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Theory of Memristive Controllers: Design and Stability Analysis for Linear Plants

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ABSTRACT
Memristive System is a class on non-linear systems with very interesting properties. It is considered to be the fourth basic circuit element like Resistors, Capacitors and Inductors. Till date most of the works on memristive systems concentrated on its applications in the field of designing super dense non volatile memory, crossbar latches, neural networks, modeling of neural synapses, nonlinear oscillators and filters. Much less work has been done in its use in the field of control theory. This paper presents groundwork in the field of using Memristive Systems for control purposes.

General Terms
Nonlinear Control Theory

Keywords
Memsristor, Non Linear Control, Describing Function, Local Stability, Region of Attraction

1. INTRODUCTION
The existence of memristor was proposed by Leon Chua [1] around four decades back. His argument was build on the fact that since there are four very fundamental variables of electromagnetism v, i, q and Φ there should be six relations relating one variable to another but only five of them were known. Among the five relationships two of them followed directly while the other three was given by the three circuit elements Resistors (R), Capacitors (C) and Inductors (L). He argued that there should be a fourth basic circuit element.

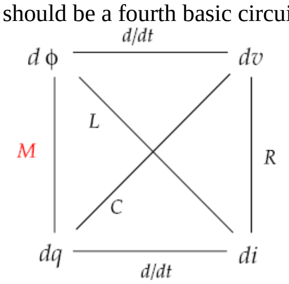


Fig 1: Relations between the four fundamental variables of electromagnetism: Flux Φ, Voltage v, Charge q and Current i. Among them five of the relationship were well known but the relationship between Flux Φ and Charge q was missing. Based on symmetry he argued the existence of memristor (M) which relates these two variables.

Not much importance was given to his work until 2008 when a group of four researchers from HP Labs headed by Stanley Williams [2] physically realized memristor. From then a lot of research has taken in the field of modeling and application of memristor [3, 4, 5, 6, 7]. The application was mainly concentrated in the field of non volatile memory, crossbar latches, neural networks and filters. Except [8,9] almost no work has been done in the field of using memristors in the field of control engineering. In this paper we lay the groundwork of using memristors or in more general memristive systems as a controller.

2. THE MEMRISTOR
According to Chua the flux Φ and the charge q should be related by a function f such that,

$$\phi = f(q) \tag{1}$$

Differentiating (1) on both the sides we get,

$$\frac{d\phi}{dt} = \frac{d}{dq} f(q) * \frac{dq}{dt} \tag{2}$$

Using Lenz's Law we get,

$$V = M(q) * i \tag{3}$$

where,

$$q = \int_0^t i dt \text{ and } M(q) = \frac{d}{dq} f(q)$$

So memristor behaves as a resistor of resistance $M(q)$ which vary with the amount of charge that passes through it. $M(q)$ is called the memristance of the memristor. The memristor synthesized by the HP Labs consist is a thin film 5 nm thick. According to the simplest model [4], the linear drift model, the memristance $M(q)$ of this memristor can be given by the following relationship:

$$M(q) = \begin{cases} R_{OFF}, & q \le q_{MIN} \\ R_o - \alpha * q, & q_{MIN} < q < q_{MAX} \\ R_{ON}, & q \ge q_{MAX} \end{cases} \tag{4}$$

where,

$$\alpha = \frac{R_{OFF} - R_{ON}}{\Delta Q} \text{ and } \Delta Q = q_{MAX} - q_{MIN}$$

In practice, $R_{OFF} >> R_{ON}$. The symbol of a memristor is shown in Figure. 2. It is a general convention that when the current enters the memristor from the terminal which is marked in bold (terminal A) the resistance of the memristor decreases while if the current enters from the other terminal (terminal B) its resistance increases.

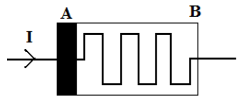


Fig 2: Memristor Symbol

3. PROBLEM DEFINATION

The work in hand consists of making a memristive controller for a linear plant. Consider a time-invariant linear plant given by the state equation:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \\ y(t) &= \mathbf{C}\mathbf{x}(t)\end{aligned} \tag{5}$$

We consider a n^{th} order, SISO system. Hence, $\mathbf{x}(t) \in \mathbf{R}^{n*1}, \mathbf{A} \in \mathbf{R}^{n*n}, \mathbf{B} \in \mathbf{R}^{n*1}$ and $\mathbf{C} \in \mathbf{R}^{1*n}$. $u(t)$ and $y(t)$ are scalars since we are dealing with a SISO system. The idea is to design a memristive controller to control the output $y(t)$ of the system.

The organization of the paper from here on is as follows. In Section 4 we summarize the controller architecture and derive the controller input output equation. Section 5 deals with stability analysis while section 6 gives a qualitative overview of the time domain performance aspects of the memristive controller. Section 7 provide simulation studies.

4. MEMRISTIVE CONTROLLER

Feedback control scheme has been chosen to control the linear plant for its widespread popularity. A feedback control loop with a memristive controller is shown in Figure. 3.

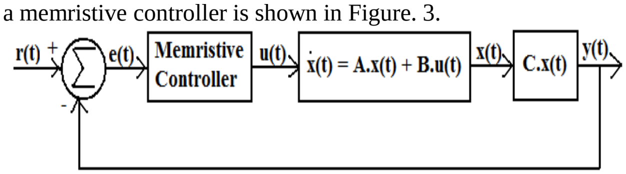
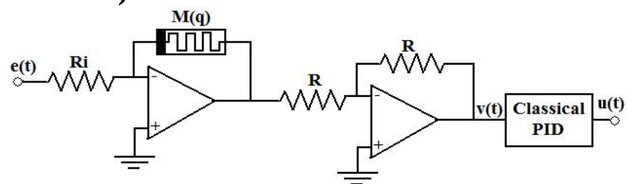


Fig 3: Linear Plant with a memristive controller in a feedback loop. r(t) is the setpoint, y(t) is the plants output or the process variable, e(t) is the error signal or the difference between plant output and the desired setpoint, u(t) is the input to the plant or the manipulated variable and x(t) are the internal states of the plant.

Classical PID controllers have been the favourite choice for controlling any process in a feedback loop. We propose two different architectures of the memristive controller which is merely a simple modification of the existing classical PID controllers.

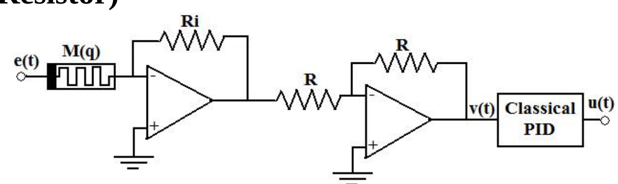
4.1 Architecture-1 (Memristor as Feedback Resistor)



The relation between $e(t)$ and $v(t)$ for the above architecture is given by,

$$v(t) = \begin{cases} \dfrac{R_{OFF}}{R_i} e(t), & \displaystyle\int_0^t \frac{e(\tau)}{R_i} d\tau \le q_{MIN} \\ \left(\dfrac{R_o}{R_i} - \dfrac{\alpha}{R_i^2} \displaystyle\int_0^t e(\tau) d\tau \right) e(t), & q_{MIN} < \displaystyle\int_0^t \frac{e(\tau)}{R_i} d\tau < q_{MAX} \\ \dfrac{R_{ON}}{R_i} e(t), & \displaystyle\int_0^t \frac{e(\tau)}{R_i} d\tau \ge q_{MAX} \end{cases} \tag{6}$$

4.2 Architecture-2 (Memristor as Input Resistor)



In this configuration the current $i(t)$ through the memristor entering from terminal A is given by,

$$i(t) = \frac{e(t)}{M(q)} \tag{7}$$

For this configuration the variation memristance as a function of time has been derived in [5] and is given by,

$$M = \sqrt{R_o^{\,2} - 2\alpha \int_0^t e(\tau) d\tau} \tag{8}$$

Substituting (8) in (7) we get,

$$i(t) = \frac{e(t)}{\sqrt{R_o^{\,2} - 2\alpha \int_0^t e(\tau) d\tau}} \tag{9}$$

The relation between e(t) and v(t) in this architecture is given by,

$$v(t) = \begin{cases} \dfrac{R_i}{R_{OFF}} e(t), & \displaystyle\int_0^t i(\tau) d\tau \le q_{MIN} \\ \dfrac{R_i}{M} e(t), & q_{MIN} < \displaystyle\int_0^t i(\tau) d\tau < q_{MAX} \\ \dfrac{R_i}{R_{ON}} e(t), & \displaystyle\int_0^t i(\tau) d\tau \ge q_{MAX} \end{cases} \tag{10}$$

For both the above architecture the input-output relationship of the classical PID is given by,

$$u(t) = v(t) + K_I \int_0^t v(\tau) d\tau + K_D \frac{dv}{dt} \tag{11}$$

where, K_I and K_D are integral and derivative time constant respectively.
Since the same performance can be achieved by using any of the above architecture just by altering the controller parameters (R_o , R_{ON} , R_{OFF} , R_i , α), we will prefer using architecture-I for its simplicity of input-output relationship.

5. STABILITY ANALYSIS

5.1 Describing Function Analysis for Limit Cycle Prediction

Describing function has been the most wildly used tool for understanding the existence of limit cycle in a feedback loop. A. Delgado [8] derived the describing function of the memristor considering its non-linearity to be static. Though the work is appreciable but approximation of the input-output relation as static nonlinearity is not acceptable as the input-output relation vary significantly with the frequency of the input signal proving that the non-linearity is dynamic not static. The following input output relation (pinched-hysteresis loop) adapted from [4] validates our conclusion.

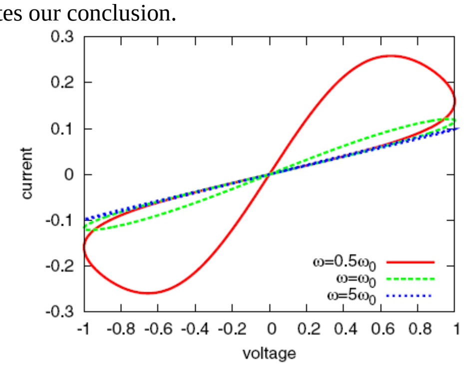


Fig 4: The pinched-hysteresis loop exhibited by a typical memristor for various frequencies of the input sinosodial signal.

We will first derive the describing function $N(a, w)$ of the memristive controller without the classical PID and then merge it with the magnitude and phase relation of the classical PID to get the final result $\psi(a, w)$.

Consider,

$$e(t) = a * Sin(wt)$$

Then the describing function is given by,

$$N(a,w) = \frac{jw}{\pi a} \int_0^{\frac{2\pi}{w}} v(t) e^{jwt} dt \qquad (12)$$

Then,

$$q(t) = \int_0^t \frac{e(\tau)}{R_i} d\tau = \frac{a}{wR_i}[1 - Cos(wt)] \qquad (13)$$

So, $q(t) \in [0, \frac{2a}{wR_i}]$.

There are two cases possible:

Case-I $\left(\frac{2a}{wR_i} \le q_{MAX} \right)$

In this case the memristor never goes in the saturation zone and the output $v(t)$ is given by:

$$v(t) = \left[\frac{R_o}{R_i} - \frac{\alpha}{R_i^2} * \frac{a}{w} (1 - Cos(wt) \right] a * Sin(wt) \qquad (14)$$

$$= \left(\frac{R_o}{R_i} - \frac{\alpha a}{wR_i^2} \right) a * Sin(wt) + \frac{\alpha a^2}{2wR_i^2} Sin(2wt) \qquad (15)$$

Substituting equation (15) in (12) and evaluating it using Mathematica 4.1 gives,

$$N(a, w) = \frac{R_o}{R_i} - \frac{\alpha a}{wR_i^2} \qquad (16)$$

Case-II $\left(\frac{2a}{wR_i} > q_{MAX} \right)$

In such cases the memristor will reach saturation at a time t_o and come back to non-saturation zone at a time given by $\frac{2\pi}{w} - t_o$, where t_o is given by,

$$q_{MAX} = \frac{a}{wR_i}[1 - Cos(wt_o)] \qquad (17)$$

Using equation (4) we get $q_{MAX} = \frac{(R_o - R_{ON})}{\alpha}$. Substituting in equation (17) we get,

$$t_o = \frac{1}{w} Cos^{-1}\left[1 - \frac{wR_i}{\alpha a}(R_o - R_{ON}) \right] \qquad (18)$$

The output $v(t)$ is given by,

$$v(t) = \begin{cases} \left[\frac{R_o}{R_i} - \frac{\alpha a}{wR_i^2}(1 - Cos(wt)) \right] a * Sin(wt), & 0 < t < t_o \\ \frac{R_{ON}}{R_i} a * Sin(wt), & t_o \le t \le \frac{2\pi}{w} - t_o \\ \left[\frac{R_o}{R_i} - \frac{\alpha a}{wR_i^2}(Cos(wt_o) - Cos(wt)) \right] a * Sin(wt), & \frac{2\pi}{w} - t_o < t < \frac{2\pi}{w} \end{cases} \qquad (19)$$

By substituting t_o from equation (18) into equation (19) and solving it yields,

$$v(t) = \begin{cases} \left(\frac{R_o}{R_i} - \frac{\alpha a}{wR_i^2} \right) a * Sin(wt) + \frac{\alpha a^2}{2wR_i^2} Sin(2wt), & 0 < t < t_o \\ \frac{R_{ON}}{R_i} a * Sin(wt), & t_o \le t \le \frac{2\pi}{w} - t_o \\ \left(\frac{R_o}{R_i} - \frac{\alpha a}{wR_i^2} \right) a * Sin(wt) + \frac{\alpha a^2}{2wR_i^2} Sin(2wt), & \frac{2\pi}{w} - t_o < t < \frac{2\pi}{w} \end{cases} \qquad (20)$$

Substituting equation (20) in (12) and evaluating it using Mathematica 4.1 gives,

$$N(a, w) = \frac{R_{ON}}{R_i} + \frac{2}{3\pi} * \frac{\alpha a}{wR_i^2} * Sin^3(wt_o) + \frac{1}{\pi}\left[\left(\frac{R_{ON} - R_o}{R_i} \right) + \frac{\alpha a}{wR_i^2} \right] \left[\frac{Sin(2wt_o)}{2} - wt_o \right] \qquad (21)$$

Hence equation (16) and (21) are the describing functions of the memristive controller for two different conditions.

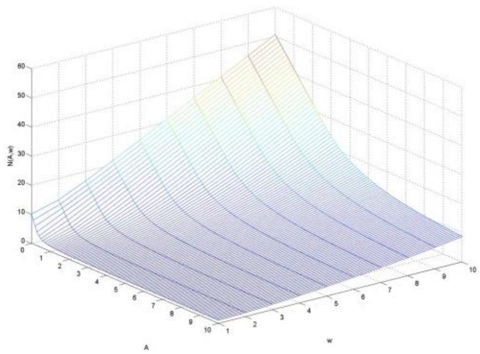


Fig 5: A plot showing variation of N(A,w) with amplitude A and frequency w.In this plot Ron=0.5,Roff=5,Ri=0.5, α =4.5

The transfer function $G(s)$ of the PID described by equation (11) is,

$$G(s)=\frac{u(s)}{v(s)}=1+\frac{K_I}{s}+sK_D \qquad (22)$$

Substituting $s = jw$ yields,

$$G(jw)=\left|G(jw)\right|e^{j\angle G(jw)} \qquad (23)$$

where,

$\left|G(jw)\right|=\sqrt{1+\left(wK_D-\frac{K_I}{w}\right)^2}$ and $\angle G(jw)=Tan^{-1}\left(wK_D-\frac{K_I}{w}\right)$

The describing function $\psi(a,w)$ of the memristive controller with the classical PID is given by,

$$\psi(a,w)=N(a,w)*G(jw) \qquad (24)$$

The closed loop transfer function $G_c(s)$ of the system shown in Figure. 3 is,

$$G_c(s)=\frac{\psi(a,w)G_p(s)}{1+\psi(a,w)G_p(s)} \qquad (25)$$

where, $G_p(s)$ is the transfer function of the linear plant.

The limit cycles and hence the stability conditions for the closed loop can be derived from the characteristic equation,

$$1+\psi(a,w)G_p(s)=0 \qquad (26)$$

5.2 Stability around and equilibrium point

Memristive controller proposed is basically a nonlinear PID. A lot of work has been done in the past three decades to study stability of nonlinear PIDs around an equilibrium point mostly using Lyapunov stability analysis. Some notable ones are by Arimoto [10], Kelly[11], Seraji[12]. Other recent works on nonlinear PID can be found in [13], [14], [15].

Throughout the rest of the paper we use the notation $\lambda_m(A)$ and $\lambda_M(A)$ to indicate smallest and largest eigenvalues of a matrix A. The norm of a matrix A is defined as $\|A\|=\sqrt{\lambda_M(A^TA)}$ and that of a vector x is given by $\|x\|=\sqrt{x^Tx}$. The notation $\mathrm{Re}(y)$ means the real part of a variable y. To analyze the stability around an equilibrium point we approach the problem in a rather indirect way by analyzing the region of attraction for a given set of controller parameters. The method used to calculate the region of attraction is similar to the one used by Khalil [16].

To simplify the work further presented in the paper we make certain assumptions:

Assumption-1:

The linear plant described by equation (5) is asymptotically stable. Mathematically speaking, the real part of the eignvalues of matrix A are negative or $\mathrm{Re}(\lambda_M(A))<0$. In other words the matrix A is Hurwitz. The differential equation notation of the n^{th} order linear plant as described by equation (5) is :

$$a_n\frac{d^ny}{dt^n}+a_{n-1}\frac{d^{n-1}y}{dt^{n-1}}+.....+a_2\frac{d^2y}{dt^2}+a_1\frac{dy}{dt}+a_0y=u \qquad (27)$$

Assumption-2:

We assume that $q_{MIN}=0$ and $q_{MAX}=\Delta Q$. Hence, $R_o=R_{OFF}$.

Assumption-3:

It is to be understood that the closed loop system with the memristive controller is a switched nonlinear system. In order to avoid lyapunov analysis of a switched nonlinear system it will be worth approximating $M(q)$ by a single function. It can be shown that the function as described by equation (4) can be approximated by the following function:

$$f(q)=A+B*Tanh[-C*(q-D)] \qquad (28)$$

where, $Tanh(x)=\frac{e^x-e^{-x}}{e^x+e^{-x}}$ and A,B,C,D are positive constants obtained by least square fit.

$$A=\frac{(R_{OFF}+R_{ON})}{2}$$
$$B=\frac{(R_{OFF}-R_{ON})}{2} \qquad (29)$$
$$C=\frac{2.4375}{\Delta Q}$$
$$D=\frac{\Delta Q}{2}$$

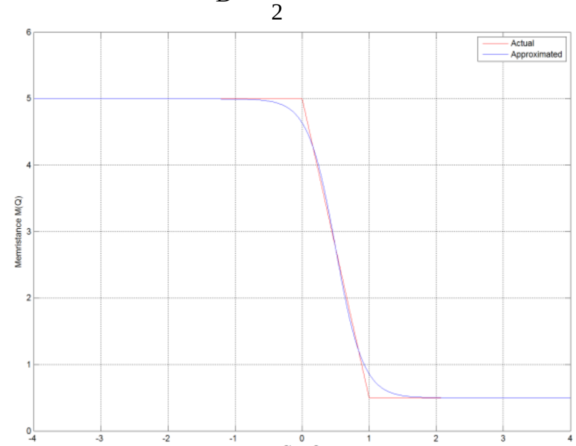


Fig 6: A typical least square fit of equation (28) and equation (4). In this plot Ron=0.5, Roff=5 and ΔQ=1.

To find the region of attraction of the closed loop system we proceed in the following way. First we derive the output of the memristive controller with the classical PID. Then we represent the closed loop system using differential equation and state space notation. It is followed by deriving conditions for local stabilization of the closed loop system and then the region of attraction for a given set of controller parameter.

5.2.1 Output of Memristive PID Controller

From hereon a variable x which is a function of time, i.e. $x(t)$ will just be represented as x. If otherwise the independent variable will be explicitly mentioned in parenthesis. The notation $\dot{x}$ represents the first derivative of x.

For architecture-I $q=\int_0^t \frac{e(\tau)}{R_i}d\tau$. For further analysis we replace variable q by z. Also we introduce a new variable $\delta=\frac{e}{R_i}$. Hence $z=\int_0^t \delta(\tau)d\tau$.

The input $v(t)$ to the classical PID is given by,

$$v(t)=f\left(\int_0^t \delta(\tau)d\tau\right)\delta \tag{30}$$

Hence the final output u of the controller is,

$$u=f\left(\int_0^t \delta(\tau)d\tau)\right)\delta+K_I\int_0^t f\left(\int_0^\tau \delta(\eta)d\eta\right)\delta(\tau)d\tau$$
$$+K_D\frac{d}{dt}\left[f\left(\int_0^t \delta(\tau)d\tau\right)\delta\right] \tag{31}$$

Solving equation (31) we get,

$$u=f(z)\delta+K_Ih(z)+K_Df(z)\dot{\delta}+K_Dg(z)\delta^2 \tag{32}$$

where,

$$h(z)=\int f(z)dz \text{ and } g(z)=\frac{d}{dz}f(z)$$

5.2.2 Closed Loop Dynamics

Substituting equation (32) in equation (27) we get the differential equation governing the closed loop dynamics,

$$\sum_{j=0}^n a_j\frac{d^jy}{dt^j}=f(z)\delta+K_Ih(z)+K_Df(z)\dot{\delta}+K_Dg(z)\delta^2 \tag{33}$$

We consider the set-point tracking problem with desired position $y=y_d$. Hence, $e=y_d-y$ or $R_i\delta=y_d-y$. Hence equation (33) can be re-written as,

$$-R_i\sum_{j=0}^n a_j\frac{d^j\delta}{dt^j}+a_0y_d=f(z)\delta+K_Ih(z)+K_Df(z)\dot{\delta}+K_Dg(z)\delta^2 \tag{34}$$

At equilibrium,

$$a_0y_d=K_Ih(z_d) \tag{35}$$

We define a new variable $\sigma=z-z_d$ and substitute it in equation (34),

$$-R_i\sum_{j=0}^n a_j\frac{d^j\delta}{dt^j}+a_0y_d=f(\sigma+z_d)\delta+K_Ih(\sigma+z_d) \tag{36}$$
$$+K_Df(\sigma+z_d)\dot{\delta}+K_Dg(\sigma+z_d)\delta^2$$

Define a state vector $X\in\mathbf{R}^{(n+1)*1}$ such that $X^T=[x_1,x_2,x_3,x_4....,x_{n+1}]=\left[\sigma,\delta,\frac{d\delta}{dt},\frac{d^2\delta}{dt^2}....,\frac{d^{n-1}\delta}{dt^{n-1}}\right]$.

The state equation of the closed loop system is,

$$\dot{x}_i=x_{i+1},\ 1\le i\le n \tag{37}$$
$$\dot{x}_i=\frac{-1}{a_n}\sum_{j=0}^{n-1}a_jx_{j+2}-\frac{1}{R_ia_n}[-a_0y_d+f(x_1+z_d)x_2+K_Ih(x_1+z_d)$$
$$+K_Df(x_1+z_d)x_3+K_Dg(x_1+z_d){x_2}^2],\ i=(n+1)$$

It is trivial to note that the equilibrium point of the above system is $X^T=0\in\mathbf{R}^{1*(n+1)}$.

5.2.3 Local Stability

To study the local stability of the system around its equilibrium point we have to linearize the non-linear system,

$$\dot{\mathbf{X}}=F(\mathbf{X}) \tag{38}$$

around its equilibrium point such that,

$$\dot{\mathbf{X}}=\mathbf{BX} \tag{39}$$

where, $\mathbf{B}\in\mathbf{R}^{(n+1)*(n+1)}$ is the Jacobian Matrix. Without loss of generality if we consider $\mathbf{X}=0$ as the equilibrium point then B is given by,

$$\mathbf{B}=\left.\frac{\partial F}{\partial\mathbf{X}}\right|_{\mathbf{X}=0} \tag{40}$$

As given in Khalil[16], a system described by equation (39) is asymptotically stable if and only if,

- All eignvalues λ_i of $\mathbf{B}$ satisfy $\mathrm{Re}(\lambda_i)$.

  OR

- There are positive definite matrix $\mathbf{P}$ and $\mathbf{Q}$ such that the lyapunov equation $\mathbf{B}^T\mathbf{P}+\mathbf{PB}=-\mathbf{Q}$ is satisfied.

We can use any of the above two conditions to prove local stability of the system. Also correctness of one of the above condition verifies the correctness of the other. Here we prove the first condition.

For the system described by state equation (37), $\mathbf{B}$ is given by the following matrix,

$$\begin{bmatrix} 0 & 1 & 0 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & 0 & . & . & . & 0 \\ 0 & 0 & 0 & 1 & . & . & . & 0 \\ 0 & 0 & 0 & 0 & . & . & . & 0 \\ . & . & . & . & & & & . \\ . & . & . & . & & & & . \\ . & . & . & . & & & & . \\ \frac{-K_IC}{a_n} & \frac{-(a_0+C)}{a_n} & \frac{-(a_1+K_DC)}{a_n} & \frac{-a_2}{a_n} & . & . & . & \frac{-a_{n-1}}{a_n} \end{bmatrix}$$

where, $C=f(z_d)/R_i$. We declare without proof (due to space limitation) that $f(z_d)=R_{ON}$

System described by equation (39) with $\mathbf{B}$ as given above can also be represented by the following differential equation,

$$\sum_{j=0}^{n} a_j \frac{d^j y}{dt^j} = \frac{R_{ON}}{R_i}\left[e(t) + K_I \int_0^t e(\tau)d\tau + K_D \frac{de}{dt} \right] \qquad (41)$$

The above differential equation is that of a linear plant described by equation (39) being controlled by a PID controller with transfer function $G_C(s)$ given by,

$$G_C(s) = \frac{R_{ON}}{R_i}\left[1 + \frac{K_I}{s} + sK_D \right] \qquad (42)$$

It is well known from root locus analysis that for a stable linear plant as assumed we will always have a set of PID parameters such that the closed loop system is stable. So there will always be a combination of K_I, K_D, R_{ON}, R_i such that the closed loop system is stable (or all the eignvalues of $\mathbf{B}$ have negative real parts). Ziegler-Nichols Closed Loop method is conventionally used to find these parameters. The ultimate gain K_u and the corresponding oscillation period T_u needed for Ziegler-Nichols Closed Loop method can be theoretically calculated using Nyquist Stability Criteria or Routh-Hurwitz Criteria.

5.2.4 Region of Attraction

According to Khalil[16], if $\mathbf{X} = 0$ is the equilibrium of a nonlinear system defined by equation (38), where $F : D \to \mathbf{R}^n$ is locally lipschitz and $D \subset \mathbf{R}^n$ is a domain containing the origin. Let $\Phi(t, \mathbf{X})$ be the solution of equation (38) that starts at initial state $\mathbf{X}$ at time $t = 0$. The region of attraction of the origin, denoted by R_A is defined by,

$$R_A = \{\mathbf{X} \in D \mid \Phi(t, \mathbf{X}) \to 0 \text{ as } t \to \infty\}$$

In other words, if $\mathbf{X} = 0$ is an asymptotically stable equilibrium point for $\dot{\mathbf{X}} = F(\mathbf{X})$, then its region of attraction R_A □ is an open, connected, invariant set whose boundary are formed by trajectories.

It is almost impossible to find the exact region of attraction. There are numerous works done to find almost a sharp estimate of the region of attraction but these methods are computationally extensive. In this paper we provide a very rough estimate of the region of attraction. The method used to estimate the region of attraction draws a part of its motivation from the method used by Khalil[16].

Let the lyapunov candidate be, $V(\mathbf{X}) = \mathbf{X}^T\mathbf{P}\mathbf{X}$ where $\mathbf{P}$ the matrix as encountered in the discussion of local stability is. Since the closed loop system was proven to be locally stable it follows that there will definitely be a positive definite matrix $\mathbf{P}$ and $\mathbf{Q}$ to satisfy the lyapunov equation. We assume $\mathbf{Q} = \mathbf{I}$ and solve the lyapunov equation to get $\mathbf{P}$. If,

$$\dot{\mathbf{X}} = F(\mathbf{X}) = \mathbf{B}\mathbf{X} + G(\mathbf{X}) \qquad (43)$$

Then,

$$\dot{V}(\mathbf{X}) = -\mathbf{X}^T\mathbf{Q}\mathbf{X} + 2\mathbf{X}^T\mathbf{P}G(\mathbf{X}) \qquad (44)$$

As adopted by Khalil[16] an estimate of R_A is the largest domain Ω_C defined by $V(\mathbf{X}) < C$ such that $\dot{V}(\mathbf{X})$ is negative definite.

We know that there is a ball $B_R = \| \mathbf{X} \| \le R$ such that $\dot{V}(\mathbf{X}) < 0$ in B_R. Let Ω_C be contained in B_R by choosing $C = \min_{\|\mathbf{X}\|=R} V(\mathbf{X})$.

Now,

$$\lambda_M(P)\| \mathbf{X} \|^2 \ge V(\mathbf{X}) \ge \lambda_m(P)\| \mathbf{X} \|^2 \qquad (45)$$

Hence choose, $C = \lambda_m(P)R^2$.

$$\lambda_M(Q)\| \mathbf{X} \|^2 \ge \dot{V}(\mathbf{X}) \ge \lambda_m(Q)\| \mathbf{X} \|^2 \qquad (46)$$

If $G(\mathbf{X})$ is locally lipschitz than for any $\gamma > 0$ there will exist an $R > 0$ such that,

$$\| G(\mathbf{X}) \| < \gamma \| \mathbf{X} \|, \ \forall \| \mathbf{X} \| < R \qquad (47)$$

Now using inequality (45), (46) and (47) we can arrive at the following conservative form of equation (44),

$$\dot{V}(\mathbf{X}) \le -[\lambda_m(\mathbf{Q}) - 2\gamma\lambda_M(\mathbf{P})]\| \mathbf{X} \|^2, \ \forall \| \mathbf{X} \| < R \qquad (48)$$

Now $\dot{V}(\mathbf{X}) < 0$ when,

$$\gamma < \frac{\lambda_m(\mathbf{Q})}{2 * \lambda_M(\mathbf{P})} \qquad (49)$$

Since $\mathbf{Q} = \mathbf{I}$, $\lambda_m(\mathbf{Q}) = 1$.

Now, γ known as the lipschitz constant is a function of R given by $\gamma = \Psi(R)$. To find the largest ball B_R over which $\dot{V}(\mathbf{X}) < 0$ we need to equate,

$$\psi(R) = \frac{1}{2 * \lambda_M(\mathbf{P})} \qquad (50)$$

Then the region of attraction as defined in Khalil[18] is given by,

$$\mathbf{X}^T\mathbf{P}\mathbf{X} < \lambda_m(P)R^2 \qquad (51)$$

The next work is to determine $\psi(R)$. Now $\psi(R)$ is determined by $G(\mathbf{X})$ where $G(\mathbf{X})^T = [y_1, y_2, y_3 ...., y_{n+1}]$.

Now,

$$y_i = 0, \ 1 \le i \le n \qquad (52)$$

$$y_i = \frac{1}{R_i a_n}[a_0 y_d - K_I h(x_1 + z_d) + K_I f(z_d)x_1 - K_D g(x_1 + z_d)x_2^{\ 2} + \{f(z_d) - f(x_1 + z_d)\}(x_2 + K_D x_3)], \ i = (n+1)$$

To find the lipschitz constant we take help of the following theorem as mentioned in Khalil[16],

If $f : [a \times b] \times D \to \mathbf{R}^m$ be continuous for some domain $D \in \mathbf{R}^n$. Suppose $\frac{\partial f}{\partial x}$ exists and is continuous on $[a \times b] \times D$.

If, for a convex subset $W \subset D$, there is a constant $L \ge 0$ such that,

$$\left\| \frac{\partial f}{\partial x}(t, x) \right\| \le L$$

on $[a \times b] \times W$, then

$$\| f(t, x) - f(t, y) \| \le L \| x - y \|$$

for all $t \in [a, b], \ x \in W, \text{ and } y \in W$.

It can be shown that,

$$\left\|\frac{\partial G}{\partial \mathbf{X}}(t,\mathbf{X})\right\| = \frac{1}{R_i a_n}\sqrt{a^2+b^2+c^2} \tag{53}$$

where,

$$a = \frac{\partial y_{n+1}}{\partial x_1} = [K_I\{f(z_d)-f(x_1+z_d)\} - g(x_1+z_d)(x_2+K_D x_3) - K_D g'(x_1+z_d){x_2}^2]$$

$$b = \frac{\partial y_{n+1}}{\partial x_2} = [\{f(z_d)-f(x_1+z_d)\} - 2K_D g(x_1+z_d)x_2]$$

$$c = \frac{\partial y_{n+1}}{\partial x_3} = [K_D\{f(z_d)-f(x_1+z_d)\}]$$

and, $g'(x) = \frac{\partial g}{\partial x}$

Suppose we are interested in calculating lipschitz constant over the convex set,

$$W = \{\mathbf{X}\in\mathbf{R}^{n+1} \mid |x_i| < r,\ \forall 1 \le i \le (n+1)\}$$

For $f(x)$ as defined by equation (28) and (29) and assuming $f(z_d) = R_{ON}$ it can be shown that,

$$\begin{aligned} &-2B \le \{f(z_d)-f(x_1+z_d)\} \le 0 \\ &-BC \le g(x_1+z_d) \le 0 \\ &\frac{-4BC^2}{3\sqrt{3}} \le g'(x_1+z_d) \le \frac{4BC^2}{3\sqrt{3}} \end{aligned} \tag{54}$$

Consider that the largest value of $\left\|\frac{\partial G}{\partial \mathbf{X}}(t,\mathbf{X})\right\|$ and hence the lipschitz constant L is given by,

$$L = \frac{1}{R_i a_n}\sqrt{{a_M}^2+{b_M}^2+{c_M}^2} \tag{55}$$

An analytical argument (whose explanation we skip) shows that,

$$L(r) = \begin{cases} 0; & r < (z_d - \Delta Q) \\ L_1; & r = (z_d - \Delta Q) \\ \max(L_1, L_2(r)); & (z_d - \Delta Q) < r < z_d \\ \max(L_1, L_2(z_d)); & r \ge z_d \end{cases} \tag{56}$$

where, L_1 and L_2 is be obtained by substituting the following values of a_M, b_M, c_M in equation (56):

For L_1 substitute,

$$\begin{aligned} a_M &= \left[BC(1+K_D)(z_d-\Delta Q) + \frac{4BC^2K_D}{3\sqrt{3}}(z_d-\Delta Q)^2\right] \\ b_M &= 2K_D BC(z_d-\Delta Q) \\ c_M &= 0 \end{aligned} \tag{57}$$

For L_2 substitute,

$$\begin{aligned} a_M &= [K_I(f(z_d)-f(r+z_d)) + BC(1+K_D)r] \\ b_M &= [f(z_d)-f(r+z_d)+2K_D BCr] \\ c_M &= K_D[f(z_d)-f(r+z_d)] \end{aligned} \tag{58}$$

It is easy to observe that if a function is lipschitz over the convex set W with lipschitz constant L it will be lipschitz over the convex set B_r with lipschitz constant L. Hence,

$$\gamma = \psi(r) = L \tag{59}$$

To solve for the value of R which satisfies equation (50) we plot the curve $\psi(r)$ and the straight horizontal line representing the RHS of equation (50) on the same graph. The minimum value of r where the two graphs intersect is the required value of R.

6. TIME DOMAIN PERFORMANCE

The superiority in performance of memristive PID controller can be explained as follows. Consider a set point tracking problem. A unit step input is given at time $t=0$. Initially $M(q)$ is high and hence the input $v(t)$ to the classical PID is large. So a large $u(t)$ is produced driving the output quickly to the set point. But as time passes q increases and hence $M(Q)$ decreases and saturates at a given value. Thereby $u(t)$ decreases and this prevents excessive overshoot.

The following steps briefly describe the tuning procedure of a memristive controller:

1) Determine the ultimate gain K_u and the corresponding oscillation period T_u using Nyquist Stability Criteria or Routh-Hurwitz Criteria for the linear plant $G_P(s)$.

2) Use Ziegler-Nichols Closed Loop tuning method to find the initial values of K_I and K_D.

3) Consider the open loop transfer function as $G_P(s)*\left(1+\frac{K_I}{s}+K_D s\right)$ and find the ultimate gain K'_U.

4) Set $R_i = 1$ and $R_{ON} = 0.5*K'_U$.

5) Try setting R_{OFF} and α by trial and error method. A high R_{OFF} will drive the process variable rapidly towards set-point but will cause overshoot and sometimes cause instability. A high α will saturate the gain of the controller to R_{ON} faster and hence prevent the overshoot.

6) To get better performance repeat step 5) for various R_{ON} keeping in mind $R_{ON} < K'_U$. Higher value of R_{ON} will give under-damped performance and vice-versa.

7. SIMULATION STUDIES

The unit step response of various linear systems when controlled by a optimal memristive PI controller and a optimal conventional PI controller has been plotted in this section. The graph in blue and magenta represents the response of the process with a conventional PI controller and memristive PI controller respectively.

Case-1 (Pitch dynamics of a passenger carrying Jet Aircraft)

$$\frac{\theta(s)}{\delta_e(s)} = \frac{1.31*(s+0.016)*(s+0.3)}{(s^2+0.0047s+0.0053)(s^2+0.806s+1.311)}$$

The above transfer function relates the pitch angle θ of a typical passenger carrying jet aircraft to its elevator displacement δ_e.

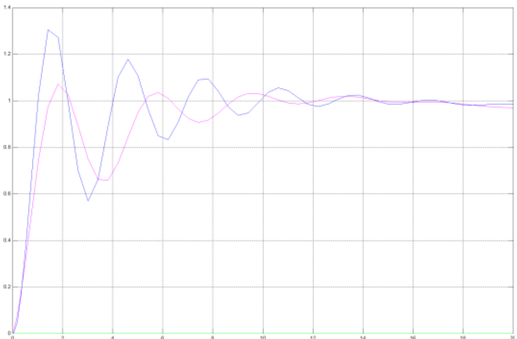


Fig 7: Conventional PI controller: Kp=2.45, Ki=0.211. Memristive PI Controller: Roff=100, Ron=1.2, Ri=1, α =20000, Ki=0.0861.

Case-2 (Hard-Disk head dynamics)

$$\frac{Y(s)}{U(s)} = \frac{K_T}{s(1+\tau_1 s)(1+\tau_2 s)} \cdot \frac{w_n^{\,2}}{s^2 + 2\xi w_n s + w_n^{\,2}}$$

The above transfer function is a typical structure which relates the dynamics of the hard-disk head position Y with the input voltage U .Here $K_T = 0.05$, $\tau_1 = 0.001$, $\tau_2 = 0.05$, $\zeta = 0.03$, $w_n = 2\pi f$ where $f = 2000$. All the values are in SI unit.

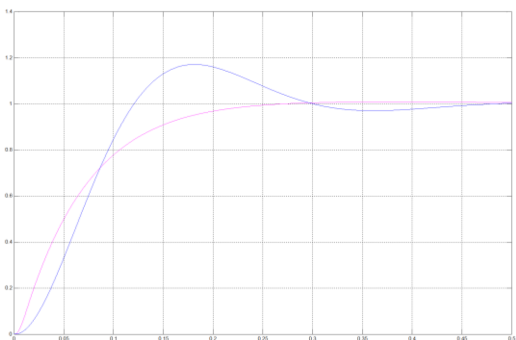


Fig 8: Conventional PI controller: Kp=400, Ki=0. Memristive PI Controller: Roff=2400, Ron=80, Ri=1, α =180000, Ki=0.

8. CONCLUSION

In this paper two memristive controller architectures have been proposed. Stability analysis and performance aspects of one of the architecture have been studied. The performance of the memristive PI controller and conventional PI controller has been studied by carrying out simulation studies with two different kinds of processes.

The emerging field of MEMS, Biomemetic Robotics etc. calls for controller in nano-scales. Since memristors are explicitly found in nano-scales the control architectures proposed in this paper can be used for used for control purposes in these fields.

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