Strong Convergence of SP Iterative Scheme for Quasi-Contractive Operators

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ABSTRACT
In this paper , we study the strong convergence of SP iterative scheme for quasi-contractive operators in Banach spaces. We show that Picard , Mann , Ishikawa , Noor, new two step and SP iterative schemes are equivalent for quasi-contractive operators. In addition, we show that the rate of convergence of SP iterative scheme is better than the other iterative schemes mentioned above for increasing and decreasing functions .

General Terms
Computational Mathematics

Keywords
SP iteration, Picard iteration, Mann iteration ,Ishikawa iteration, Noor iteration , new two step iteration , Strong convergence, Quasi-contractive operators

1. INTRODUCTION AND PRELIMINARIES
There is a close relationship between the problem of solving a nonlinear equation and that of approximating fixed points of a corresponding contractive type operator. Consequently , there is a theoretical and practical interest in approximating fixed points of various contractive type operators. Let (X, d) be a complete metric space and $T : X \longrightarrow X$ a selfmap of X. Suppose that $F_T = \{ p \in X, Tp = p\}$ is the set of fixed points of T. There are several iterative processes in the literature for which the fixed points of operators have been approximated over the years by various authors. In a complete metric space, the Picard iterative process $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n \; n = 0, 1, \ldots \tag{1.1}$$

has been employed to approximate the fixed points of mappings satisfying the inequality

$$d(Tx, Ty) \le \alpha\, d(x, y) \tag{1.2}$$

for all $x , y \in X$ and $\alpha \in [0, 1)$.

Condition (1.2) is called the Banach's contraction condition. Any operator satisfying(1.2) is called a strict contraction.

In 1953, W.R. Mann defined the Mann iteration [5] as

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n Tu_n \ , \tag{1.3}$$

where $\{ \alpha_n \}$ is a sequences of positive numbers in [0,1].

In 1974, S. Ishikawa defined the Ishikawa iteration [4] as

$$s_{n+1} = (1-\alpha_n)s_n + \alpha_n Tt_n$$
$$t_n = (1-\beta_n)s_n + \beta_n Ts_n \ , \tag{1.4}$$

where $\{ \alpha_n \}$ and $\{ \beta_n \}$ are sequences of positive numbers in [0,1].

In 2008, S. Thianwan defined the new two step iteration [19] as

$$v_{n+1} = (1-\alpha_n)w_n + \alpha_n Tw_n$$
$$w_n = (1-\beta_n)v_n + \beta_n Tv_n \ , \tag{1.5}$$

where $\{ \alpha_n \}$ and $\{ \beta_n \}$ are sequences of positive numbers in [0,1].

In 2001, M. A .Noor defined the three step Noor iteration [6] as

$$p_{n+1} = (1-\alpha_n)p_n + \alpha_n Tq_n$$
$$q_n = (1-\beta_n)p_n + \beta_n Tr_n$$
$$r_n = (1-\gamma_n)p_n + \gamma_n Tp_n \ , \tag{1.6}$$

where $\{ \alpha_n \}$, $\{ \beta_n \}$ and $\{ \gamma_n \}$ are sequences of positive numbers in [0,1].

Recently , Phuengrattana and Suantai defined the SP iteration [7] as

$$x_{n+1} = (1-\alpha_n)y_n + \alpha_n Ty_n$$
$$y_n = (1-\beta_n)z_n + \beta_n Tz_n$$
$$z_n = (1-\gamma_n)x_n + \gamma_n Tx_n \ , \tag{1.7}$$

where $\{ \alpha_n \}$, $\{ \beta_n \}$ and $\{ \gamma_n \}$ are sequences of positive numbers in [0,1].

Remarks:

1. If $\gamma_n = 0$, then (1.6) reduces to the Ishikawa iteration (1.4).
2. If $\beta_n = \gamma_n = 0$,then (1.6) reduces to the Mann iteration (1.3).
3. If $\beta_n = 0$, then (1.5) reduces to the Mann iteration (1.3).
4. If $\beta_n = \gamma_n = 0$,then (1.7) reduces to the Mann iteration (1.3).
5. If $\gamma_n = 0$,then (1.7) reduces to the new two step iteration (1.5).

In 1972, Zamfirescu [21] obtained the following interesting fixed point theorem:

Theorem 1.1. Let (X, d) be a complete metric space and $T : X \longrightarrow X$ a mapping for which there exists real numbers a, b and

c satisfying $a \in (0,1)$, b , $c \in (0, \frac{1}{2})$ such that for each pair $x, y \in X$ at least one of the following conditions hold

(i) $d(Tx,Ty) \le a\, d(x,y)$

(ii) $d(Tx,Ty) \le b[d(x,Tx) + d(y,Ty)]$

(iii) $d(Tx,Ty) \le c[d(x,Ty) + d(y,Tx)]$ (1.8)

Then T has a unique fixed point p and the Picard iteration $\{x_n\}$ defined by

$$x_{n+1} = Tx_n,\ n = 0, 1, \ldots$$

converges to p for any arbitrary but fixed $x_0 \in X$.

The operators satisfying the condition (1.8) are called Zamfirescu operators.

Berinde[1] introduced a new class of operators on an arbitrary Banach space satisfying

$$d(Tx,Ty) \le 2\delta\, d(x,Tx) + \delta d(x,y) \quad (1.9)$$

$\forall$ x, y $\in X$ and $\delta \in [0,1)$ .He proved that this class is wider than the class of Zamfirescu operators and used the Ishikawa iteration process to approximate fixed points of this class of operators in an arbitrary Banach space given in the form of following theorem.

Theorem 1.2[1] Let K be a nonempty closed convex subset of an arbitrary Banach space X and $T : K \longrightarrow K$ a mapping satisfying (1.9). Let $\{s_n\}_{n=0}^{\infty}$ be defined through the Ishikawa iteration (1.4) and $x_0 \in K$, where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences of positive numbers in [0,1] with $\{\alpha_n\}$ satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then $\{s_n\}_{n=0}^{\infty}$ converges strongly to the fixed point of T .

Rafiq [8] studied the convergence of the three step iteration process for quasi-contractive operators.

Several authors [3,10-18, 20, 22] have studied the equivalence between various iterative schemes. S.M. Solutz[17,18] proved that Picard, Mann ,Ishikawa and Noor iterations are equlivalent for quasi-contractive operators. In 2009, Isa Yildirim et.al. [23] proved that Picard ,new two step, Mann and Ishikawa iterations are equivalent for quasi- contractive operators .

Fixed point iterative procedures are designed to be applied in solving equations arising in physical formulation but there is no systematic study of numerical aspects of these iterative procedures . In computational mathematics , it is of vital interest to know which of the given iterative procedures converge faster to a desired solution , commonly known as rate of convergence . B. E. Rhoades [9] compared the Mann and Ishikawa iterative procedures concerning their rate of convergence . he illustrated the difference in the rate of convergence for increasing and for deceasing functions . Indeed he used computer programs , perhaps for the first time , to compare the Mann and Ishikawa iterations through examples . S.L. Singh[16] extended the work of Rhoades.

We shall need the following Lemma:

Lemma 1. [2] If δ is a real number such that $0 \le \delta < 1$, and $\{\epsilon_n\}_{n=0}^{\infty}$ is a sequence of positive numbers such that $\lim_{n\to\infty} \epsilon_n = 0$, then for any sequence of positive numbers $\{u_n\}_{n=0}^{\infty}$ satisfying

$$u_{n+1} \le \delta u_n + \epsilon_n, \quad n = 0,1,2,\ldots.$$

we have $\lim_{n\to\infty} u_n = 0$.

The intérest of this paper is to study the strong convergence of SP iterative scheme and show equivalence between the SP iterative scheme and other above mentioned iterative schemes for quasi-contractive operators . Also, with the help of C++ programming ,we show that for increasing functions and decreasing functions , the SP iterative scheme converges faster than the other iterative schemes mentioned earlier.

2. RESULT ON STRONG CONVERGENCE

Theorem 2.1. Let K be a nonempty closed convex subset of an arbitrary Banach space X and $T : K \longrightarrow K$, a mapping satisfying (1.9) . Let $\{x_n\}_{n=0}^{\infty}$ be defined through the SP iteration (1.7) and $x_0 \in K$, where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences of positive numbers in [0,1] with $\{\alpha_n\}$ satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the fixed point of T .

Proof : Theorem 1.1 shows that T has a unique fixed point in K, say p.

From (1.7) , we have

$$\| x_{n+1} - p \| \le (1-\alpha_n) \| y_n - p \| + \alpha_n \| Ty_n - p \| \quad (2.1)$$

It follows from (1.9) and (2.1) that

$$\| x_{n+1} - p \| \le (1-\alpha_n) \| y_n - p \| + \alpha_n \delta \| y_n - p \| + 2\alpha_n \delta \| p - Tp \|$$
$$= [1-\alpha_n(1-\delta)] \| y_n - p \| \quad (2.2)$$

Similarly from (1.7), we have the following estimates:

$$\| z_n - p \| \le (1-\gamma_n) \| x_n - p \| + \gamma_n \delta \| x_n - p \|$$
$$= [1-\gamma_n(1-\delta)] \, \| x_n - p \| \quad (2.3)$$

$$\| y_n - p \| \le (1-\beta_n) \| z_n - p \| + \beta_n \delta \| z_n - p \|$$
$$= [1-\beta_n(1-\delta)] \, \| z_n - p \| \quad (2.4)$$

Using (2.3) and (2.4) in (2.2), we have

$$\| x_{n+1} - p \| \le [1-\alpha_n(1-\delta)][1-\beta_n(1-\delta)][1-\gamma_n(1-\delta)] \, \| x_n - p \|$$
$$\le [1-\alpha_n(1-\delta)] \, \| x_n - p \|$$
$$\le \prod_{k=0}^{n} [1-\alpha_k(1-\delta)] \, \| x_0 - p \|$$
$$\le e^{-(1-\delta)\sum_{k=0}^{\infty} \alpha_k} \left\| x_0 - p \right\| \quad (2.5)$$

Since $0 \le \delta < 1$, $\alpha_k \in [0,1]$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, so $e^{-(1-\delta)\sum_{k=0}^{n} \alpha_k} \to 0$ as $n \to \infty$. Hence ,it follows from (2.5) that $\lim_{n\to\infty} \left\| x_{n+1} - p \right\| = 0$.

Therefore $\{x_n\}_{n=0}^{\infty}$ converges strongly to p.

3. EQUIVALENCE BETWEEN SP AND OTHER ITERATIVE SCHEMES

Theorem 3.1 Let K be a nonempty closed convex subset of an arbitrary Banach space X and $T : K \to K$ a mapping satisfying (1.9) . If $u_0 = x_0$ and $\alpha_n \geq A > 0, \forall\ n \in N,$ then the following are equivalent:

1.The Mann iteration (1.3) converges to p ,

2.The SP iteration (1.7) converges to p.

Proof: Let the Mann iteration (1.3) converges to p i.e. $u_n \to p$. We shall show the SP iteration (1.7) also converges to p i.e. $x_n \to p$.

It follows from (1.3) ,(1.7) and (1.9), that

$$\begin{aligned}\| x_{n+1} - u_{n+1} \| &\leq (1-\alpha_n) \| y_n - u_n \| + \alpha_n \| Ty_n - Tu_n \| \\ &\leq (1-\alpha_n) \| y_n - u_n \| + \alpha_n \delta \| y_n - u_n \| + 2\alpha_n \delta \| u_n - Tu_n \| \\ &\leq [1-\alpha_n(1-\delta)] \| y_n - u_n \| + 2\alpha_n \delta \| u_n - Tu_n \| \qquad (3.1)\end{aligned}$$

Using (1.7) and (1.9), we have

$$\begin{aligned}\|y_n - u_n\| &\leq (1-\beta_n)\|z_n - u_n\| + \beta_n \|Tz_n - u_n\| \\ &\leq (1-\beta_n)(1-\gamma_n)\|x_n - u_n\| + (1-\beta_n)\gamma_n \|Tx_n - u_n\| \\ &\quad + \beta_n\{\|Tz_n - Tu_n\| + \|u_n - Tu_n\|\} \\ &\leq (1-\beta_n)(1-\gamma_n)\|x_n - u_n\| + (1-\beta_n)\gamma_n \|Tx_n - Tu_n\| \\ &\quad + (1-\beta_n)\gamma_n \|Tu_n - u_n\| + \beta_n \delta \|z_n - u_n\| \\ &\quad + 2\beta_n \delta \|u_n - Tu_n\| + \beta_n \|u_n - Tu_n\| \\ &\leq (1-\beta_n)[1-\gamma_n(1-\delta)]\|x_n - u_n\| + 2\delta(1-\beta_n)\gamma_n \|u_n - Tu_n\| \\ &\quad + [\gamma_n(1-\beta_n) + (2\delta+1)\beta_n]\|Tu_n - u_n\| + \beta_n \delta \|z_n - u_n\| \qquad (3.2)\end{aligned}$$

Using (1.9) and (1.7) we have the following estimate :

$$\begin{aligned}\| z_n - u_n \| &\leq (1-\gamma_n) \| x_n - u_n \| + \gamma_n \| Tx_n - u_n \| \\ &\leq (1-\gamma_n) \| x_n - u_n \| + \gamma_n \{\| Tx_n - Tu_n \| + \| Tu_n - u_n \|\} \\ &\leq (1-\gamma_n) \| x_n - u_n \| + \gamma_n \delta \| x_n - u_n \| + 2\gamma_n \delta \| Tu_n - u_n \| + \gamma_n \| Tu_n - u_n \| \\ &= [1-\gamma_n(1-\delta)] \| x_n - u_n \| + (2\delta+1)\gamma_n \| Tu_n - u_n \| \qquad (3.3)\end{aligned}$$

Substituting (3.3) in (3.2) we have

$$\begin{aligned}\| y_n - u_n \| &\leq [1-\beta_n(1-\delta)][1-\gamma_n(1-\delta)] \| x_n - u_n \| \\ &\quad + [(2\delta+1)\beta_n + (1-\beta_n)\gamma_n \\ &\quad + 2\delta(1-\beta_n)\gamma_n + 2\delta^2\beta_n\gamma_n + \delta\ \beta_n\gamma_n] \| Tu_n - u_n \| \qquad (3.4)\end{aligned}$$

Also,

$$\begin{aligned}\| u_n - Tu_n \| &\leq \| u_n - p \| + \| p - Tu_n \| \\ &= (1+\delta) \| u_n - p \| \qquad (3.5)\end{aligned}$$

Using (3.4) and (3.5) in (3.1), we have

$$\begin{aligned}\| x_{n+1} - u_{n+1} \| &\leq [1-\alpha_n(1-\delta)][1-\beta_n(1-\delta)][1-\gamma_n(1-\delta)] \| x_n - u_n \| \\ &\quad + \{[1-\alpha_n(1-\delta)][(2\delta+1)\beta_n + (1-\beta_n)\gamma_n + 2\delta(1-\beta_n)\gamma_n \\ &\quad + 2\delta^2\beta_n\gamma_n + \delta\ \beta_n\gamma_n] + 2\delta\alpha_n\}(\delta+1) \| u_n - p \| \\ &\leq h\|x_n - u_n\| + l_n\|u_n - p\| , \qquad (3.6)\end{aligned}$$

where $h = [1-\alpha_n(1-\delta)][1-\beta_n(1-\delta)][1-\gamma_n(1-\delta)] < 1$ if $\alpha_n \geq A > 0$ and

$$\begin{aligned}l_n &= [1-\alpha_n(1-\delta)][(2\delta+1)\beta_n + (1-\beta_n)\gamma_n + 2\delta(1-\beta_n)\gamma_n \\ &\quad + 2\delta^2\beta_n\gamma_n + \delta\ \beta_n\gamma_n] + 2\delta\alpha_n\}(\delta+1) \cdot\end{aligned}$$

Using $u_n \to p$ as $n \to \infty$ and Lemma 1.1, (3.6) yields

$$\|x_n - u_n\| \to 0 \text{ as } n \to \infty.$$

In addition

$$\|x_n - p\| \leq \|x_n - u_n\| + \|u_n - p\|$$

and this implies that $x_n \to p$ as $n \to \infty$.

Conversely, we prove that $x_n \to p$ implies $u_n \to p$.

Using (1.3) , (1.7) and (1.9) , we have

$$\begin{aligned}\| x_{n+1} - u_{n+1} \| &\leq (1-\alpha_n) \| y_n - u_n \| + \alpha_n \| Ty_n - Tu_n \| \\ &\leq (1-\alpha_n) \| y_n - u_n \| + \alpha_n \delta \| y_n - u_n \| + 2\alpha_n \delta \| y_n - Ty_n \| \\ &\leq [1-\alpha_n(1-\delta)] \| y_n - u_n \| + 2\alpha_n \delta \| y_n - Ty_n \| \qquad (3.7)\end{aligned}$$

Using (1.7) and (1.9), we have

$$\begin{aligned}\|y_n - u_n\| &\leq (1-\beta_n)\|z_n - u_n\| + \beta_n \|Tz_n - u_n\| \\ &\leq (1-\beta_n)(1-\gamma_n)\|x_n - u_n\| + (1-\beta_n)\gamma_n \|Tx_n - u_n\| \\ &\quad + \beta_n\{\|Tz_n - x_n\| + \|x_n - u_n\|\} \\ &\leq (1-\beta_n)(1-\gamma_n)\|x_n - u_n\| \\ &\quad + (1-\beta_n)\gamma_n\{\|Tx_n - x_n\| + \|x_n - u_n\|\} \\ &\quad + \beta_n \|Tz_n - x_n\| + \beta_n \|x_n - u_n\| \\ &\leq \|x_n - u_n\| + (1-\beta_n)\gamma_n\{\|Tx_n - p\| + \|x_n - p\|\} \\ &\quad + \beta_n \|Tz_n - p\| + \beta_n \|x_n - p\| \\ &\leq \|x_n - u_n\| + (1-\beta_n)\gamma_n(1+\delta)\|x_n - p\| \\ &\quad + \beta_n \delta[1-\gamma_n(1-\delta)]\|x_n - p\| + \beta_n \|x_n - p\| \\ &= \|x_n - u_n\| + [\gamma_n\{1+\delta-\beta_n(1+\delta)\} + \beta_n\{\delta(1-\gamma_n) \\ &\quad + \gamma_n\delta^2 + 1\}]\|x_n - p\| \qquad (3.8)\end{aligned}$$

Again, using (1.7) and (1.9), we have the following estimate:

$$\begin{aligned}\| y_n - Ty_n \| &\leq \| y_n - p \| + \| Ty_n - p \| = (1+\delta) \| y_n - p \| \\ &\leq (1+\delta)(1-\beta_n) \| z_n - p \| + \delta(1+\delta)\beta_n \| z_n - p \| \\ &= (1+\delta)[1-\beta_n(1-\delta)] \| z_n - p \| \\ &\leq (1+\delta)[1-\beta_n(1-\delta)][1-\gamma_n(1-\delta)] \| x_n - p \| \qquad (3.9)\end{aligned}$$

Substituting (3.8) and (3.9) in (3.7), we have

$$\begin{aligned}\| x_{n+1} - u_{n+1} \| &\leq [1-\alpha_n(1-\delta)] \| x_n - u_n \| \\ &\quad + \{[1-\alpha_n(1-\delta)][\gamma_n\{1+\delta-\beta_n(1+\delta)\} \\ &\quad + \beta_n\{\delta(1-\gamma_n) + \gamma_n\delta^2 + 1\}]\|x_n - p\| \\ &\quad + 2\alpha_n\delta(1+\delta)[1-\beta_n(1-\delta)][1-\gamma_n(1-\delta)] \| x_n - p \| \qquad (3.10)\end{aligned}$$

Since $\alpha_n \geq A > 0, \forall\ n \in N$, so $0 \leq 1-\alpha_n(1-\delta) < 1, \forall\ n \in N.$ Also $x_n \to p$ as $n \to \infty$ .Hence using Lemma 1.1 , (3.10) yields $\|x_n - u_n\| \to 0$ as $n \to \infty$.

In addition $\|u_n - p\| \leq \|x_n - u_n\| + \|x_n - p\|$ and this implies that $u_n \to p$ as $n \to \infty$. Hence the result.

Keeping in mind Solutz's results [17,18] and Isa Yildirim et al's results [20], Theorem 3.1 leads to the following corollary:

Corollary 3.2. Let K be a nonempty closed convex subset of a Banach space X and $T : K \rightarrow K$ a mapping satisfying (1.9). Ifs the initial point is same for all itérations , $\alpha_n \geq A > 0, \forall\ n \in N$, then the following are equivalent :

(i) The Picard iteration (1.1) converges to the fixed point p of T
(ii) The Mann iteration (1.3) converges to p .
(iii) The Ishikawa iteration (1.4) converges to p .
(iv) The new two step iteration (1.5) converges to p .
(v) The Noor iteration (1.6) converges to p .
(vi) The SP iteration (1.7) converges to p .

4. EXPERIMENTS

In this section, with the help of computer programming in C++ , we study the nature of convergence of Picard ,Mann, Ishikawa , new two step ,Noor and SP iterative schemes to locate a fixed point of increasing function $2x^3–7x^2 + 8x–2$ and decreasing function $(1-x)^m$. The outcome is listed in the form of Tables1 and Table 2 by taking initial approximation x_0=0.8 and $\alpha_n=\beta_n=\gamma_n=\frac{1}{(1+n)^{\frac{1}{2}}}$ for all iterative schemes.

For detailed study, these programs are again executed after changing the parameters and the readings are recorded (discussed in the next section).

5. OBSERVATIOINS

Decreasing function $(1-x)^m$

1.For m = 8 and x_o= 0.8, Picard scheme never converges (oscillates between 0 and 1), Mann scheme converges in 9 iterations, Ishikawa scheme converges in 35 iterations, Noor scheme converges in 10 iterations , the new two step scheme converges in 8 iterations and the SP scheme converges in 7 iterations.

2. For m=30 and x_o=0.8 , Picard scheme converges in 13 iterations, Ishikawa scheme converges in 40 iterations, Noor scheme converges in 12 iterations , the new two step scheme converges in 11 iterations and the SP scheme converges in 9 iterations.

3.Taking initial guess x_o = 0.2 (nearer to the fixed point), Picard scheme never converges (oscillates between 0 and 1), Mann scheme converges in 10 iterations, Ishikawa scheme converges in 34 iterations, Noor scheme converges in 11 iterations, the new two step scheme converges in 8 iterations and the SP scheme converges in 8 iterations.

4.Taking $\alpha_n=\beta_n=\gamma_n=\frac{1}{(1+n)^{\frac{1}{4}}}$ and x_o= 0.8 , we obtain that the Mann scheme converges in 23 iterations, the Ishikawa iteration converges in 56 iterations, Noor scheme converges in 21 iterations , the new two step scheme converges in 18 iterations while the SP scheme converges in 15 iterations.

Increasing function $2x^3–7x^2 + 8x–2$

1. For x_o=0.8 ,The Picard scheme converges to a fixed point in 4 iterations , Mann scheme converges in 36 iterations, Ishikawa scheme converges in 22 iterations, Noor scheme converges in 5 iterations, the new two step scheme converges in 8 iterations and the SP scheme converges in 2 iterations.

2.Taking initial guses x_o=0.6 (away from the fixed point), Picard scheme converges to a fixed point in 5 iterations, the Mann scheme converges in 52 iterations, the Ishikawa scheme converges in 40 iterations, Noor scheme converges in 30 iterations, the new two step scheme converges in 13 iterations and the SP scheme converges in 4 iterations.

3.Taking $\alpha_n=\beta_n=\gamma_n=\frac{1}{(1+n)^{\frac{1}{4}}}$, x_o= 0.8 we obtain that Mann scheme converges in 13 iterations, the Ishikawa iteration converges in 9 iterations, Noor scheme converges in 3 iterations, the new two step scheme converges in 4 iterations and the SP scheme converges in 2 iterations.

6. CONCLUSIONS

For ***Decreasing function***, we observe the following :

1. Picard iteration never converges and increasing order of rate of convergence for other iterative schemes is Ishikawa ,Noor, Mann , new two step and SP iteration.

2. On increasing the value of m, rate of convergence of each iterative scheme decreases i.e each iterative scheme takes more number of iterations to converge while their order of convergence remains same.

3. For initial guess nearer to the fixed point , Mann , Noor and SP schemes shows an increase , Ishikawa scheme shows an decrease in the number of iterations while the new two step scheme shows no change.

4. The speed of iterative schemes depends on α_n and β_n . If we increase the value of α_n and β_n ,the fixed point is obtained in more number of iterations for all schemes.

For ***Increasing function***, we observe the following :

1. Increasing order of rate of convergence for iterative schemes is Mann, Ishikawa , new two step , Noor , Picard and SP scheme.

2. For initial guess away from the fixed point , the number of iterations increases in each iterative scheme. Hence , closer the initial guess to the fixed point ,quicker the result is achieved.

3. If we increase the value of α_n and β_n , the fixed point is obtained in less number of iterations for all schemes (except SP scheme ,which remains almost unaffected).

Table 1 (Decreasing Function)

SP iteration			Noor iteration		Picard iteration		Mann iteration		Ishikawa iteration		New two step iteration	
n	fx_n	x_{n+1}	fx_n	x_{n+1}	fx_n	x_{n+1}	fx_n	x_{n+1}	fx_n	x_{n+1}	fx_n	x_{n+1}
0	2.56e-06	3.094628e-38	2.56e-06	3.094628e-38	2.56e-06	2.56e-06	2.56e-06	2.56e-06	2.56e-06	0.99998	2.56e-06	0.99998
1	1	0.171088	1	0.70689	0.99998	0.99998	0.99998	0.707093	3.094628e-38	0.337084	3.094628e-38	0.129982
2	0.222878	0.193474	5.448075e-05	0.321518	3.094628e-38	3.094628e-38	5.41798e-05	0.298884	0.037296	0.280216	0.328266	0.164626
3	0.179038	0.18796	0.044906	0.205813	1	1	0.058387	0.178636	0.072047	0.246237	0.237163	0.183586
4	0.189069	0.188356	0.158264	0.187899	0	0	0.207151	0.191388	0.104203	0.225137	0.197371	0.187965
5	0.188332	0.188348	0.189183	0.188327	1	1	0.182777	0.187873	0.129956	0.211699	0.189059	0.188337
6	**0.188348**	**0.188348**	0.188386	0.188345	0	0	0.189232	0.188386	0.14912	0.203085	0.188367	0.188348
7	**0.188348**	**0.188348**	0.188353	0.188347	1	1	0.188276	0.188347	0.162666	0.197582	**0.188348**	**0.188348**
8	**0.188348**	**0.188348**	0.188349	0.188347	0	0	**0.188348**	**0.188348**	0.171873	0.194094	**0.188348**	**0.188348**
9	**0.188348**	**0.188348**	**0.188348**	**0.188348**	1	1	**0.188348**	**0.188348**	0.177941	0.191903	**0.188348**	**0.188348**
10	**0.188348**	**0.188348**	**0.188348**	**0.188348**	0	0	**0.188348**	**0.188348**	0.181848	0.190539	**0.188348**	**0.188348**
11	**0.188348**	**0.188348**	**0.188348**	**0.188348**	1	1	**0.188348**	**0.188348**	0.184318	0.189695	**0.188348**	**0.188348**
12	**0.188348**	**0.188348**	**0.188348**	**0.188348**	0	0	**0.188348**	**0.188348**	0.185862	0.189175	**0.188348**	**0.188348**
13	**0.188348**	**0.188348**	**0.188348**	**0.188348**	1	1	**0.188348**	**0.188348**	0.186817	0.188856	**0.188348**	**0.188348**
14	**0.188348**	**0.188348**	**0.188348**	**0.188348**	0	0	**0.188348**	**0.188348**	0.187406	0.188661	**0.188348**	**0.188348**
15	**0.188348**	**0.188348**	**0.188348**	**0.188348**	1	1	**0.188348**	**0.188348**	0.187767	0.188541	**0.188348**	**0.188348**
16	**0.188348**	**0.188348**	**0.188348**	**0.188348**	0	0	**0.188348**	**0.188348**	0.187989	0.188467	**0.188348**	**0.188348**
17	**0.188348**	**0.188348**	**0.188348**	**0.188348**	1	1	**0.188348**	**0.188348**	0.188126	0.188422	**0.188348**	**0.188348**
18	**0.188348**	**0.188348**	**0.188348**	**0.188348**	0	0	**0.188348**	**0.188348**	0.18821	0.188394	**0.188348**	**0.188348**
19	**0.188348**	**0.188348**	**0.188348**	**0.188348**	1	1	**0.188348**	**0.188348**	0.188262	0.188377	**0.188348**	**0.188348**
20	**0.188348**	**0.188348**	**0.188348**	**0.188348**	0	0	**0.188348**	**0.188348**	0.188294	0.188366	**0.188348**	**0.188348**
21	**0.188348**	**0.188348**	**0.188348**	**0.188348**	1	1	**0.188348**	**0.188348**	0.188314	0.188359	**0.188348**	**0.188348**
22	**0.188348**	**0.188348**	**0.188348**	**0.188348**	0	0	**0.188348**	**0.188348**	0.188326	0.188355	**0.188348**	**0.188348**
23	**0.188348**	**0.188348**	**0.188348**	**0.188348**	1	1	**0.188348**	**0.188348**	0.188334	0.188352	**0.188348**	**0.188348**
24	**0.188348**	**0.188348**	**0.188348**	**0.188348**	0	0	**0.188348**	**0.188348**	0.188339	0.188351	**0.188348**	**0.188348**
25	**0.188348**	**0.188348**	**0.188348**	**0.188348**	1	1	**0.188348**	**0.188348**	0.188342	0.18835	**0.188348**	**0.188348**
26	**0.188348**	**0.188348**	**0.188348**	**0.188348**	0	0	**0.188348**	**0.188348**	0.188344	0.188349	**0.188348**	**0.188348**
27	**0.188348**	**0.188348**	**0.188348**	**0.188348**	1	1	**0.188348**	**0.188348**	0.188345	0.188348	**0.188348**	**0.188348**
28	**0.188348**	**0.188348**	**0.188348**	**0.188348**	0	0	**0.188348**	**0.188348**	0.188346	0.188348	**0.188348**	**0.188348**
29	**0.188348**	**0.188348**	**0.188348**	**0.188348**	1	1	**0.188348**	**0.188348**	0.188347	0.188348	**0.188348**	**0.188348**
30	**0.188348**	**0.188348**	**0.188348**	**0.188348**	0	0	**0.188348**	**0.188348**	0.188347	0.188348	**0.188348**	**0.188348**
31	**0.188348**	**0.188348**	**0.188348**	**0.188348**	1	1	**0.188348**	**0.188348**	0.188347	0.188348	**0.188348**	**0.188348**
32	**0.188348**	**0.188348**	**0.188348**	**0.188348**	0	0	**0.188348**	**0.188348**	0.188347	0.188348	**0.188348**	**0.188348**
33	**0.188348**	**0.188348**	**0.188348**	**0.188348**	1	1	**0.188348**	**0.188348**	0.188347	0.188348	**0.188348**	**0.188348**
34	**0.188348**	**0.188348**	**0.188348**	**0.188348**	0	0	**0.188348**	**0.188348**	**0.188348**	**0.188348**	**0.188348**	**0.188348**
35	**0.188348**	**0.188348**	**0.188348**	**0.188348**	1	1	**0.188348**	**0.188348**	**0.188348**	**0.188348**	**0.188348**	**0.188348**
36	**0.188348**	**0.188348**	**0.18834**	**0.188348**	0	0	**0.188348**	**0.188348**	**0.188348**	**0.188348**	**0.188348**	**0.188348**
37	**0.188348**	**0.188348**	**0.188348**	**0.188348**	1	1	**0.188348**	**0.188348**	**0.188348**	**0.188348**	**0.188348**	**0.188348**
38	**0.188348**	**0.188348**	**0.188348**	**0.188348**	0	0	**0.188348**	**0.188348**	**0.188348**	**0.188348**	**0.188348**	**0.188348**
39	**0.188348**	**0.188348**	**0.188348**	**0.188348**	1	1	**0.188348**	**0.188348**	**0.188348**	**0.188348**	**0.188348**	**0.188348**

Table 2 (Increasing function)

SP iteration			Noor iteration		Picard iteration		Mann iteration		Ishikawa iteration		New two step iteration	
n	fx_n	x_{n+1}	fx_n	x_{n+1}	fx_n	x_{n+1}	fx_n	x_{n+1}	fx_n	x_{n+1}	fx_n	x_{n+1}
0	0.944	0.999988	0.944	0.999988	0.944	0.944	0.944	0.944	0.944	0.996513	0.944	0.996513
1	**1**	**1**	1	0.999996	0.996513	0.996513	0.996513	0.981132	0.99988	0.998978	0.999988	0.999698
2	**1**	**1**	1	0.999998	0.999988	0.999988	0.999631	0.991812	0.999999	0.999568	1	0.999946
3	**1**	**1**	1	0.999999	**1**	**1**	0.999932	0.995872	1	0.999784	1	0.999986
4	**1**	**1**	**1**	**1**	**1**	**1**	0.999983	0.99771	1	0.999881	1	0.999996
5	**1**	**1**	**1**	**1**	**1**	**1**	0.999995	0.998643	1	0.999929	1	0.999999
6	**1**	**1**	**1**	**1**	**1**	**1**	0.999998	0.999155	1	0.999956	1	0.999999
7	**1**	**1**	**1**	**1**	**1**	**1**	0.999999	0.999454	1	0.999972	**1**	**1**
8	**1**	**1**	**1**	**1**	**1**	**1**	1	0.999636	1	0.999981	**1**	**1**
9	**1**	**1**	**1**	**1**	**1**	**1**	1	0.999751	1	0.999987	**1**	**1**
10	**1**	**1**	**1**	**1**	**1**	**1**	1	0.999826	1	0.999991	**1**	**1**
11	**1**	**1**	**1**	**1**	**1**	**1**	1	0.99876	1	0.999994	**1**	**1**
12	**1**	**1**	**1**	**1**	**1**	**1**	1	0.999911	1	0.999995	**1**	**1**
13	**1**	**1**	**1**	**1**	**1**	**1**	1	0.999934	1	0.999997	**1**	**1**
14	**1**	**1**	**1**	**1**	**1**	**1**	1	0.999951	1	0.999997	**1**	**1**
15	**1**	**1**	**1**	**1**	**1**	**1**	1	0.999964	1	0.999998	**1**	**1**
16	**1**	**1**	**1**	**1**	**1**	**1**	1	0.999972	1	0.999999	**1**	**1**
17	**1**	**1**	**1**	**1**	**1**	**1**	1	0.999979	1	0.999999	**1**	**1**
18	**1**	**1**	**1**	**1**	**1**	**1**	1	0.999984	1	0.999999	**1**	**1**
19	**1**	**1**	**1**	**1**	**1**	**1**	1	0.999987	1	0.999999	**1**	**1**
20	**1**	**1**	**1**	**1**	**1**	**1**	1	0.99999	1	0.999999	**1**	**1**
21	**1**	**1**	**1**	**1**	**1**	**1**	1	0.999992	**1**	**1**	**1**	**1**
22	**1**	**1**	**1**	**1**	**1**	**1**	1	0.999994	**1**	**1**	**1**	**1**
23	**1**	**1**	**1**	**1**	**1**	**1**	1	0.999995	**1**	**1**	**1**	**1**
24	**1**	**1**	**1**	**1**	**1**	**1**	1	0.999996	**1**	**1**	**1**	**1**
25	**1**	**1**	**1**	**1**	**1**	**1**	1	0.999997	**1**	**1**	**1**	**1**
26	**1**	**1**	**1**	**1**	**1**	**1**	1	0.999997	**1**	**1**	**1**	**1**
27	**1**	**1**	**1**	**1**	**1**	**1**	1	0.999998	**1**	**1**	**1**	**1**
28	**1**	**1**	**1**	**1**	**1**	**1**	1	0.999998	**1**	**1**	**1**	**1**
29	**1**	**1**	**1**	**1**	**1**	**1**	1	0.999999	**1**	**1**	**1**	**1**
30	**1**	**1**	**1**	**1**	**1**	**1**	1	0.999999	**1**	**1**	**1**	**1**
31	**1**	**1**	**1**	**1**	**1**	**1**	1	0.999999	**1**	**1**	**1**	**1**
32	**1**	**1**	**1**	**1**	**1**	**1**	1	0.999999	**1**	**1**	**1**	**1**
33	**1**	**1**	**1**	**1**	**1**	**1**	1	0.999999	**1**	**1**	**1**	**1**
34	**1**	**1**	**1**	**1**	**1**	**1**	1	0.999999	**1**	**1**	**1**	**1**
35	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
36	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
37	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
38	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
39	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**

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8. REFERENCES

[1] Berinde , V., On the convergence of the Ishikawa iteration in the class of quasi-contractive operators, Acta Math. Univ. Comenianae LXXIII(2004), 119-126.

[2] Berinde, V., Iterative Approximation of Fixed Points, Springer-Verlag, Berlin Heidelberg, 2007.

[3] Berinde, V., Picard iteration converges faster than Mann iteration for a class of quasi-contractive operators, Fixed Point Theory and Applications, volume 2(2004), 97-105.

[4] Ishikawa, S. , Fixed points by a new iteration method , Proc. Amer. Math. Soc. 44(1974), 147-150.

[5] Mann , W.R., Mean value methods in iteration, Proc. Amer. Math. Soc. 4(1953), 506-510.

[6] Noor, M. A., New approximation schemes for general variational inequalities, J. Math. Anal. Appl. 251(2000), 217-229.

[7] Phuengrattana , Withunand , Suantai, Suthep , On the rate of convergence of Mann Ishikawa, Noor and SP iterations for continuous functions on an arbitrary interval, Journal of Computational and Applied Mathematics, 235(2011), 3006- 3014.

[8] Rafiq, A.,On the convergence of the three- step iteration process in the class of quasi-contractive operators , Acta Mathematica Academiae Paedagogicae Nyiregyhaziens 22(2006), 305-309.

[9] Rhoades , B. E., Comments on two fixed point iteration methods, J. Math. Anal. Appl. 56(1976), 741-750.

[10] Rhoades , B. E. and Soltuz , S¸.M., On the equivalence of Mann and Ishikawa iteration methods, Int. J. Math. Math. Sci. 2003(2003), 451-459.

[11] Rhoades ,B. E. and Soltuz , S¸.M., The equivalence of the Mann and Ishikawa iteration for non-Lipschitzian operators, Int. J. Math. Math. Sci. 42(2003), 2645-2652.

[12] Rhoades ,B. E. and Soltuz , S¸.M., The equivalence between the convergences of Ishikawa and Mann iterations for asymptotically pseudocontractive map, J. Math. Anal. Appl. 283(2003), 681-688.

[13] Rhoades , B. E. and Soltuz , S¸.M., The equivalence of Mann and Ishikawa iteration for a Lipschitzian psi-uniformly pseudocontractive and psi-uniformly accretive maps, Tamkang J. Math., 35(2004), 235-245.

[14] Rhoades , B. E. and S¸oltuz , S¸.M., The equivalence between the convergences of Ishikawa and Mann iterations for asymptotically nonexpansive in the intermediate sense and strongly successively pseudocontractive maps, J. Math. Anal. Appl. 289(2004), 266-278.

[15] Rhoades , B. E. and Soltuz , S¸.M., The equivalence between Mann-Ishikawa iterations and multistep iteration, Nonlinear Analysis 58(2004), 219-228.

[16] Singh , S. L., A new approach in numerical Praxis, Progress Math.(Varansi) 32(2) (1998),75-89.

[17] Soltuz , S¸.M., The equivalence of Picard, Mann and Ishikawa iterations dealing with quasi-contractive operators, Math. Comm. 10(2005), 81-89.

[18] Soltuz ,S¸. M., The equivalence between Krasnoselskij , Mann, Ishikawa ,Noor and multistep iterations, Mathematical Communications 12(2007),53-61.

[19] Thiainwan, S., Common fixed points of new iterations for two asymptotically nonexpansive nonself mappings in Banach spaces, Journal of Computational and Applied Mathematics,2009,688-695.

[20] Yildirim ,Isa, Ozdemir, Murat and Kiziltunc ,Hukmi, On the convergence of the new two-step iteration process in the class of quasi-contractive operators , Int. Journal of Math Analysis,Vol 3(2009),no. 38 ,1881-1892.

[21] Zamfirescu, T., Fixed point theorems in metric spaces, Arch. , Math., 23(1972), 292-298.

[22] Zhiqun, Xue , Remarks on equivalence among Picard, Mann and Ishikawa iterations in Normed spaces. Fixed Point Theory and Applications, volume 2007, 5 pages.