Combined Adaptive Observer-Controller for Lipschitz Nonlinear Systems

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ABSTRACT
A combination between an adaptive sliding mode observer and a backstepping sliding mode controller is designed for a Lipschitz nonlinear system. This combination guaranties the tracking of trajectory, estimation of both the unmeasured state and the unknown parameters. A parameter variation margin is defined for that the combination is robust. The simulation results prove the combination robustness when the parameters are constants or varied in a defined margin.

Keywords
backstepping, sliding mode, observer, controller, adaptation law, lipschitz systems.

1. INTRODUCTION
The sliding mode technique is known as a robust technique in presence of parameter uncertainties and perturbations ([1], [2], [3]). For that it is applied to develop nonlinear observers and controllers ([5], [6], [7]). In [4] the author proved that the adaptive sliding mode observer is robust to estimate the state and the unknown parameter when the parameter variations verify a defined margin of variation.

The disadvantage of sliding mode control is the chattering phenomenal. To eliminate this problem, some researchers have extended the sliding mode to another form, like high order sliding mode [8], or combined the sliding mode technique to another algorithm such as backstepping ([9], [10]). The major advantage of the backstepping controller is the feasibility to construct the lyapunov function and the control law. The sliding mode backstepping controller and its adaptive form showed a best performance in trajectory tracking and in elimination of the chattering phenomena for uncertain nonlinear systems transformed in a semi strict feedback form ([11], [12]).

The constructed control laws depend on all the state. To improve the controller performance an observer estimating the unmeasured state is needed to be combined to a controller.

The Lipschtiz nonlinear systems are much studied to develop a nonlinear observer. The observer convergence is guaranty under some condition. When the parameter vector is unknown, the nonlinear observer is extended to an adaptive form. The adaptation law didn't depend on observer architecture (only the synthesis technique affect the adaptation law form).

In literature, the purpose of the combination between observer and controller is to ensure the observer performance to estimate the unmeasured state and the unknown input ([13], [14]). In [14] sufficient conditions are determined to define the unknown input observer designed to lipschitz nonlinear systems which is the same as the case of linear unknown input observer design.

The combination between controller and observer is considered to guaranty a trajectory tracking and a convergence of the unmeasured state. The study presented in ([15], [16]) show the robustness of the observer based controller for a class of nonlinear systems to tracking trajectory and estimate the unmeasured state. In ([17], [18]), the combination between an observer and a controller is designed to ensure the convergence of the unmeasured state and the stabilization of the nonlinear observer.

The unknown parameters affect the controller performance. The unknown parameters are estimated by an adaptation law determined trough out the controller or the observer.

In this work, an adaptive observer is combined to a backstepping sliding mode controller. The adaptation law is determined trough out the observer. The combination is studied for lipschitz nonlinear systems. The combination is designed to guaranty the robustness to estimate state and parameters and also to ensure the trajectory tracking when the nominal parameters are constants or varied in accordance with a defined margin parameter variation. The combination robustness is tested trough out an academic example.

2. ADAPTIVE SLIDING MODE OBSERVERS
Many adaptation laws are constructed for lipschtiz nonlinear systems. The adaptation laws depend only on the observer analysis techniques. In literature, adaptive observers are more studied in the case of classical approaches. Due to the robustness of the sliding mode technique, the authors in [19] compared the classical observer and the high order sliding mode adaptive observer in the case of parameters varying constantly in time. They concluded that the second architecture allow to converge rapidly to the true state than the first one. A systematic approach to synthesis adaptive observer is developed by [20] for Lipschitz nonlinear systems.

In [4], the authors developed an adaptive sliding mode observer for a class of lipschtiz nonlinear systems. The authors proved the robustness of the proposed adaptive sliding mode observer when the nominal constant unknown parameter vector varied linearly.

Consider the nonlinear systems:

$$\begin{cases} \dot{x} = Ax + \phi(x) + f(x,u)\theta + bu \\ y = Cx \end{cases} \quad (1)$$

With:

$x \in R^n ; \phi \in R^n ; f = diag(f_1, f_2, \cdots f_n) \in R^{n*p}$;
$\theta \in R^p ; u \in R$ and $y \in R^m$

Where ϕ and f are two Lipschitz matrices such that:

$$\| f(x,u) - f(\hat{x},u) \| < \alpha_1 \| x - \hat{x} \|$$

$$\| \phi(x) - \phi(\hat{x}) \| < \alpha_2 \| x - \hat{x} \|$$

The architecture of the adaptive sliding mode observer is:

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + \phi(\hat{x},u) + f(\hat{x},u)\hat{\theta} + L(y - C\hat{x}) - \\ \quad \gamma\, sign(e_y) \\ \quad \dot{\hat{\theta}} = \frac{1}{\rho} f^T(\hat{x},u)(y - C\hat{x}) \end{cases} \quad (2)$$

This adaptive observer is stable and converges to the desired state if:

$$L \geq \gamma$$

With/ $L \in R^n, \rho = diag(\rho_1, \rho_2 \cdots \rho_n)$ and $\gamma \in R^n$ are constants vectors.

The stability and robustness conditions are determined via the following lyapunov function:

$$V_o = \frac{1}{2} e^T e + \frac{1}{2} \rho \tilde{\theta}^T \tilde{\theta}$$

Which: $\tilde{\theta} = \theta - \hat{\theta}$ and $e = x - \hat{x}$

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3. BACKSTEPPING SLIDING MODE CONTROLLER ALGORITHMS

A backstepping technique is a systematic method to construct a lyapunov function and a controller law. The backstepping controllers guaranty the trajectory tracking performance.

The sliding mode controller is known as a robust controller in presence of uncertainties and perturbations its problem is the chattering phenomenon. To eliminate this problem a combination between the backstepping and sliding mode technique is made. A new controller called a backstepping sliding mode controller (BSMC), designed to uncertain nonlinear systems transformed in a semi strict feedback form (SSF), is présented. In ([11], [12]) the authors show that an adaptive version of the BSMC studied to a SSF system is robust to trajectory tracking. In this section, a BSMC is constructed for a lipschtiz nonlinear system.

The backstepping algorithm is described by the following step:

Step 1: Considering the variable error $z_1 = x_1 - y_r$

Then

$$\dot{z}_1 = \dot{x}_1 = x_2 + f_1^T(x)\theta + \phi_1(x) - \dot{y}_r \quad (3)$$

Considering (3) as a subsystem which is stabilizable and the lyapunov function is:

$$V_1(z_1, \theta) = \frac{1}{2} z_1^2$$

The derivative of the lyapunov function V_1 is:

$$\dot{V}_1(z_1, \theta) \leq -c_1 z_1^2 + z_1 z_2$$

With c_1 is a positive constant.

Step2: Considering the variable error $z_2 = x_2 - \dot{y}_r - \alpha_1$

With $\alpha_1 = -c_1 z_1 - \phi_1(x) - f_1^T(x)\theta$

The z_2 derivative is:

$$\dot{z}_2 = \dot{x}_2 - \ddot{y}_r + c_1 \dot{z}_1 - \frac{\partial \alpha_1}{\partial x_1} \dot{x}_1 - \frac{\partial \alpha_1}{\partial y_r} \dot{y}_r - \frac{\partial \alpha_1}{\partial \theta} \dot{\theta}$$

Or $\dot{z}_1 = z_2 - c_1 z_1$

Then

$$\dot{z}_2 = \dot{x}_2 - \ddot{y}_r + c_1(z_2 - c_1 z_1) - \frac{\partial \alpha_1}{\partial x_1} \dot{x}_1 - \frac{\partial \alpha_1}{\partial y_r} \dot{y}_r - \frac{\partial \alpha_1}{\partial \theta} \dot{\theta}$$

$$V_2(z_2, \theta) = V_1(z_1, \theta) + \frac{1}{2} z_2^2$$

After derivation V_2 becomes:

$$\dot{V}_2(z_2, \theta) = \dot{V}_1(z_1, \theta) + z_2 \dot{z}_2$$
$$\dot{V}_2(z_2, \hat{\theta}) \leq -c_1 z_1^2 - c_2 z_2^2 + z_2 z_3$$

With:

$$z_3 = x_3 - \ddot{y}_r - \alpha_2$$

$$\alpha_2 = c_1^2 z_1 - z_1 - (c_2 + c_1) z_2 - f_2^T \theta + \frac{\partial \alpha_1}{\partial x_1} x_2 +$$
$$\frac{\partial \alpha_1}{\partial x_1} f_1^T \theta + \frac{\partial \alpha_1}{\partial x_1} \phi_1 + \frac{\partial \alpha_1}{\partial y_r} \dot{y}_r + \frac{\partial \alpha_1}{\partial \theta} \dot{\theta} - \phi_2$$

Step i ($1 < i \leq n-1$): Defining $z_i = x_i - \alpha_{i-1} - y_r^{(i-1)}$

$$\alpha_i(x_1,\dots x_i,\theta)=-\sum_{j=1}^{i-1}c_j z_i-(-1)^{i-1}\sum_{k=1}^{i-2}\left(\sum_{j=0}^{k}c_1^{k-j}c_2^{j}\right)z_k$$
$$+c_{i-1}\sum_{j=1}^{i-1}c_j z_{i-1}+\sum_{j=1}^{i-2}c_2^{j}z_{j+1}-\sum_{j=1}^{i-2}\left[z_{i-1}-c_j z_{i-2}\right]+$$
$$\sum_{j=0}^{i-1}c_1^{i-1-j}c_2^{j}z_{i-1}-c_1^{i}z_1-f_i^{T}\theta+\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}(x_{j+1}+\phi_j)+$$
$$\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\theta}\dot{\theta}+\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}f_j^{T}\theta+\phi_i \quad (4)$$

With $c_i > 0$ and the variable error derivative z_i is:

$$\dot{z}_i=x_{i+1}-y_r^{(i)}+\sum_{j=1}^{i-1}c_j z_i+(-1)^{i-1}\sum_{k=1}^{i-2}\left(\sum_{j=0}^{k}c_1^{k-j}c_2^{j}\right)z_k$$
$$-c_{i-1}\sum_{j=1}^{i-1}c_j z_{i-1}-\sum_{j=1}^{i-2}c_2^{j}z_{j+1}+\sum_{j=1}^{i-2}\left[z_{i-1}-c_j z_{i-2}\right]-$$
$$\sum_{j=0}^{i-1}c_1^{i-1-j}c_2^{j}z_{i-1}+c_1^{i}z_1+f_i^{T}\theta-\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}(x_{j+1}+\phi_j)-$$
$$\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\theta}\dot{\theta}-\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_j}f_j^{T}\theta-\sum_{j=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial y_r^{(j-1)}}y_r^{(j)}+\phi_i \quad (5)$$

$$V_i=V_{i-1}+\frac{1}{2}z_i^2$$
$$\dot{V}_i=\dot{V}_{i-1}+z_i\dot{z}_i \quad (6)$$
$$\dot{V}_i\le-\sum_{j=1}^{i}c_j z_j^2+z_i z_{i+1}$$

Step n:

Defining $z_n=x_n-\alpha_{n-1}-y_r^{(n-1)}$ (7)

With α_{n-1} has the expression (4) when $n=i$.

The derivative of z_n is:

$$\dot{z}_n=\sum_{j=1}^{i-1}c_j z_i-c_{i-1}\sum_{j=1}^{i-1}c_j z_{i-1}-\sum_{j=1}^{i-2}c_2^{j}z_{j+1}+$$
$$(-1)^{i-1}\sum_{k=1}^{i-2}\left(\sum_{j=0}^{k}c_1^{k-j}c_2^{j}\right)z_k+\sum_{j=1}^{i-2}\left[z_{i-1}-c_j z_{i-2}\right]-$$
$$\sum_{j=0}^{i-1}c_1^{i-1-j}c_2^{j}z_{i-1}+c_1^{i}z_1+\sum_{i=1}^{n}a_i x_i+g_n(x)u+$$
$$f_n^{T}(x)\theta+\phi_n-\sum_{i=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_i}(x_{i+1}+\phi_i)-\frac{\partial\alpha_{n-1}}{\partial\theta}\dot{\theta}-$$
$$\sum_{i=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial y_r^{(i-1)}}y_r^{(i)}-y_r^{(n)}-\sum_{i=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_i}f_i^{T}\theta \quad (8)$$

Considering the lyapunov function:

$$V_n=V_{n-1}+\frac{1}{2}z_n^2+\frac{1}{2}\tilde{\theta}^{T}\Gamma\tilde{\theta} \quad (9)$$

The V_n derivative is:

$$\dot{V}_n\le-\sum_{i=1}^{n}c_i z_i^2+(-1)^{i-1}\sum_{k=1}^{i-2}\left(\sum_{j=0}^{k}c_1^{k-j}c_2^{j}\right)z_k z_n(\sum_{j=1}^{i-1}c_j z_i-$$
$$c_{i-1}\sum_{j=1}^{i-1}c_j z_{i-1}-\sum_{j=1}^{i-2}c_2^{j}z_{j+1}+\sum_{j=1}^{i-2}\left[z_{i-1}-c_j z_{i-2}\right]-$$
$$\sum_{j=0}^{n-1}c_1^{n-1-j}c_2^{j}z_{n-1}+c_1^{n}z_1+g_n(x)u+f_n^{T}(x)\theta+\phi_n+$$
$$c_n z_n-y_r^{(n)}-\sum_{i=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_i}(x_{i+1}+\phi_i)-\sum_{i=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_i}f_i^{T}(x)\theta-$$
$$\sum_{i=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial y_r^{(i-1)}}y_r^{(i)}-\frac{\partial\alpha_{n-1}}{\partial\theta}\dot{\theta}+f_n(x)) \quad (10)$$

The backstepping sliding mode controller u has the following form:

$$u=\bar{u}+sign(z_1)(kz_1+\lambda) \quad (11)$$

$\bar{u}$: The backstepping sliding mode control law

The backstepping sliding mode controller is stable and robust if:

$$\dot{V}_n\le-\sum_{i=1}^{n}c_i z_i^2$$

$$\dot{V}_n\le-Z^{T}MZ \quad (12)$$

With $c_i > 0$,

$$\mathrm{M} = [c_1\ c_2 \ldots\ldots c_n]^T,\ Z = [z_1\ z_2 \ldots\ldots z_n]^T$$

- The controller is :

$$u = \frac{1}{g_n(x)}\left[\begin{array}{l} \sum_{j=1}^{n-1} c_j z_i - c_{n-1}\sum_{j=1}^{n-1} c_j z_{n-1} - \sum_{j=1}^{n-2} c_2^j z_{j+1} + \\ (-1)^{i-1}\sum_{k=1}^{i-2}\left(\sum_{j=0}^{k} c_1^{k-j} c_2^j\right) z_k \\ + \sum_{j=1}^{n-2}\left[z_{i-1} - c_j z_{i-2}\right] - \sum_{j=0}^{n-1} c_1^{n-1-j} c_2^j z_{n-1} + \\ c_1^n z_1 - \sum_{i=1}^{n} a_i x_i - f_n^T(x)\theta + \\ \sum_{i=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_i}(x_{i+1} + \phi_i) + \sum_{i=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_i} f_i^T(x)\theta + \\ \sum_{i=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial y_r^{(i-1)}} y_r^{(i)} + \frac{\partial\alpha_{n-1}}{\partial\theta}\dot{\theta} - \phi_n + y_r^{(n)} \end{array}\right] + (k z_1 + \lambda)\, sign(z_1) \quad (13)$$

$$\lambda = -k|z_1| \quad (14)$$

4. COMBINED ADAPTIVE OBSERVER-CONTROLLER

In literature a controller based observer is presented to lipsctiz nonlinear systems. The proposed controller is applied as a system and observer input which the separation principal is not usually verified. To ensure a trajectory tracking and estimation of unmeasured state and unknown constant parameters, an adaptive observer-controller combination, designed for a lipchitz nonlinear system which, is still robust for a defined parameter variation margin.

Theorem: For a Lipchitz nonlinear system, the adaptive sliding mode observer combined to the backstepping sliding mode controller is robust and stable if:

- The backstepping sliding mode controller architecture is:

$$u = \frac{1}{g_n(x)}\left[\begin{array}{l} \sum_{j=1}^{n-1} c_j z_i - c_{n-1}\sum_{j=1}^{n-1} c_j z_{n-1} - \sum_{j=1}^{n-2} c_2^j z_{j+1} + \\ (-1)^{i-1}\sum_{k=1}^{i-2}\left(\sum_{j=0}^{k} c_1^{k-j} c_2^j\right) z_k \\ + \sum_{j=1}^{n-2}\left[z_{i-1} - c_j z_{i-2}\right] - \sum_{j=0}^{n-1} c_1^{n-1-j} c_2^j z_{n-1} + \\ c_1^n z_1 - \sum_{i=1}^{n} a_i x_i - f_n^T(x)\theta + \\ \sum_{i=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_i}(x_{i+1} + \phi_i) + \sum_{i=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial x_i} f_i^T(x)\theta + \\ \sum_{i=1}^{n-1}\frac{\partial\alpha_{n-1}}{\partial y_r^{(i-1)}} y_r^{(i)} + \frac{\partial\alpha_{n-1}}{\partial\theta}\dot{\theta} - \phi_n + y_r^{(n)} \end{array}\right] + (k z_1 + \lambda)\, sign(z_1)$$

- The adaptive sliding mode observer is:

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + \phi(\hat{x}) + f(\hat{x},u)\hat{\theta} + bu + L(y - C\hat{x}) - \gamma\, sign(e_y) \\ \dot{\hat{\theta}} = \frac{1}{\rho} f^T(\hat{x},u)(y - C\hat{x}) \end{cases}$$

The combinaison is robuste if:

- $L \geq \gamma$
- $\lambda = -k|z_1|$

Proof:

See Appendix

The combination robustness is tested throughout an academic example.

5. SIMULATION RESULTS

Considering the nonlinear system

$$\begin{cases} \dot{x}_1 = x_2 - \theta_1 x_1^3 \\ \dot{x}_2 = -x_1 - \theta_2 \exp(x_1) - 2x_2 + u \\ y = x_1 \end{cases}$$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix};\ \phi(\hat{x}) = 0;$$

$$f(x,u) = \begin{bmatrix} -x_1^3 & 0 \\ 0 & -\exp(x_1) \end{bmatrix};\ \theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix};\ b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$e_y = y - \hat{y};\ e_1 = x_1 - \hat{x}_1;\ e_2 = x_2 - \hat{x}_2$$

The sliding mode observer is:

$$\begin{cases} \dot{\hat{x}}_1 = \hat{x}_2 - \hat{\theta}_1 \hat{x}_1^3 + L_1 e - \gamma_1 sign(e_y) \\ \dot{\hat{x}}_2 = -\hat{x}_1 - \hat{\theta}_2 \exp(\hat{x}_1) - 2\hat{x}_2 + L_2 e - \gamma_2 sign(e_y) + u \\ \hat{y} = \hat{x}_1 \end{cases}$$

The adaptation law is:

$$\dot{\hat{\theta}} = \begin{cases} -\dfrac{1}{\rho_1} e_1 \hat{x}_1^3 \\ -\dfrac{1}{\rho_2} e_2 \exp(\hat{x}_1) \end{cases}$$

The controller parameters are:

$$z_1 = \hat{x}_1 - y_r$$

$$z_2 = \hat{x}_2 - \dot{y}_r + c_1 z_1 - \hat{\theta}_1 \hat{x}_1^3$$

$$u = c_1^2 z_1 - (c_2 + c_1) z_2 + \hat{x}_1 + 2\hat{x}_2 + \hat{\theta}_2 \exp(\hat{x}_1) +$$
$$3\hat{x}_1^3 \hat{x}_2 \hat{\theta}_1 - 3\hat{\theta}_1^2 \hat{x}_1^5 + (kz_1 + \lambda) sign(z_1) + \ddot{y}_r$$

$$\lambda \le -k_1 z_1$$

The simulation parameters are:

$$L_1 = 1;\ L_2 = 1;\ \gamma_1 = 10^{-3};\ \gamma_2 = 10^{-2};\ \hat{\theta}_1(0) = 0.3;$$

$$\hat{\theta}_2(0) = 0.1;\ \hat{x}_1(0) = 2;\ \hat{x}_2(0) = 2;\ x_1(0) = x_2(0) = 0;$$

$$\theta_1 = 1.1;\ \theta_2 = 0.1;\ \frac{1}{\rho_1} = 0.7;\ \frac{1}{\rho_2} = 5,\ c_1 = 0.2;\ c_2 = 10^{-4};$$

$$k = -8;\ \lambda = 9$$

The reference signal is:

$$y_r = \begin{cases} 0.1 & si\ 0 \le t \le 8s \\ 0.5 & si\ 8s \le t \le 14s \\ 1 & si\ t \ge 14s \end{cases}$$

If the nominal parameter is constant, the combination between an adaptive sliding mode observer and a backstepping sliding mode controller is robust to trajectory tracking (fig.1.a), estimation the unmeasured state. The state variables converge to the desired values after the a short time (fig.1.c,d). This combination is also robust to estimate the unknown parameter (fig.1.e, f). This is due to the control signal absolute amplitude is the same for all the reference amplitude varied (fig.1.b).

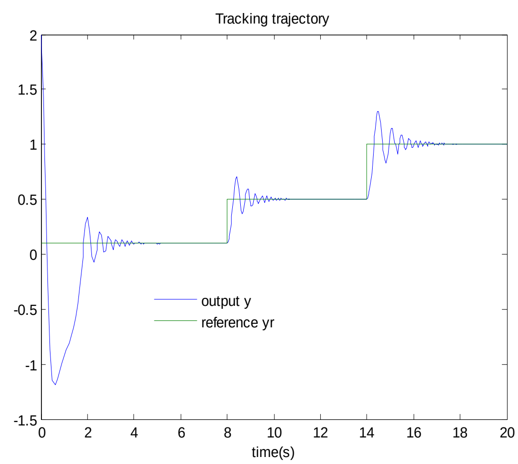

Fig.1.a The trajectory tracking

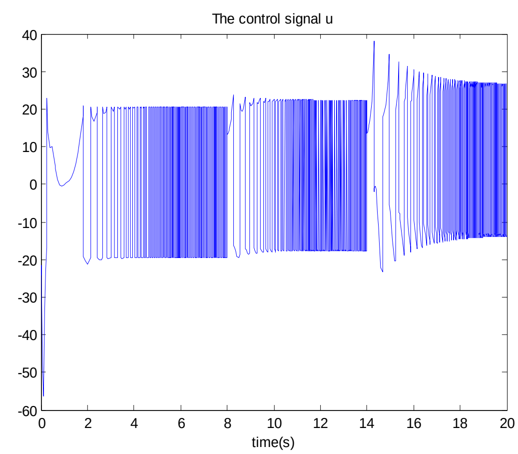

Fig.1.b The signal control u

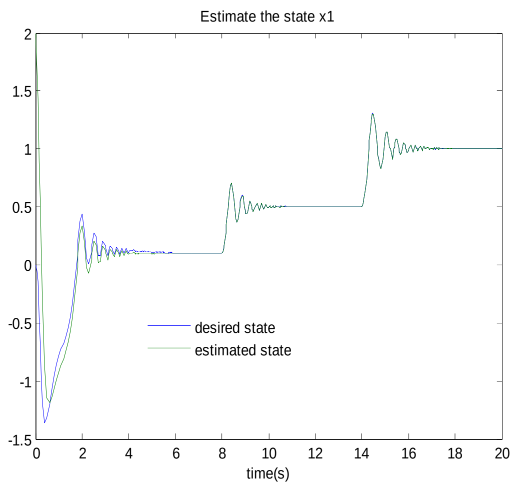

Fig.1.c Estimation of the state x_1

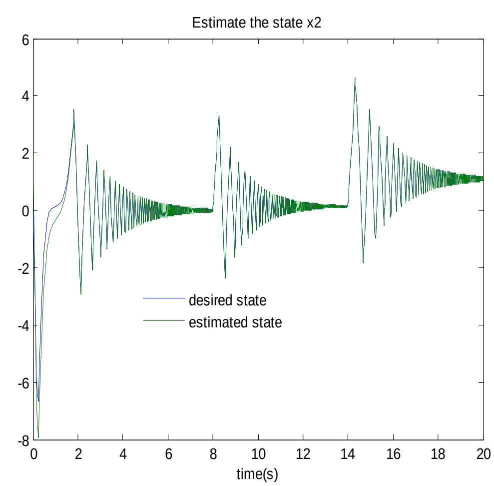

Fig.1.d Estimation of the state x_2

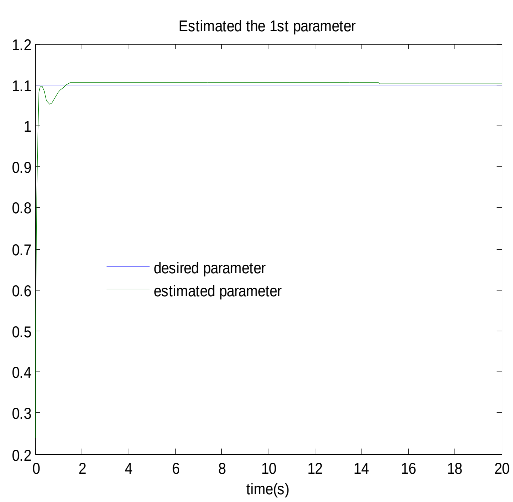

Fig.1.e Estimation of the parameter θ_1

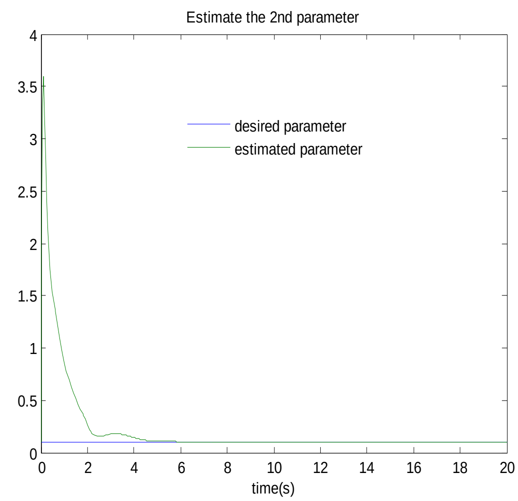

Fig.1.f Estimation of the parameter θ_2

Fig.1. Trajectory tracking and parameter and state estimation with constants parameter

The combination between the backstepping sliding mode control and the adaptive sliding mode observer is robust to state and parameter estimation and trajectory tracking which is different from literature works ([13], [14]) where the designed approaches permit only to improve the state estimation performance and build the unknown input.

6. CONCLUSION

In this paper, an adaptive sliding mode observer is combined to a backstepping sliding mode controller for a class of lipschitz nonlinear systems. This combination is designed to ensure the trajectory tracking, estimation of the unmeasured state and the unknown parameter when the parameters are constants or varied linearly. The simulation results show that the adaptive sliding mode observer combined to a backstepping sliding mode controller is robust to trajectory tracking and the estimation of state and parameters if the nominal parameter are constants.

7. APENDIX

Proof:

To study the combination robustness and stability, a lyapunov function is considered as:

$$V = V_o + V_n \qquad \text{(A.1)}$$

The derivative of V is:

$$\dot{V} = \dot{V}_o + \dot{V}_n \qquad \text{(A.2)}$$

Construction of the lyapunov function $\dot{V}_n$

Step 1: Considering the variable error $z_1 = \hat{x}_1 - y_r$

With/

$$\dot{z}_1 = \dot{\hat{x}}_1 = \hat{x}_2 + f_1^T(\hat{x})\hat{\theta} + \phi_1(\hat{x}) - \dot{y}_r \qquad \text{(A.3)}$$

When $\hat{\theta}$ is the estimate of θ

We have: $\dot{z}_1 = \dot{\hat{x}}_1 = \hat{x}_2 + f_1^T(\hat{x})\hat{\theta} + \phi_1(\hat{x}) - \dot{y}_r$

The subsystem (A.3) is stable and the lyapunov function is:

$$V_1(z_1, \hat{\theta}) = \frac{1}{2} z_1^2$$

After derivation, development and simplification, the expression of V_1 becomes:

$$\dot{V}_1(z_1, \hat{\theta}) \le -cz_1^2 + z_1 z_2$$

With c_1 is a positive constant

$$z_2 = \hat{x}_2 - \dot{y}_r - \alpha_1$$

$$\alpha_1 = -c_1 z_1 - \phi_1(\hat{x}) - f_1^T(\hat{x})\hat{\theta}$$

The z_2 derivative is:

$$\dot{z}_2 = \dot{\hat{x}}_2 - \ddot{y}_r - c_1\dot{z}_1 - \frac{\partial \alpha_1}{\partial \hat{x}_1}\dot{\hat{x}}_1 - \frac{\partial \alpha_1}{\partial y_r}\dot{y}_r - \frac{\partial \alpha_1}{\partial \hat{\theta}}\dot{\hat{\theta}}$$

$$V_2(z_2, \hat{\theta}) = V_1(z_1, \hat{\theta}) + \frac{1}{2} z_2^2$$

The lyapunov function V_2 derivative is:

$$\dot{V}_2(z_2, \hat{\theta}) = \dot{V}_1(z_1, \hat{\theta}) + z_2\dot{z}_2$$

$$\dot{V}_2(z_2, \hat{\theta}) \le -c_1 z_1^2 - c_2 z_2^2 + z_2 z_3$$

With: $z_3 = \hat{x}_3 - \ddot{y}_r - \alpha_2$

$$\alpha_2 = -z_1 + c_1^2 z_1 - (c_2 + c_2) z_2 - f_2^T\hat{\theta} + \frac{\partial \alpha_1}{\partial \hat{x}_1}\hat{x}_2 + \frac{\partial \alpha_1}{\partial \hat{x}_1} f_1^T\hat{\theta} + \frac{\partial \alpha_1}{\partial \hat{x}_1}\phi_1 + \frac{\partial \alpha_1}{\partial y_r}\dot{y}_r + \frac{\partial \alpha_1}{\partial \hat{\theta}}\dot{\hat{\theta}} - \phi_2$$

Step i ($1 < i \le n-1$):

Defining $z_i = \hat{x}_i - \alpha_{i-1} - y_r^{(i-1)}$

$$\alpha_i(\hat{x}_1, \ldots \hat{x}_i, \hat{\theta}) = -\sum_{j=1}^{i-1} c_j z_i - c_1^i z_1 - (-1)^{i-1}\sum_{k=1}^{i-2}\left(\sum_{j=0}^{k} c_1^{k-j} c_2^j\right) z_k - \sum_{j=1}^{i-2}\left[z_{i-1} - c_j z_{i-2}\right] + \sum_{j=0}^{i-1} c_1^{i-1-j} c_2^j z_{i-1} - f_i^T\hat{\theta} + \sum_{j=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial \hat{x}_j}(\hat{x}_{j+1} + \phi_j) + \sum_{j=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial \hat{\theta}}\dot{\hat{\theta}} + \sum_{j=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial \hat{x}_j} f_j^T\hat{\theta} - \phi_i + c_{i-1}\sum_{j=1}^{i-1} c_j z_{i-1} + \sum_{j=1}^{i-2} c_2^j z_{j+1} \qquad \text{(A.4)}$$

With $c_i > 0$ and the variable error z_i derivative is:

$$\dot{z}_i = \dot{\hat{x}}_{i+1} - \sum_{j=1}^{i-1} c_j z_i + c_{i-1}\sum_{j=1}^{i-1} c_j z_{i-1} + \sum_{j=1}^{i-2} c_2^j z_{j+1} - (-1)^{i-1}\sum_{k=1}^{i-2}\left(\sum_{j=0}^{k} c_1^{k-j} c_2^j\right) z_k - \sum_{j=1}^{i-2}\left[z_{i-1} - c_j z_{i-2}\right] + f_i^T\hat{\theta} - \sum_{j=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial \hat{x}_j}(\hat{x}_{j+1} + \phi_j) - \sum_{j=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial \hat{\theta}}\dot{\hat{\theta}} - \sum_{j=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial \hat{x}_j} f_j^T\hat{\theta} + \phi_i \qquad \text{(A.5)}$$

$$V_i = V_{i-1} + \frac{1}{2} z_i^2$$
$$\dot{V}_i = \dot{V}_{i-1} + z_i \dot{z}_i \qquad \text{(A.6)}$$
$$\dot{V}_i \le -\sum_{j=1}^{i} c_j z_j^2 + z_i z_{i+1}$$

Step n:

Defining $z_n = x_n - \alpha_{n-1} - y_r^{(n-1)}$ (A.7)

With α_{n-1} is the same as the equation (A.4) when $n = i$.

The derivative of the variable z_n is:

$$\dot{z}_n = \sum_{j=1}^{n-1} c_j z_n - c_{n-1} \sum_{j=1}^{n-1} c_j z_{i-1} - \sum_{j=1}^{n-2} c_2^j z_{j+1} + (-1)^{n-1} \sum_{k=1}^{n-2} \left(\sum_{j=0}^{k} c_1^{k-j} c_2^j \right) z_k + \sum_{j=1}^{n-2} \left[z_{n-1} - c_j z_{n-2} \right] + \sum_{i=1}^{n} a_i x_i + g_n(x) u + f_n^T(\hat{x}) \hat{\theta} + \phi_n - \sum_{i=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \hat{x}_i} (\hat{x}_{i+1} + \phi_i) - \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} - y_r^{(n)} - \sum_{i=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \hat{x}_i} f_i^T \hat{\theta} \qquad \text{(A.8)}$$

Considering the lyapunov function:

$$V_n = V_{n-1} + \frac{1}{2} z_n^2 \qquad \text{(A.9)}$$

The V_n derivative is:

$$\dot{V}_n \le -\sum_{i=1}^{n} c_i z_n^2 + z_n \Big(\sum_{j=1}^{n-1} c_j z_n - c_{n-1} \sum_{j=1}^{n-1} c_j z_{n-1} - \sum_{j=1}^{n-2} c_2^j z_{j+1} + (-1)^{n-1} \sum_{k=1}^{n-2} \left(\sum_{j=0}^{k} c_1^{k-j} c_2^j \right) z_k + \sum_{j=1}^{n-2} \left[z_{n-1} - c_j z_{n-2} \right] + g_n(\hat{x}) u + f_n^T(\hat{x}) \hat{\theta} + \phi_n + c_n z_n - y_r^{(n)} - \sum_{i=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \hat{x}_i} (\hat{x}_{i+1} + \phi_i) - \sum_{i=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \hat{x}_i} f_i^T(\hat{x}) \hat{\theta} - \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} + f_n(\hat{x}) \Big) \qquad \text{(A.10)}$$

For the observer, choosing the following Lyapunov function:

$$V_o = \frac{1}{2} e^T e + \frac{1}{2} \rho \tilde{\theta}^T \tilde{\theta} \qquad \text{(A.11)}$$

Where $\tilde{\theta} = \theta - \hat{\theta}$ and $e = x - \hat{x}$

The dynamical error is:

$$\dot{e} = A(x - \hat{x}) + \phi(x) - \phi(\hat{x}) + f(x,u)\theta - f(\hat{x},u)\hat{\theta} - L(y - C\hat{x}) + \gamma \; sign(e_y)$$

The derivative of the function V is then:

$$\Rightarrow \dot{V}_o = e^T \dot{e} - \rho \dot{\hat{\theta}}^T \tilde{\theta}$$

$$\dot{V}_o = e^T A e + e^T (\phi(x) - \phi(\hat{x})) + e^T (f(x,u)\theta - f(\hat{x},u)\hat{\theta}) - e^T L C e + e^T \gamma \; sign(e_y) - \rho \dot{\hat{\theta}}^T \tilde{\theta}$$

Or $e_y = C e$ and $sign(e_y) = C \, sign(e)$

$$\dot{V}_o = e^T A e + e^T (\phi(x) - \phi(\hat{x})) + e^T (f(x,u)\theta - f(\hat{x},u)(\theta - \tilde{\theta})) - e^T L C e + e^T \gamma \; C sign(e) - \rho \dot{\hat{\theta}}^T \tilde{\theta}$$

$$\dot{V}_o \le e^T A e + e^T \alpha_2 e + e^T \alpha_1 e\theta + e^T f(\hat{x},u)\tilde{\theta} - e^T L C e + e^T \gamma C\, sign(e) - \rho \dot{\hat{\theta}}^T \tilde{\theta}$$

$$sign(e) = \frac{e}{|e|}$$

$$\dot{V}_o \le e^T (A - LC + \alpha_2 + \alpha_1 \theta + \gamma C) e + e^T f(\hat{x},u)\tilde{\theta} - \rho \dot{\hat{\theta}}^T \tilde{\theta} \tag{A.12}$$

Introducing the equation (A.12) and (A.10) into the expression (A.2), it becomes:

$$\dot{V} \le -\sum_{i=1}^{n} c_i z_i^2 + z_n \Big(\sum_{j=1}^{n-1} c_j z_n + (-1)^{n-1} \sum_{k=1}^{n-2} \Big(\sum_{j=0}^{k} c_1^{k-j} c_2^j\Big) z_k - c_{n-1}\sum_{j=1}^{n-1} c_j z_{n-1} - \sum_{j=1}^{n-2} c_2^j z_{j+1} + \sum_{j=1}^{n-2}\big[z_{n-1} - c_j z_{n-2}\big] + g_n(\hat{x})u + f_n^T(\hat{x})\hat{\theta} + \phi_n + c_n z_n - y_r^{(n)} - \sum_{i=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \hat{x}_i}(\hat{x}_{i+1} + \phi_i) - \sum_{i=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \hat{x}_i} f_i^T(\hat{x})\hat{\theta} - \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}}\dot{\hat{\theta}} + f_n(\hat{x})\Big) + e^T (A - LC + \alpha_2 + \alpha_1 \theta + \gamma C) e + e^T f(\hat{x},u)\tilde{\theta} - \rho \dot{\hat{\theta}}^T \tilde{\theta}$$

The backstepping sliding mode controller u has the following form:

$$u = \bar{u} + sign(z_1)(kz_1 + \lambda)$$

To satisfy, the Lyapunov condition $\dot{V} \le 0$, we can just write:

1) $A - LC + \alpha_2 + \alpha_1 \theta + \gamma C \le 0$

That is: $LC \ge A + \alpha_2 + \alpha_1 \theta + \gamma C \ge \gamma C$

Where $\alpha_i \in R^{nxn}$ is a diagonal matrix for all $i = 1,2$ and $\theta \in R^p$

Then $L \ge \gamma$

2) $e^T f(\hat{x},u)\tilde{\theta} - \rho \dot{\hat{\theta}}^T \tilde{\theta} = 0$

That is $\dot{\hat{\theta}}^T = \frac{1}{\rho} e^T f(\hat{x},u)$ or $\frac{1}{\rho}$ is a scalar

Than $\dot{\hat{\theta}} = \frac{1}{\rho} f^T(\hat{x},u) e$

3) The backstepping sliding mode controller is

$$u = \frac{1}{g_n(\hat{x})}\left[\begin{array}{l} \sum_{j=1}^{n-1} c_j z_n - c_{n-1}\sum_{j=1}^{n-1} c_j z_{n-1} - \sum_{j=1}^{n-2} c_2^j z_{j+1} + \\ (-1)^{n-1}\sum_{k=1}^{n-2}\Big(\sum_{j=0}^{k} c_1^{k-j} c_2^j\Big) z_k + \\ \sum_{j=1}^{n-2}\big[z_{n-1} - c_j z_{n-2}\big] - \sum_{i=1}^{n} a_i \hat{x}_i - f_n^T(\hat{x})\hat{\theta} + \\ \sum_{i=1}^{n-1}\frac{\partial \alpha_{n-1}}{\partial \hat{x}_i}(\hat{x}_{i+1} + \phi_i) + \\ \sum_{i=1}^{n-1}\frac{\partial \alpha_{n-1}}{\partial \hat{x}_i} f_i^T(\hat{x})\hat{\theta} + \sum_{i=1}^{n-1}\frac{\partial \alpha_{n-1}}{\partial y_r^{(i-1)}} y_r^{(i)} + \\ \frac{\partial \alpha_{n-1}}{\partial \hat{\theta}}\dot{\hat{\theta}} - \phi_n + y_r^{(n)} \end{array}\right] + (kz_1 + \lambda) sign(z_1)$$

$$\lambda = -k|z_1|$$

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