Analysis of Stresses and Strains in a Rotating Homogeneous Thermoelastic Circular Disk by using Finite Element Method

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ABSTRACT
This study focuses on the finite element analysis of thermoelastic field in a thin circular disk subjected to a thermal load and an inertia force due to rotation of the disk. Based on the two dimensional thermoelastic theories the axisymmetric problem is formulated in terms of second order ordinary differential equation which is solved by FEM. Further the temperature profiles have been modeled with the help of heat conduction equation. Some numerical results of thermoelastic field are presented and discussed for Aluminum (Al) circular disk.

Keywords—FEM; Circular disk; Axisymmetric; Heat Conduction

1. INTRODUCTION
Rotating disks are one of the fundamental components of many machines and mechanisms and are often subjected to loading or excitation in the transverse (out-of-plane) direction. Afsar and Go [1] analysed thermoelastic characteristics of a thin circular FGM rotating disk having a concentric hole and subjected to a thermal load. Yongdong *et al.*. [2] and Zhong and Yu [3] established a mechanical model for the functionally gradient material (FGM) beam with rectangular cross section and also discussed the effects of the non-homogeneity parameter on the distribution of the normal stress and on the position of the natural axis for several different loading cases. The first linear analysis of rotating disks was performed by Lamb and Southwell [4], they obtained the solution for complete disk of uniform thickness, accounting for the effects of centrifugal and bending stresses. Mote [5] has used the Rayleigh-Ritz technique to investigate the free vibration characteristics of centrally clamped, variable thickness axisymmetric disks with axisymmetric in-plane stress distributions. An analytical solution in order to study deformation of a rotating disk composed of a linear, elastic, isotropic and homogeneous material by application of mechanical and thermal load have been proposed by Timoshenko and Goodier [6].

Cole and Benson [7] proposed a technique for determining the forced response to spatially fixed loads using an eigen function approach that predicts, in advance, the modes that dominate disk deflections. Benson [8] discussed the steady deflection of a transversely loaded, extremely flexible spinning disk using a hybrid of membrane and late theories when bending stiffness of the disk is small. Chen and Bogy [9] obtained the derivatives of the eigen values of a flexible spinning disk with a stationary load with respect to certain parameters in the system. Further, in another study Mote [10] has also been developed finite element method (FEM) procedures for plates with significant membrane stresses. Nigh and Olson [11] presented a FEM formulation for analysis of disks either in a body-fixed or a space-fixed co-ordinate system.

Finite Element Method (FEM) is one of the most successful and dominant numerical method in the last century. It is extensively used in modeling and simulation of engineering and science due to its versatility for complex geometries of solids and structures and its flexibility for many non-linear problems. The FEM is regarded as relatively accurate and versatile numerical tool for solving differential equations that model physical phenomenon [12-13]. The FEM is closely related to the classical variational concept of the Rayleigh Ritz method [14, 15]. The mathematical theory of the finite element has been developed and promoted by many researchers (Strang and Fax [16], Babuska *et al.*. [17] Oden & Reddy [18] and Ouria et.al.. [19].Abundant literatures related to disk problems can be found over the years. The motive behind the entire disk related problems are their industrial applications like circular saw blades, computer memory disks and disc brakes. It is noticed that many of these applications involves a rotating annular disk subjected to stationary load acted transversely. In the present paper finite element technique is used to evaluate stresses, strains and displacement for various cases of temperature distributions for homogenous rotating circular disk made up of Aluminum material.

2. MATHEMATICAL MODEL
We considered a circular disk with a concentric circular hole as shown in Fig. 1. The disk is assumed to be rotating with angular frequency ω. The origin of the polar coordinate system $r-\theta$ is assumed to be located at the center of the disk and hole.

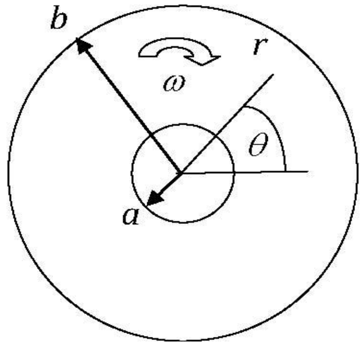

Figure 1. Schematic diagram of a rotating disk with concentric circular hole

3. BOUNDARY CONDITIONS

The disk considered in the present study is subjected to a temperature gradient field. The inner surface of the disk is assumed to be fixed to a shaft so that isothermal conditions can be prevailed on it. The outer surface of the disk is free from any mechanical load and maintained at uniform temperature gradient. Thus, the boundary conditions of the problem are given by

(i) $r = a,\ u_r = 0, T = 0$

(ii) $r = b,\ \sigma_r = 0, \dfrac{dT}{dr} = T_0$ (1)

where u_r and σ_r denote displacement and stress along the radial direction.

4. FORMULATION OF THE PROBLEM

When a material is subjected to a temperature gradient field, it experiences a stress arising from an incompatible eigen-strain. Eigenstrains (Dhaliwal and Singh [20]) are non-elastic strains or free expansion strains that develop in a body due to various reasons, such as phase transformation, precipitation, temperature change, etc. in the present study, the eigenstrain is associated with the thermal expansion of the disk. Since the material of the disk is isotropic, the thermal eigenstrain at a point is the same in all directions which can be given by

$$\varepsilon^* = \alpha(r)T(r) \tag{2}$$

Where $T(r)$ is the change in temperature at any distance r.

The total strain is the sum of the elastic strain and the eigen strain. Thus, the components of the total strain are given by

$$\varepsilon_r = e_r + \varepsilon^*,\ \varepsilon_\theta = e_\theta + \varepsilon^* \tag{3}$$

Where ε_r and ε_θ are the radial and circumferential component of the total strain and e_r and e_θ are the radial and circumferential components of the elastic strain. The elastic strains are related to stresses by Hooke's law. Thus

$$\varepsilon_r = \frac{1}{E}\left(\sigma_r - \nu\sigma_\theta\right) + \varepsilon^*;$$

$$\varepsilon_\theta = \frac{1}{E}\left(\sigma_\theta - \nu\sigma_r\right) + \varepsilon^* \tag{4}$$

Where σ_r and σ_θ are the radial and circumferential stress components, respectively. The two dimensional equilibrium equation in polar coordinates the inertia force due to rotation of the disk is given by

$$\frac{\partial\sigma_r}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{\sigma_r - \sigma_\theta}{r} + \rho\omega^2 r = 0$$

$$\frac{\partial\tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta} + \frac{2\tau_{r\theta}}{r} = 0 \tag{5}$$

where Ω is angular rotation of the disk and ω is being the angular frequency of the vibration modes in the disk. Because of symmetry, $\tau_{r\theta}$ vanishes and σ_r, σ_θ are independent of θ. Thus, the second Eq. (5) is identically satisfied and the first equilibrium equation is reduced to

$$\frac{d}{dr}(r\sigma_r) - \sigma_\theta + \rho\omega^2 r = 0 \tag{6}$$

Now, substitute $F = r\sigma_r$ into Eqs. (6) and (4) gives

$$\sigma_\theta = \frac{dF}{dr} + \rho\omega^2 r^2$$

$$\varepsilon_r = \frac{1}{E}\left(\frac{F}{r} - \nu\frac{dF}{dr}\right) - \frac{\nu\rho}{E}\omega^2 r^2 + \varepsilon^*;$$

$$\varepsilon_\theta = \frac{1}{E}\left(\frac{dF}{dr} - \frac{\nu F}{r}\right) + \frac{\rho}{E}\omega^2 r^2 + \varepsilon^* \tag{7}$$

Strain-displacement relations for the axisymmetric problem are

$$\varepsilon_r = \frac{du_r}{dr},\ \varepsilon_\theta = \frac{u_r}{r} \tag{8}$$

From Eq. (8), it is seen that two strain components are related by $\varepsilon_r = \dfrac{d}{dr}(r\varepsilon_\theta)$. By making use of Eq. (7) into this relation, we get

$$\frac{d^2F}{dr^2} + \frac{1}{r}\frac{dF}{dr} - \frac{1}{r^2}F + \rho\omega^2 r(3+\nu) + E\alpha\frac{dT}{dr} = 0 \tag{9}$$

The heat conduction equation for a dynamic coupled thermoelastic solid is given by [Dhaliwal and Singh [20]]

$$K\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)T - \rho C_e\frac{\partial T}{\partial t} = \frac{E\alpha T_0(\dot{e}_r + \dot{e}_\theta)}{1-\nu} \tag{10}$$

Where K is the thermal conductivity, C_e -Specific heat at constant strain and T_0 being uniform reference temperature The equations (9) and (10) constitute the mathematical model consisting of second order differential equations which provides us the function F and the components of stress.

5. TEMPERATURE FIELD

We shall consider following three cases of thermal variations in the disk:

Case I: Disk having uniform temperature distribution.

In this case, we have

$$T(r) = T_0, \frac{dT}{dr} = 0 \tag{11}$$

Case II: Disk at steady state temperature distribution.

In this case $\dfrac{\partial}{\partial t} \cong 0$, so that the heat conduction equation (10) takes the form

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)T = 0$$

Upon solving this equation with the help of thermal conditions (4), we obtain

$$T(r) = bT_0\log(r/a), \frac{dT}{dr} = \frac{bT_0}{r} \tag{12}$$

Case III: Non-heat conducting disk.

In this case thermal conductivity $K = 0$, so that eq. (10), leads to the temperature relation given by

$$T = \frac{-T_0\alpha}{\rho Ce(1-\nu)}(e_r + e_\theta) \tag{13}$$

Upon substituting the values of e_r and e_θ from eqs. (3) into eq. (13) and rearranging the terms, we get

$$T(r) = \frac{-T_0\alpha}{\rho Ce}\left(\frac{dF}{dr} + \frac{1}{r}F + \rho\omega^2 r^2\right) \tag{14}$$

$$\frac{dT}{dr} = \frac{-T_0\alpha}{\rho Ce}\left(\frac{d^2F}{dr^2} + \frac{1}{r}\frac{dF}{dr} - \frac{1}{r^2}F + 2\rho\omega^2 r\right) \tag{15}$$

These temperature distributions given by Eqs.(11), (12) and (14) holds in the domain $a \le r \le b$.

6. FINITE ELEMENT FORMULATION

Substituting the values of $\frac{dT}{dr}$ from equations (11), (12) and (14) in Eq. (9) and following a standard finite element discretization approach, the domain of the disk is divided radially into N number of elements of equal size and above Eq. (9) can be transformed to the following system of simultaneous algebraic equations:

$$\sum_{j=1}^{2} K_{ij}^e F_j^e = L_i^e;\ i = 1,2 : e = 1,2,...,N \tag{16}$$

where

$$K_{ij}^e = \int_{r_e}^{r_{e+1}} \frac{d\phi_i^e}{dr}\frac{d\phi_j^e}{dr}dr - \int_{r_e}^{r_{e+1}} \frac{1}{r}\phi_i^e \frac{d\phi_j^e}{dr}dr + \int_{r_e}^{r_{e+1}} \frac{1}{r^2}\phi_i^e\phi_j^e dr$$

$$L_i^e = \int_{r_e}^{r_{e+1}} \phi_i^e f(r)dr + \phi_i^e(r_{e+1})\frac{d\phi_j^e}{dr}(r_{e+1}) - \phi_i^e(r_e)\frac{d\phi_j^e}{dr}(r_e)$$

$f(r) = \rho\omega^2 r(\nu+3)$; (Case-I)

$f(r) = E\alpha\frac{bT_0}{r} + \rho\omega^2 r(\nu+3)$; (Case-II)

$f(r) = \rho\omega^2 r\left[\frac{(\nu+3)\rho Ce - 2ET_0\alpha^2}{\rho Ce - T_0 E\alpha^2}\right]$; (Case-III)

$$\phi_1^e = \frac{r_{e+1} - r}{r_{e+1} - r_e};\ \phi_2^e = \frac{r - r_e}{r_{e+1} - r_e} \tag{17}$$

The symbol e used in the above equation indicates the element number which is used to discretize the domain of the disk. Once the value of F calculated, various components of stress, strain and displacement can be easily calculated by the following relations:

$$\sigma_r = \frac{1}{r}\sum_{j=1}^{2} F_j^e\phi_j^e \quad , \quad \sigma_\theta = \sum_{j=1}^{2} F_j^e\frac{d\phi_j^e}{dr} + \rho\omega^2 r^2$$

$$\varepsilon_r = \frac{1}{E}\sum_{j=1}^{2}\left[\frac{F_j^e\phi_j^e}{r} - \nu F_j^e\frac{d\phi_j^e}{dr}\right] - \frac{\nu\rho\omega^2 r^2}{E} + \varepsilon^*$$

$$\varepsilon_\theta = \frac{1}{E}\sum_{j=1}^{2}\left[F_j^e\frac{d\phi_j^e}{dr} - \frac{\nu}{r}F_j^e\phi_j^e\right] + \frac{\nu\rho\omega^2 r^2}{E} + \varepsilon^*$$

$$u_r = \frac{r}{E}\sum_{j=1}^{2}\left[F_j^e\frac{d\phi_j^e}{dr} - \frac{\nu}{r}F_j^e\phi_j^e\right] + \frac{\nu\rho\omega^2 r^3}{E} + r\varepsilon^*$$

7. RESULT AND DISCUSSION

In this section, some numerical results of thermoelastic field i.e. different components of stress, strain and displacement are presented for Al disk. The mechanical and thermal properties of the disk are same as that of Afsar and Go [1]. The Poisson's ratio is taken as 0.3 which is constant throughout the material. The accuracy of the finite element results is verified by varying the element size used to discretize the disk and it is found that all the results converge very well when the element size is 1mm for a disk of a = 15 mm and b = 150 mm.

To examine the effect of temperature distribution profiles on the components of stress, strain and displacement, three different cases are considered. Further the angular speed ω= 1rad/s and the ratio of the outer radius to the inner radius b / a = 10 .

Fig. 2 illustrates the variation of radial stress versus $(r - a)/(b - a)$ for various cases of temperature distribution. It is observed that the radial stress is maximum in magnitude for Case-II for the lower range of $(r - a)/(b - a)$ value. But as the ratio $(r - a)/(b - a)$ increases the radial stress behavior also get changed to Case-III > Case-II > Case-I.

The variation of circumferential stress with $(r - a)/(b - a)$ ratio for different cases is shown in Fig. 3. It is observed from the figure that circumferential stress is positive for the whole range of $(r - a)/(b - a)$ ratio for case I and increases slightly in the lower range of $(r - a)/(b - a)$ ratio i.e. from 0-0.6 and then increases gradually from 0.6-1.0. For case II, circumferential stress in negative in lower range of $(r - a)/(b - a)$ ratio i.e. from 0-0.3approx., and changes its sign as we move towards the higher values of $(r - a)/(b - a)$ ratio. In case III, it is observed that the circumferential stress is negative for all lower values of $(r - a)/(b - a)$ ratio and is almost constant for. 0-0.6 $(r - a)/(b - a)$ ratio. A dip is observed for circumferential stress for this case when the value of $(r - a)/(b - a)$ ratio reaches about 0.7, and after that a sudden increase in the value of circumferential stress is noticed towards the positive end. The trend can be written to show the variation of circumferential stress with the $(r - a)/(b - a)$ ratio for the lower range is Case III >Case II > Case I (in magnitude), but the trend changes for the higher range as Case III> Case I> Case II. Fig. 4 shows the variation of radial strain along the $(r - a)/(b - a)$ ratio for three different cases. It is observed that radial strain is almost constant for Case I. And for Case II and Case III the behavior of radial strain is almost similar for lower range of $(r - a)/(b - a)$ ratio as in both cases radial strain is strictly changes its sign from positive to negative values. Fig. 5 shows the variation of circumferential strain for various cases of temperature distributions. It is observed that circumferential strain increases linearly up to $(r - a)/(b - a)$ =0.2 then remains constant and positive for Case I for whole range of $(r - a)/(b - a)$ ratio. In case II, circumferential strain shows a fluctuating behavior across the whole the range of $(r-a)/(b-a)$ ratio. In Case III, a quite different behavior is observed as circumferential strain is negative up to $(r-a)/(b-a)=0.8$ and after that it suddenly increases and become positive. The variation of displacement with $(r-a)/(b-a)$ ratio is reported in Fig. 6 for various cases of temperature distributions. A positive displacement with a steady behavior is observed for Case I and the Case II shows a similar behavior with opposite sign. Displacement shows a fluctuating behavior in lower range of $(r-a)/(b-a)$ ratio (from 0-0.6), then increases sharply in the higher range of $(r-a)/(b-a)$ ratio.

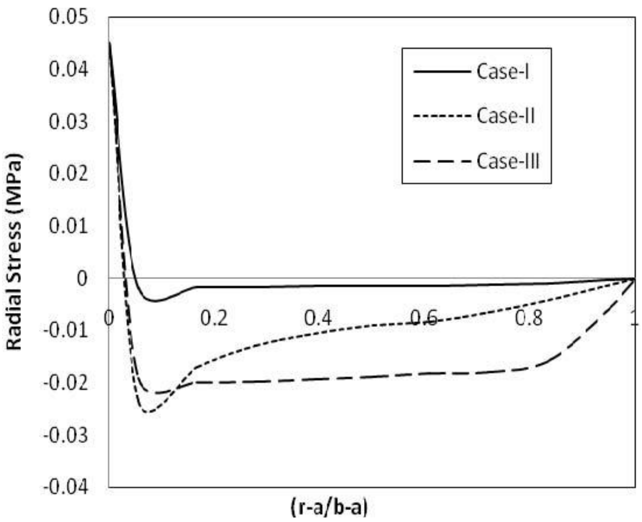

Fig. 2: Radial Stress versus $(r-a)/(b-a)$ for various cases of temperature distribution.

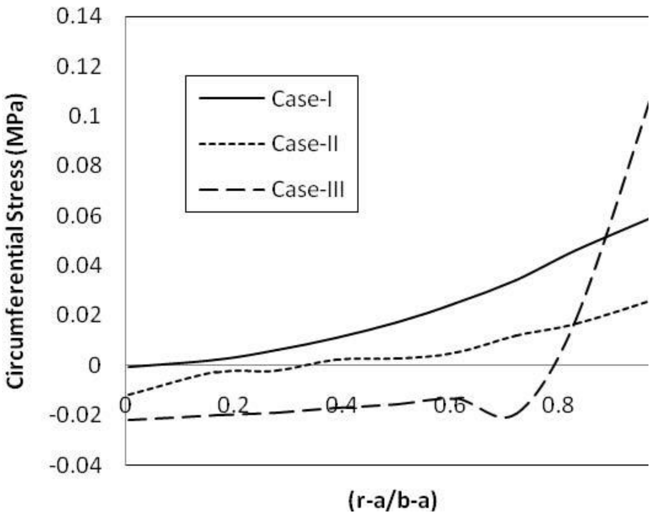

Fig. 3: Circumferential Stress versus $(r-a)/(b-a)$ for various cases of temperature distribution.

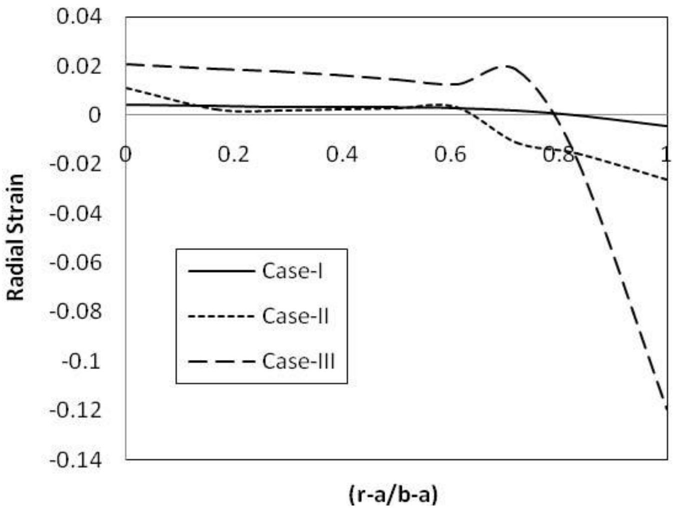

Fig. 4: Radial Strain versus $(r-a)/(b-a)$ for various cases of temperature distribution.

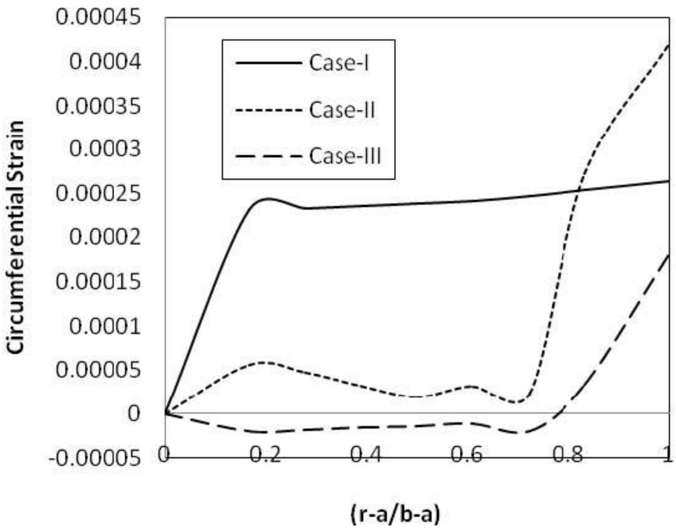

Fig. 5: Circumferential Strain versus $(r-a)/(b-a)$ for various cases of temperature distribution.

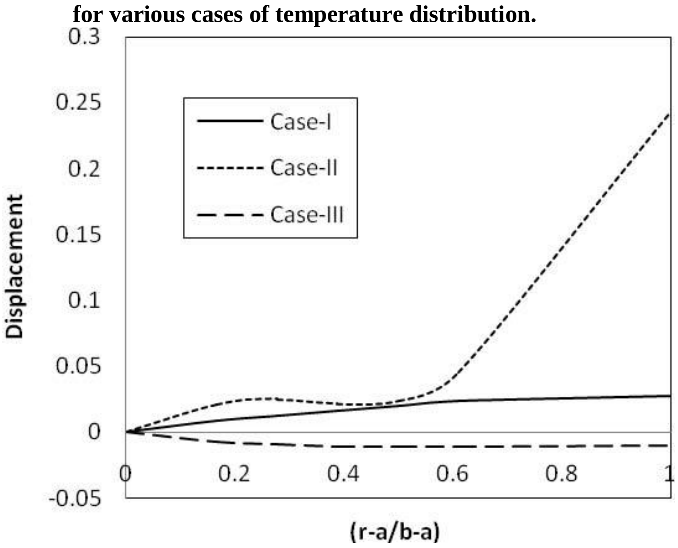

Fig. 6: Displacement versus $(r-a)/(b-a)$ for various cases of temperature distribution.

8. CONCLUSION

An analytical solution for thermoelasticity equilibrium equations of a thin axisymmetric rotating disk made of an isotropic material is presented. The variation of different components of stress, strain and displacement in radial direction is measured by applying thermal load with the help of finite element method. It is found that the thermoelastic field in disk is significantly influenced by the temperature distribution profile. Thus, the thermoelastic field in a disk can be controlled and optimized by controlling these parameters. Finally the model is helpful in designing circular cutter or grinding disk.

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