{d\eta} \right)_{\eta=-\frac{1}{2}} = \frac{Gr}{m_1^2 - m_2^2} \left[\left(m_1 \tanh \frac{m_1}{2} - m_2 \tanh \frac{m_2}{2} \right) \left(\frac{1}{2} - \theta_0 \right) - \frac{1}{2} \left(m_1 \coth \frac{m_1}{2} - m_2 \coth \frac{m_2}{2} \right) \right], \quad (18)$$

and

$$\left(\frac{du_1}{d\eta} \right)_{\eta=\frac{1}{2}} = \frac{Gr}{m_1^2 - m_2^2} \left[\left(m_2 \tanh \frac{m_2}{2} - m_1 \tanh \frac{m_1}{2} \right) \left(\frac{1}{2} - \theta_0 \right) + \frac{1}{2} \left(m_2 \coth \frac{m_2}{2} - m_1 \coth \frac{m_1}{2} \right) \right]. \quad (19)$$

Numerical values of shear stresses at the cold wall ($\eta = -1/2$) and hot wall ($\eta = 1/2$) are presented in Table 1 for various values of θ_0 and M^2 taking $Ec = 1$ and $Gr = 10$. Table 1 shows that the shear stress at the cold wall ($\eta = -1/2$) decreases with increase in either M^2 or θ_0 . On the other hand, the magnitude of the shear stress at the hot wall ($\eta = 1/2$) decreases with increase in either M^2 or θ_0 .

Table 1: Shear stress at the walls due to the flow for $Ec = 1, Gr = 10$

$M^2 \setminus \theta_0$	τ_{x_1}			$-\tau_{x_2}$		
	0.0	0.2	0.4	0.0	0.2	0.4
5	1.59383	0.90656	0.21929	1.84251	1.15525	0.46798
10	1.22406	0.66260	0.10115	1.58321	1.02176	0.46031
15	1.07334	0.59067	0.10799	1.34003	0.85736	0.37468
20	0.95792	0.52967	0.10143	1.18330	0.75506	0.32681

As θ_0 decreases from the maximum value $\theta_0 = \frac{1}{2}$, we arrive to the value $\theta_0 = (\theta_0)_{\text{crit.}}$ for which there is an incipient flow reversal near the cold wall. The condition for incipient flow reversal can be obtained by letting $\left(\frac{du_1}{d\eta} \right)_{\eta=-\frac{1}{2}} = 0$ which in term gives the equation for the critical value of θ_0 as

$$(\theta_0)_{\text{crit.}} = \frac{1}{2} - \frac{m_2 \coth \frac{m_2}{2} - m_1 \coth \frac{m_1}{2}}{2 \left(m_2 \tanh \frac{m_2}{2} - m_1 \tanh \frac{m_1}{2} \right)}. \quad (20)$$

The values of $(\theta_0)_{\text{crit.}}$ are entered in the Table 2 for different values of M^2 and Gr . It is observed that the critical

$(\theta_0)_{\text{crit.}}$ decreases with increase in either M^2 or Gr .

Table 2: Critical values of θ_0 at the cold wall ($\eta = -1/2$) for $Ec = 1, \theta_0 = 0.2$

$M^2 \setminus Gr$	2	4	6	8	10
3	0.30101	0.29879	0.29659	0.29441	0.29225
6	0.27662	0.27471	0.27281	0.27093	0.26907
9	0.25630	0.25463	0.25298	0.25134	0.24971
12	0.23914	0.23767	0.23621	0.23476	0.23333
15	0.22447	0.22317	0.22187	0.22059	0.21931

The rate of volume flux is given by

$$Q = \frac{Gr}{m_1^2 - m_2^2} \left[\left(\frac{1}{2} - \theta_0 \right) \left(\frac{\tanh \frac{m_2}{2}}{m_2} - \frac{\tanh \frac{m_1}{2}}{m_1} \right) \right], \quad (21)$$

where m_1 and m_2 are given by equation (16). The above equation comes only from the symmetric portion of the velocity distribution. The numerical values of the rate of flow Q are given

in the Table 3 for different values of the Hartmann number M , the Grashof number Gr and the temperature parameter θ_0 . Table 3 shows that the rate of volume flux decreases with increase in either M^2 or θ_0 . On the other hand, the rate of volume flux increases with increase in Grashof number Gr as expected since the velocity increases with increase in Gr .

Table 3 : The rate of volume flux $10^{-1}Q$ for $Ec = 1$

M^2	Gr and $\theta_0 = 0.2$			θ_0 and $Gr = 10$		
	2	6	10	0.0	0.2	0.4
3	0.1895	0.5518	0.8933	1.4889	0.8933	0.2978
6	0.1547	0.4530	0.7372	1.2287	0.7372	0.2457
9	0.1308	0.3845	0.6281	1.0468	0.6281	0.2094
12	0.1134	0.3342	0.5475	0.9125	0.5475	0.1825

Equation (27) shows that if $\theta_0 = \frac{1}{2}$ then the rate of volume flux $Q = 0$, which means that the cavity is closed. On the other hand, the maximum rate of volume flux occurs at $\theta_0 = 0$ and is given by

$$Q = \frac{Gr}{2(m_1^2 - m_2^2)} \left(\frac{\tanh \frac{m_2}{2}}{m_2} - \frac{\tanh \frac{m_1}{2}}{m_1} \right). \quad (22)$$

We shall now discuss the case when the applied magnetic field is weak, i.e. for $M^2 = 1$. In this case the velocity field, temperature distribution and the flow rate become

$$u_1(\eta) = Gr \left[\left\{ \frac{1}{2} \left(\frac{1}{2} - \theta_0 \right) \left(\frac{1}{4} - \eta^2 \right) + \frac{1}{6} \left(\frac{1}{4} - \eta^2 \right) \eta \right\} - \frac{M^2}{384} (5 - 24\eta^2 + 16\eta^4) \left(\frac{1}{2} - \theta_0 \right) - \frac{M^2}{5760} (7\eta - 40\eta^3 + 48\eta^5) \right], \quad (23)$$

$$\theta(\eta) = \eta + \left(\frac{1}{2} - \theta_0 \right) - EcGr \left[\frac{1}{384} (5 - 24\eta^2 + 16\eta^4) \left(\frac{1}{2} - \theta_0 \right) \right]$$

$$+ \frac{1}{5760} (7\eta - 40\eta^3 + 48\eta^5) + \frac{M^2}{46080} (64\eta^6 - 240\eta^4 + 300\eta^2 - 61) \left(\frac{1}{2} - \theta_0 \right) + \frac{M^2}{967680} (192\eta^7 - 336\eta^5 + 196\eta^3 - 31\eta) \right], \quad (24)$$

$$Q = Gr \left(\frac{1}{2} - \theta_0 \right) \left(\frac{1}{12} + \frac{M^2}{120} \right). \quad (25)$$

Taking the limit $M \rightarrow 0$, the equations (23) - (25) become

$$u_1(\eta) = \frac{Gr}{6} \left(\frac{1}{4} - \eta^2 \right) \left[\eta + 3 \left(\frac{1}{2} - \theta_0 \right) \right], \quad (26)$$

$$\theta(\eta) = \eta + \left(\frac{1}{2} - \theta_0 \right) - EcGr \left[\frac{1}{384} (5 - 24\eta^2 + 16\eta^4) \left(\frac{1}{2} - \theta_0 \right) + \frac{1}{5760} (7\eta - 40\eta^3 + 48\eta^5) \right], \quad (27)$$

$$Q = \frac{Gr}{12} \left(\frac{1}{2} - \theta_0 \right). \quad (28)$$

Equations (26) - (28) identical with the equations (21) - (23) of Weidman [8]. Further, if $Ec = 0$, the temperature distribution given by equation (27), becomes

$$\theta(\eta) = \eta + \left(\frac{1}{2} - \theta_0 \right). \quad (29)$$

Again, the equation (29) identical with the equation (22) of Weidman [8]. The equations (23) - (25) give the conduction regime solution as reported by Bühler [7].

4 CONCLUSION

The fully developed free convective steady MHD channel flow of an incompressible, electrically conducting, viscous fluid has been analyzed in the presence of an external transverse magnetic field. The flow has been assumed to be parallel and each of the two boundary walls of the vertical slot have been considered as asymmetric heating. Numerical results are presented to account the effect of the magnetic field on the leading flow behavior. It is found that the fluid velocity field and temperature distribution are strongly affected by the magnetic field intensity. A limiting consideration of the flow has been verified.

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