Homogenization of a Composite Periodic Structure in the Case of Composite Plate

Saleh Alsubari
Faculty of Science and Technology. Department physics, Equips CMAO, FST University Hassan II –Mohammedia-Casablanca, Morocco

Hassan Chaffoui
Faculty of Science and Technology. Department physics, Equips CMAO, FST University Hassan II Mohammedia-Casablanca, Morocco

ABSTRACT
This work is consecrated to the investigation of mechanical behavior of a composite plate containing some periodic distributions and no symmetrical with regard to the average plan. The choice of this model is characterized by two important parameters: thickness of the plate and the size of the period. It is supposed that the thickness is smaller compared to the period dimension. The obtained results indicate that the homogenisation technique is able to predict the behaviour of periodic composites. The equivalent elasticity coefficients and micro-constraints were analytically calculated, then by finite elements in the basic cell level. We have shown that the complexity of numerical modelling can be solved by choosing a plan model, which gives the same results as a three-dimensional model

Keywords
Homogenisation Method, Finite element method, Composite plate

1. INTRODUCTION
The composite plates are used in the majority of the high mechanical performances structures. Nowadays their presence in all the technical fields, justifies this importance. However, the analysis of their mechanical behavior raises large difficulties linked to their strong heterogeneity and the diversity of their geometrical forms. In fact, the mechanical properties of the laminated plates depend on the nature of the layers according to a certain angles variation law according to the thickness and the nature of the reinforcement which can be in the form of wire, of wavy mates or fabrics. The modelling of the laminated structures and sandwiches is based on the thick plates theory, introduced by E. Reissner, 1945 and R. D. Mindlin, 1951, in the case of the isotropic homogeneous contexts. Considering that this theory is badly adapted for the study of the composite plates, many authors [4, 5, 6, 9, 10, 11, 12], have suggested its improvement by giving, more or less a refined field displacements. We notice that the refined theory takes into account the warping of the transverse segment and gives a good approximation constraints. The distribution of stresses resulting from transverse, can take a parabolic form in the thickness of the plate [12]. Another approach based on the homogenisation technique [1, 2, 3, 7, 13] has been suggested when it is about a periodic structure composite . This technique, as we have shown in this study, allows to calculate, not only the total behavior law of a composite, but also the mechanical contribution of its basic components. However, these studies are interested only in situations of simple geometry where the plate consists of one-dimensional fibres, circular perforations or rectangular [1, 7, 14]. Generally, the basic geometry represent a symmetry in relation to the average plan. The calculation of the elasticity coefficients of the equivalent context is analytically done by uncoupling membrane and flexion effects. In this work, we suggest to apply the homogenisation technique by asymptotic development in the case of non-symmetrical and periodic structure plates in relation to the average plan [8, 14, 15]. Two significant parameters characterize the choice of this model, the thickness of the plate and the size of the period. We suppose that the thickness h is smaller compared to the period dimension εY. This leads to consider a plate whose average plan is covered by a set of identical periods $Y = [0, Y_1]* [0, Y_2]$. (Figure1). Consequently, the plate behaviour is characterized by elasticity coefficients perfectly determined over the Y period and stretched by periodicity over the whole space. We notice these coefficients:

$$c^{\varepsilon}_{ijkh}(x) = c_{ijkh}(x, \frac{x}{\varepsilon}), \quad x \in \Omega \text{ et } \frac{x}{\varepsilon} = y \in Y$$

$i, j, k, h = 1, 2, 3$

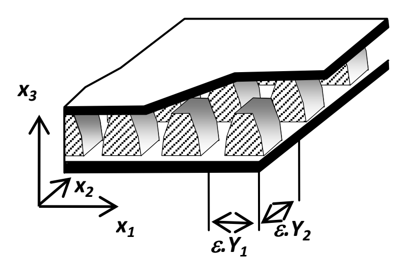

Fig 1: Plats periodic geometry

2. STATE OF THE PROBLEM.
We consider the frame work of the thick plates theory with linear elasticity, called the natural theory and we choose the situation where plates thickness h is smaller compared to the size $\varepsilon.Y$ of the period Y. Thus, we write the displacements field in the following form:

$$u^{\varepsilon}_i(x_1, x_2, x_3) = v^{\varepsilon}_i(x_1, x_2) + x_3 l^{\varepsilon}_i(x_1, x_2)$$
$$u^{\varepsilon}_3(x_1, x_2, x_3) = w^{\varepsilon}(x_1, x_2)$$
$$(x_1, x_2) \in \Omega, \ x_3 \in I_h = [-h/2, h/2], \ i = 1, 2 \tag{1}$$

The equilibrium equations are obtained by applying the principle of virtual powers taking into account the plate elastic behaviour, the external forces f_i and the moments m_i

$$-\frac{\partial}{\partial x_j}[a^{\varepsilon 0}_{ijkh}(y)\frac{\partial v^{\varepsilon}_k}{\partial x_h}+a^{\varepsilon 1}_{ijkh}(y)\frac{\partial l^{\varepsilon}_k}{\partial x_h}]=f^{\varepsilon}_i \quad (2)$$

$$a^{\varepsilon 0}_{i3j3}[l^{\varepsilon}_j+\frac{\partial w^{\varepsilon}}{\partial x_j}]-\frac{\partial}{\partial x_j}[a^{\varepsilon 1}_{ijkh}\frac{\partial v^{\varepsilon}_k}{\partial x_h}+a^{\varepsilon 2}_{ijkh}\frac{\partial l^{\varepsilon}_k}{\partial x_h}]=m^{\varepsilon}_i \quad (3)$$

$$\frac{\partial}{\partial x_i}[a^{\varepsilon 0}_{i3j3}(y)(l^{\varepsilon}_j+\frac{\partial w^{\varepsilon}}{\partial x_j})]=f^{\varepsilon}_3 \quad (4)$$

With the boundary conditions *i, j = 1, 2*

$$l^{\varepsilon}_i=0, w^{\varepsilon}=v^{\varepsilon}_i=0 \text{ on } \partial\Omega_0$$

$$N^{\varepsilon}_{ij}.n_j=0 \ , \ M^{\varepsilon}_{ij}.n_j=0 \ , Q^{\varepsilon}_i.n_i=0 \text{ on } \partial\Omega_1$$

3. HOMOGENEISATION BY ASYMPTOTIC DEVELOPMENT

The homogenisation method by asymptotic development consists of studying the limits of u^{ε} and $\sigma^{\varepsilon}_{ij}$ when ε tends towards zero. We have two scales. The first one is linked to the average plan and allows to describe the behaviour in the direction system (*0, x_1,x_2*.) The second, linked to the period *Y*, allows to reach the micro-constraints which develop around the point $x \in \Omega$.

3.1. Asymptotic Development

For each positive value of ε an asymptotic development of the displacement field shall be performed, in the form:

$$u^{\varepsilon}_i(x)=u^0_i(x)+\varepsilon u^1_i(x,y)+\varepsilon^2 u^2_i(x,y)+\ldots \quad (5)$$

where the displacement components $u^n_i(x,y)$ (n = 1, 2, 3, ...) are periodic in *Y*. With the equilibrium equations, we associate the operators: $A^{\varepsilon}_{ij}, B^{\varepsilon}_{ij}, C^{\varepsilon}_{ij}, D^{\varepsilon}_{ij}$ et E^{ε}_i such as:

$$A^{\varepsilon}_{ij}=a^{\varepsilon 0}_{ijkh}v^{\varepsilon}_{k,h},\ B^{\varepsilon}_{ij}=a^{\varepsilon 2}_{ijkh}l^{\varepsilon}_{k,h}, C^{\varepsilon}_{ij}=a^{\varepsilon 1}_{ijkh}v^{\varepsilon}_{k,h},$$
$$D^{\varepsilon}_{ij}=a^{\varepsilon 1}_{ijkh}l^{\varepsilon}_{k,h},\ E^{\varepsilon}_i=a^0_{i3j3}(l^{\varepsilon}_j+w^{\varepsilon}_{,j}) \quad (6)$$

The asymptotic development of these operators, is identical to that of $u^{\varepsilon}_i(x)$. For example the operator A^{ε}_{ij}, will be written in the form:

$$A^{\varepsilon}_{ij}(x)=A^0_{ij}(x,y)+\varepsilon A^1_{ij}(x,y)+\varepsilon^2 A^2_{ij}(x,y)+\ldots \quad (7)$$

Taking into account the variable changing $y=\frac{x}{\varepsilon}$, these operators are written:

$$A^n_{ij}=a^0_{ijkh}(\frac{\partial v^n_k}{\partial x_h}+\frac{1}{\varepsilon}\frac{\partial v^{n+1}_k}{\partial y_h}) \quad (8)$$

$$B^n_{ij}=a^2_{ijkh}(\frac{\partial l^n_k}{\partial x_h}+\frac{1}{\varepsilon}\frac{\partial l^{n+1}_k}{\partial y_h}) \quad (9)$$

$$C^n_{ij}=a^1_{ijkh}(\frac{\partial v^n_k}{\partial x_h}+\frac{1}{\varepsilon}\frac{\partial v^{n+1}_k}{\partial y_h}) \quad (10)$$

$$D^n_{ij}=a^1_{ijkh}(\frac{\partial l^n_k}{\partial x_h}+\frac{1}{\varepsilon}\frac{\partial l^{n+1}_k}{\partial y_h}) \quad (11)$$

$$E^n_i=a^0_{i3j3}(l^n_j+\frac{\partial w^n}{\partial x_j}+\frac{1}{\varepsilon}\frac{\partial w^{n+1}}{\partial y_j}) \quad (12)$$

3.2. Equivalents Coefficients.

By introducing the relations (8-12) into the equilibrium equations (2-4) and by identifying the terms of the order zero ε, the study of the limit when ε tends towards zero, allows us to obtain the microscopic equations system which enable to determine the equivalent coefficients:

$$-\frac{\partial}{\partial y_j}[a^0_{ijkh}\frac{\partial v^1_k}{\partial y_h}+a^1_{ijkh}\frac{\partial l^1_k}{\partial y_h}]=\frac{\partial a^0_{ijkh}}{\partial y_j}\frac{\partial v^0_k}{\partial x_h}+\frac{\partial a^1_{ijkh}}{\partial y_j}\frac{\partial l^0_k}{\partial x_h} \quad \text{in Y} \quad (13)$$

$$-\frac{\partial}{\partial y_j}[a^1_{ijkh}\frac{\partial v^1_k}{\partial y_h}+a^2_{ijkh}\frac{\partial l^1_k}{\partial y_h}]=\frac{\partial a^1_{ijkh}}{\partial y_j}\frac{\partial v^0_k}{\partial x_h}+\frac{\partial a^2_{ijkh}}{\partial y_j}\frac{\partial l^0_k}{\partial x_h} \quad in\text{Y} \quad (14)$$

$$-\frac{\partial}{\partial y_i}[a^0_{i3j3}\frac{\partial w^1}{\partial y_j}]=\frac{\partial a^0_{i3j3}}{\partial y_i}(l^0_j+\frac{\partial w^0}{\partial x_j}) \quad in\text{Y} \quad (15)$$

In these equations, the unknown factors v^1_i, l^1_i, w^1 check the conditions of periodicity boundary in y. And v^0_i, l^0_i, w^0 supposed known, depend only on *x*, and the microscopic equations (13-15) admit particular solutions in the form:

$$v^1_i=\phi^{rs}_i\frac{\partial v^0_r}{\partial x_s}+\varphi^{rs}_i\frac{\partial l^0_r}{\partial x_s}+\widetilde{V}_i(x) \quad (16)$$

$$l^1_i=\xi^{rs}_i\frac{\partial v^0_r}{\partial x_s}+\eta^{rs}_i\frac{\partial l^0_r}{\partial x_s}+\widetilde{L}_i(x) \quad (17)$$

$$w^1=\Phi^i(l^0_i+\frac{\partial w^0}{\partial x_i})+\widetilde{W}(x) \quad i,\ j,\ r,\ s\ =\ 1,\ 2 \quad (18)$$

The functions $\phi^{rs}_i, \varphi^{rs}_i, \xi^{rs}_i, \eta^{rs}_i$ and Φ^i check the periodicity boundary conditions and they are cellular equations solutions, defined on the period Y, as follows:

$$-\frac{\partial}{\partial y_j}\left(A_{ijkh}.\frac{\partial H^{rs}_k}{\partial y_h}\right)=\frac{\partial A_{ijrs}}{\partial y_j} \quad (19)$$

$$\frac{\partial}{\partial y_i}\left(a^0_{i3j3}\frac{\partial \Phi^m}{\partial y_j}\right)=\frac{\partial a^0_{i3m3}}{\partial y_i} \quad (20)$$

i, j, k, h, r, s, m = 1, 2

With the matrix notations :

$$A_{ijkh} = \begin{bmatrix} a^0_{ijkh} & a^1_{ijkh} \\ a^1_{ijkh} & a^2_{ijkh} \end{bmatrix},\quad H^{rs}_i = \begin{bmatrix} \phi^{rs}_i & \varphi^{rs}_i \\ \xi^{rs}_i & \eta^{rs}_i \end{bmatrix} \qquad (21)$$

By taking into account the equilibrium equations and by integrating according to the thickness h of the plate , the equations homogenized in the form:

$$-\frac{\partial}{\partial x_j}\left(a^{0H}_{ijkh}\frac{\partial v^0_k}{\partial x_h} + a^{1H}_{ijkh}\frac{\partial l^0_k}{\partial x_h}\right) = \tilde{f}_i \qquad (22)$$

$$a^{0H}_{i3j3}\left(l^0_j + \frac{\partial w^0}{\partial x_j}\right) - \frac{\partial}{\partial x_j}\left(a^{1H}_{ijkh}\frac{\partial v^0_k}{\partial x_h} + a^{2H}_{ijkh}\frac{\partial l^0_k}{\partial x_h}\right) = \tilde{m}_i \qquad (23)$$

$$-\frac{\partial}{\partial x_j}\left[a^{0H}_{i3j3}\left(l^0_j + \frac{\partial w^0}{\partial x_j}\right)\right] = \tilde{f}_3 \qquad (24)$$

Finally, the calculation of the homogenized coefficients requires analytical or numerical resolution of the various cellular problems (19,20), they are written as:

$$A^H_{ijkh} = \frac{1}{Y}\int_Y \left([A_{ijkh}] - [A_{ijrs}]\frac{\partial}{\partial y_s}[H^{kh}_r]\right).dy \qquad (25)$$

$$A^H_{i3j3} = \frac{1}{Y}\int_Y \left(a^0_{i3j3} - a^0_{i3k3}\frac{\partial \Phi^j}{\partial y_k}\right).dy \qquad (26)$$

3.3. Micro-constraints Expressions

The Micro-constraints expressions are obtained by considering the asymptotic development of the internal efforts :

$$N^\varepsilon_{ij} = N^0_{ij}(x,y) + \varepsilon N^1_{ij}(x,y) + \varepsilon^2 N^2_{ij}(x,y) + .. \qquad (27)$$

$$M^\varepsilon_{ij} = M^0_{ij}(x,y) + \varepsilon M^1_{ij}(x,y) + \varepsilon^2 M^2_{ij}(x,y).. \qquad (28)$$

$$Q^\varepsilon_i = Q^0_i(x,y) + \varepsilon Q^1_i(x,y) + \varepsilon^2 Q^2_i(x,y) + .. \qquad (29)$$

Thus, by taking into account the obtained results after the resolution of all the problems of cellular equations and by identifying terms of the same power of ε , according to [14], the first terms of the internal efforts N^0_{ij}, M^0_{ij} and Q^0_i are written as follows:

$$\sigma_{ij} = \frac{12M^0_{ij}}{h^3}x_3 + \frac{N^0_{ij}}{h} \qquad (30)$$

$$\sigma_{i3} = \frac{12Q^0_i}{h} \qquad (31)$$

These are the expressions which will be used for the numerical quantification of the micro-constraints inside the basic cell.

4.APPLICATIONS

For the validation of numerical calculations by finite elements method, considering the particular case sandwich plates reinforced by form trapezoidal veins (shown in Figure 3.a.1) and circular (Figure 3.a.2) The geometrical parameters are adjusted in order to satisfy a material cost of 65% (35% of the void). The mechanical characteristics of the used material are:

- Young Modulus: $E = 20.000\ MPa$,
- Poisson's ratio: $\nu = 0,3$

Figure 2 represents the geometry of the quarter of the basic cell relating to each plate reinforced by form circular veins.These cells are generated by three zones, noted A (known as zone of pillar), B, C (known as hollow zones).

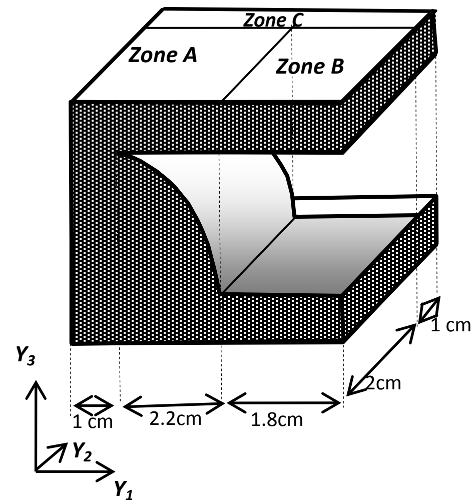

Fig 2. The quarter of the basic cells and corresponding zones in the average plan

We have treated the problem numerical calculations by finite element, two-dimensional using triangular element has three-nodes (see mesh figure 3.b.1 and 3.b. 2) for trapezoidal and circular geometry thus three-dimensional has six-nodes (see mesh figure 4.a. for trapezoidal and Figure 4.b for circular). The results from these two models are compared to analytical calculation (see table 1).

The elasticity coefficients and equivalent micro-constraints were analytically calculated then by finite element of the basic cell level. However, the symmetry of the basic cell relative to the axes (0,y1,y2) allows reducing the area of mesh to the quarter of the period. In contrast, the analytical solution requires the use of any cross-section (Figure 3.c.1 and 3.c.2) for trapezoidal and circular geometry. By applying on the plate a macroscopic traction equal to 100 MPa, we calculated the values of micro-constraints dimensional induced in the corresponding period. The results presented in Figure 5. For each vein form has macroscopic stress tensor. its rotation components and shear are constant throughout the medium , and have the values:

$$\tilde{\sigma}_{22} = \frac{6\tilde{M}_{22}}{h^2} = 100\ MPa \text{ and } \tilde{\sigma}_{23} = \frac{\tilde{Q}_2}{h} = 100\ MPa \qquad (32)$$

The rescaled micro strains are chosen as:

$$Q_i = \frac{Q^0_i}{\tilde{Q}_2},\quad N_{ij} = \frac{N^0_{ij}}{h.\tilde{\sigma}_{22}},\quad M_{ij} = \frac{M^0_{ij}}{\tilde{M}_{22}},\ i,\ j = 1,2 \qquad (33$$

Table 1 : Coefficients of elasticity of the plate reinforced by circular flanges.

	$E_1(MPa)$	$E_2(MPa)$	$G_{12}(MPa)$	$G_{13}(MPa)$	$G_{23}(MPa)$	ν_{12}
Analytical	11838	13822	4489,3	4504,9	5316,0	0,257
Finite Element(2D)	11864	13842	4499,6	4515,9	5324,0	0,257
Finite Element(3D)	11864	13842	4499,6	4515,9	5324,0	0,257

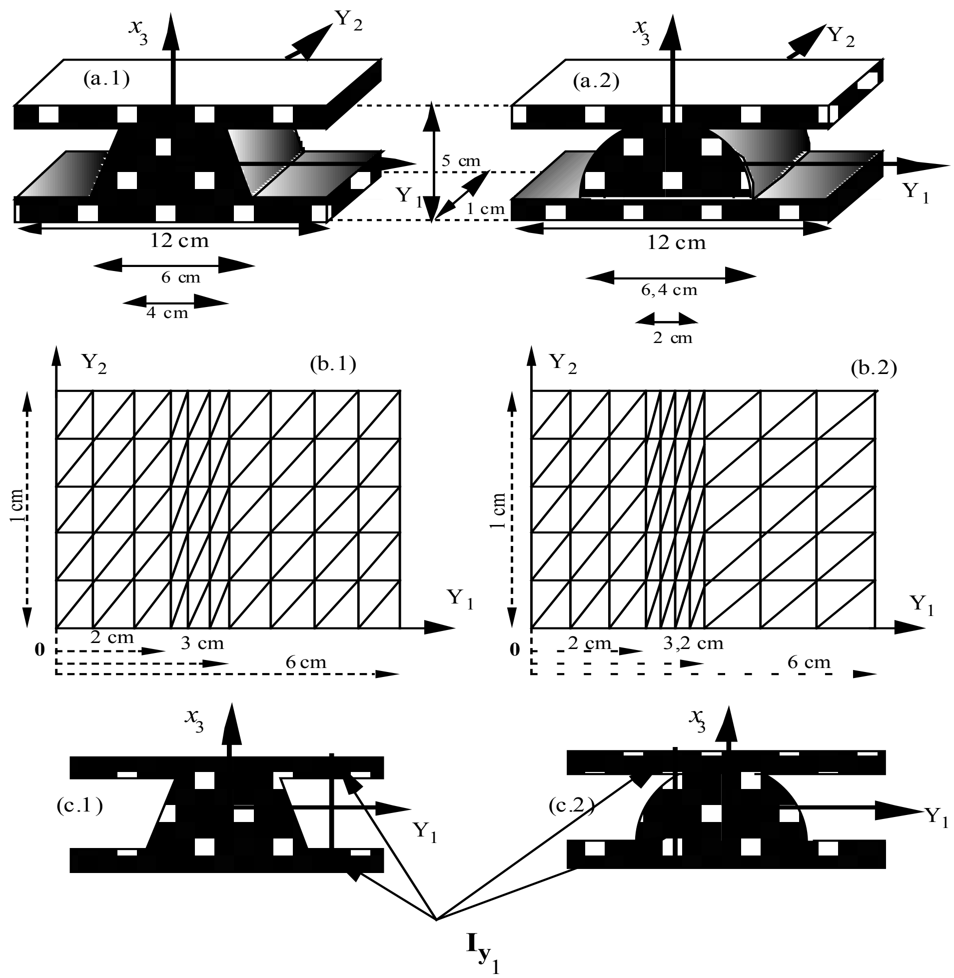

Fig 3 : Geometry of the plate with trapezoidal and circular flange

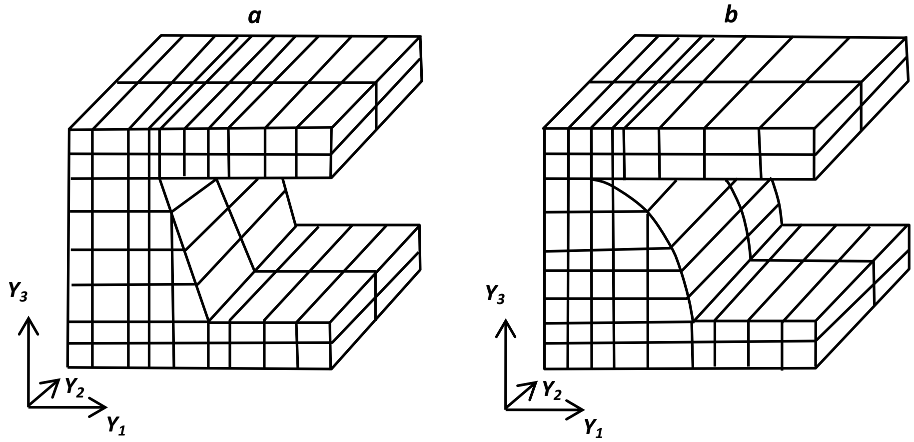

Fig 4 : Geometry of base cells(mesh size a quarter of the necessary period for three-dimensional)

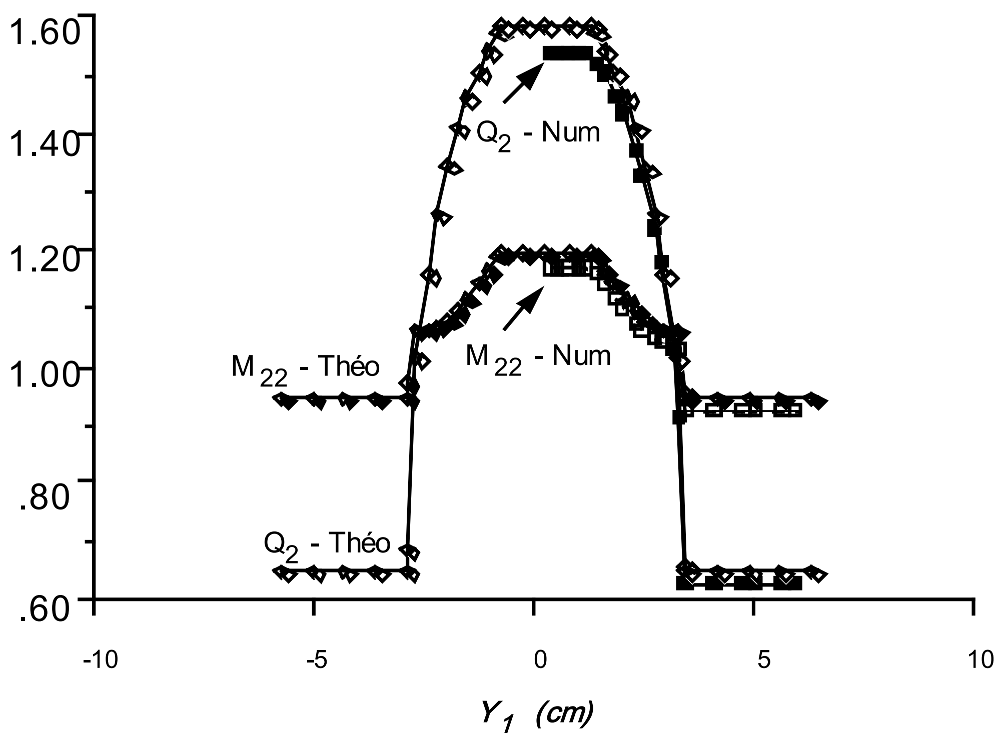

Fig 5: Distribution of rescaled micro-constraints in the basic cell relating to the circular flanges

5. CONCLUSION

In this study we have presented the results that treat the problem relative to a plate reinforced by form circular. The equivalent elasticity coefficients and the micro-constraints were analytically calculated then by finite elements method in the basic cell level. In addition, we showed that the complexity of numerical modelling can be solved by choosing a plan that gives the same results as a three-dimensional model. The results indicate that the homogenization technique is able of predict the behaviour of periodic structure plates.

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