On Some Reliability Properties of Mean Inactivity Time under Weighing

Neeraj Gandotra
Jaypee University of Information Technology, Waknaghat, H.P., INDIA
PIN – 173 234

Rakesh Kumar Bajaj
Jaypee University of Information Technology, Waknaghat, H.P., INDIA
PIN – 173 234

Nitin Gupta
Jaypee University of Information Technology, Waknaghat, H.P., INDIA
PIN – 173 234

ABSTRACT
In the present communication, we discuss some reliability properties of mean inactivity time order and study the preservation of mean inactivity time under some weight function. Weighted Distribution methods arise in the context of research related to reliability, data gathering, ecology, inference, modeling, bio-medicine and several other areas. In general, weighted distributions have wide role for the analysis of life time data.

General Terms
Reliability, Stochastic Orders, Stochastic aging

Keywords
Mean Inactivity Time, shifted likelihood ratio order, residual life

1. INTRODUCTION
By the method of ascertainment, the concept of weighted distribution has been introduced by C.R. Rao in 1963 (see Rao, [1]). Weighted distribution have been widely used as a tool in various practical problems in the selection of appropriate models for observed data drown without a proper frame, analysis of data relating to human populations and wild life management, investigation of human heredity, line transcend sampling and renewal theory, study of statistical ecology, albinism and reliability modeling. The properties of weighted random sample corresponding to original random sample is required to study when the observations can be recorded as a weighted random sample with some weight attached to the original random sample.

Let $f(\cdot)$ be the probability density function of original random variable *X*. Let $w_1(\cdot): R \to [0,\infty)$, where $R=(-\infty,\infty)$ and the recovered random variable be X_{w_1} with the probability density function given by

$$f_{w_1}(x) = \frac{w_1(x) f(x)}{w_1};\ x \in R;$$

where $w_1 = E(w_1(X)) > 0.$

The random variable X_{w_1} is called the weighted version of *X* and its distribution in relation to that of *X* is called the weighted distribution of *X* with weighted function $w_1(\cdot)$. Jain et. al. [2], Nand & Jain [3], Misra et. al. [4], Barlow et. al. [5], and Bartoszewicz et.al. [6] have studied the reliability properties of weighted distributions in relation to corresponding reliability measures of parent distributions. Here, we further derive some results on preservation of mean inactivity time order and mean inactivity time ordering by weighted distributions.

Let w_1, w_2 be two functions where $w_i : R \to R^+$, $i = 1,2$ such that $0 < E[w_1(X)] < \infty$, $0 < E[w_2(Y)] < \infty$ and $w_1 = E[w_1(X)]$, $w_2 = E[w_2(Y)]$. Let X_{w_1} and Y_{w_2} be the weighted versions of *X* and *Y* with weight functions $w_1(\cdot)$ and $w_2(\cdot)$ respectively. Then X_{w_1} and Y_{w_2} have probability density functions given by

$$f_{w_1}(x) = \frac{w_1(x) f(x)}{w_1}, \quad x \in R$$

and

$$g_{w_2}(x) = \frac{w_2(x) g(x)}{w_2}, \quad x \in R$$

respectively.

Let $F_1(x)(G_2(x)),\ x \in R$ be the distribution function of $X_{w_1}(Y_{w_2})$ and let

$$\overline{F}_1(x) = 1 - F_1(x)(\overline{G}_2(x) = 1 - G_2(x)), \quad x \in R$$

be the survival function of $X_{w_1}(Y_{w_2})$.

Let us consider a random variable X with absolutely continuous distribution function

$$F(x) = P(X \le x),\ x \in R,$$

survival (reliability) function

$$\overline{F}(x) = 1 - F(x),\ x \in R$$

and the probability density function $f(x),\ \ x \in R$, to provide definitions of notions of reliability classes, statistical dependence, stochastic orders etc.

Let the reverse failure rate (rfr) of a random variable X is given by

$$r_X(x) = \frac{f(x)}{F(x)},\ x > 0.$$

The residual life of random variable X with fixed age/time t $(t > 0)$ is

$$X_t = (X - t \mid X > t),$$

and inactivity time of random variable X with fixed age/time s $(s > 0)$ is

$$X_{(s)} = (s - X \mid X \le s).$$

The conditional distribution of $X - t$ given $X > t$ and $s - X$ given $X \le s$ are the distributions of X_t and $X_{(s)}$ respectively. The mean residual life function and mean inactivity time function when the random variable X has finite mean are defined as

$$m_X(t) = E(X_t) = \frac{\int_t^{\infty} \overline{F}(u)du}{\overline{F}(t)}$$

and

$$\mu_X(s) = E(X_{(s)}) = \frac{\int_{-\infty}^{s} F(u)du}{F(s)}$$

respectively.

Next, we introduce following definitions which are standard in literature (Refer Shaked, M. and Shanthikumar [7] and Ahmad and Kayid [8]) :

Definitions:

(a) Let $\Omega = (a,b) \subseteq R$ where $-\infty \le a < b \le \infty$ and $h : \Omega \to R^+$. Then the function $h(\cdot)$ is said to be log-concave (log-convex) on Ω if, $\forall\, x, y \in \Omega$ and $\forall\, \alpha \in (0,1)$,
$h(\alpha x + (1-\alpha)y) \ge (\le)\ (h(x))^{\alpha}\ (h(y))^{1-\alpha}$.

(b) Random variable X is said to have decreasing (increasing) reversed failure rate (DRFR (IRFR)) if $F(\cdot)$ is log-concave (log-convex) on $0,\infty$, or equivalently if the reversed failure rate function $r_X(\cdot)$ is decreasing (increasing) on $0,\infty$.

(c) Random variable X is said to have decreasing (increasing) mean residual life (DMRL (IMRL)) if $\int_x^{\infty} \overline{F}(t)\,dt$ is log-concave (log-convex) on $0,\infty$.

(d) Random variable X is said to have increasing (decreasing) mean inactivity time (IMIT (DMIT)) if $\int_0^{x} F(u)du$ is log-concave (log-convex) on $0,\infty$ or equivalently if the mean inactivity time function $\mu_X(\cdot)$ is decreasing (increasing) on $0,\infty$.

(e) Random variable X is said to be smaller then Y in the down shifted likelihood ratio order (written as $X \le_{lr\downarrow} Y$) if $\frac{g(t+x)}{f(t)}$ is increasing in $t \ge 0$ for all $x \ge 0$, where f and g denote the density functions of X and Y respectively.

(f) Random variable X is said to be smaller than random variable Y in the reversed failure rate ordering (written as $X \le_{rfr} Y$) if

$$G(s)F(t) \le G(t)F(s),$$

whenever

$$-\infty < s < t < \infty,$$

or equivalently, if

$$r_X(x) \le r_Y(x),\ \forall\ x \in (0,\infty).$$

(g) Random variable X is said to be smaller then random variable Y in the mean inactivity time ordering $(X \le_{mit} Y)$ if

$$\left[\int_{-\infty}^{s} G(u)du\right]\left[\int_{-\infty}^{t} F(u)du\right] \le \left[\int_{-\infty}^{s} F(u)du\right]\left[\int_{-\infty}^{t} G(u)du\right]$$

whenever

$$-\infty < s < t < \infty ,$$

or equivalently,

$$\mu_X(t) \ge \mu_Y(t), \ \forall \ t \in (0,\infty) .$$

2. AGING PROPERTIES AND STOCHASTIC DEPENDENCE

Let X and Y be non-negative random variables.

Lemma 2.1 (Misra et. al. [4], Theorem 2.3(a)).

If X has DRFR, $w_1(\cdot)$ *is decreasing on* $0,\infty$ *and log-concave on* $0,\infty$ *, then* X_{w_1} *has DRFR.*

Lemma 2.2 (Misra et. al. [4], Theorem 3.2(c)).

If $X \le_{rfr} Y$, $w_1(t)$ *is decreasing on* $0,\infty$ *and* $w_2(t)/w_1(t)$ *is increasing on* $0,\infty$ *, then* $X_{w_1} \le_{rfr} Y_{w_2}$.

Here we present the condition on weight function $w\ \cdot$ under which such preservation of property of IMIT under weighing is possible.

Theorem 2.1:

X_{w_1} *has IMIT if X has IMIT and* $A_1(\cdot)$ *is decreasing and log-convex on* $0,\infty$ *where* $A_1(x) = E\ \ w_1(x) \mid X \le x$.

Proof:

Let Z_1 and Z_2 be random variables having probability density function

$$f_{Z_1}(x) = \frac{F(x)}{\int_0^\infty F(u)du},$$

and

$$f_{Z_2}(x) = \frac{F_{w_1}(x)}{\int_0^\infty F_{w_1}(u)du},$$

respectively.

Z_1 has weighted version Z_2 with weight function $A_1(\cdot)$ .The random variable Z_1 has DRFR since *X* has IMIT. Under the premise of the contention, using Lemma 2.1, it follows that random variable Z_2 has DRFR which follows that X_{w_1} has IMIT.

Consider

$$F_1(x) = \frac{A_1(x)F(x)}{w_1};$$

where

$$A_1(x) = E[w_1(X) \mid X \le x].$$

From above, it is clear that X_{w_1} has DRFR (IRFR), if *X* has DRFR (IRFR).

The following theorem provides conditions on the weight function $w_1(\cdot)$ and the mean inactivity time function $\mu_X(t)$, under which a random variable *X* having IMIT, yield a weighted version which is DRFR (and hence IMIT).

Theorem 2.2:

X_{w_1} *has DRFR if X has IMIT,* $w_1(\cdot)$ *is decreasing and log-concave on* $0,\infty$ *and the mean inactivity time function* $\mu_X(t)$ *is log-convex on* $0,\infty$.

Proof:

In view of Lemma 2.1, it is enough to show that $X \le_{rfr} Z_1$ where Z_1 has probability density function $f(t-\theta)$, i.e., *X* has DRFR and mean inactivity time function $\mu_{Z_1}(t) = \mu_X(t-\theta)$.

Consider

$$r_{Z_1}(t) - r_X(t) = \frac{1-\mu'_{Z_1}(t)}{\mu_{Z_1}(t)} - \frac{1-\mu'_X(t)}{\mu_X(t)}$$
$$= \left[\frac{\mu'_X(t)}{\mu_X(t)} - \frac{\mu'_X(t-\theta)}{\mu_X(t-\theta)}\right]$$
$$+ \left[\frac{1}{\mu_X(t-\theta)} - \frac{1}{\mu_X(t)}\right]$$
$$\geq 0,$$

since $\mu_X(t)$ is log-convex on $0,\infty$ and *X* has IMIT. Therefore, $X \leq_{rfr} Z_1$ and hence *X* has DRFR. Now contention follows using Lemma 2.1.

Theorem 2.3:

$X_{w_1} \leq_{mit} Y_{w_2}$ if $X \leq_{mit} Y$, $A_1(\cdot)$ is decreasing and $A_2(\cdot)/A_1(\cdot)$ is increasing on $0,\infty$, where

$$A_1(x) = E\ \ w_1(X) \mid X \leq x$$

and $$A_2(x) = E\ \ w_2(Y) \mid Y \leq x\ .$$

Proof:

We have

$$F_{w_1}(x) = \frac{A_1(x)F(x)}{w_1}$$

and

$$G_{w_2}(x) = \frac{A_2(x)G(x)}{w_2}.$$

Let X^* and Y^* be random variables with probability density functions given by

$$f_{X^*}(x) = \frac{F(x)}{\int_0^\infty F(u)\,du}$$

and

$$f_{Y^*}(x) = \frac{G(x)}{\int_0^\infty G(u)\,du}$$

respectively.

Now $X \leq_{mit} Y$ implies that $X^* \leq_{rfr} Y^*$. Let $X^*_{A_1}$ and $Y^*_{A_2}$ be weighted version of X^* and Y^* with weight functions $A_1(\cdot)$ and $A_2(\cdot)$ respectively. Hence by Lemma 2.2,

$$X^* \leq_{rfr} Y^*.$$

Also, $X^* \leq_{rfr} Y^*$, if and only if $X_{w_1} \leq_{mit} Y_{w_2}$.

Proposition:

$A_1(\cdot)$ is increasing (decreasing) if $w_1(\cdot)$ is increasing (decreasing).

Proof:

Consider

$$F(x)\,w(x) - \int_0^x w(t)\,f(t)\,dt \geq (\leq)\ 0$$

if and only if $A_1'(x) \geq (\leq)\ 0$.

It may be noted that $w_1(\cdot)$ is increasing (decreasing) implies that $A_1(\cdot)$ is increasing (decreasing). Hence, the result follows by using the above argument.

Corollary: If $X \leq_{mit} Y$, $w_1(\cdot)$ is decreasing and $w_2(\cdot)$ is increasing on $0,\infty$, then $X_{w_1} \leq_{mit} Y_{w_2}$.

Example: If $f(x) = e^{-x}$ and $w_1 = x^{\alpha_1 - 1}$ where $\alpha_1 > 1$, $w_2 = x^{\alpha_2 - 1}$ where $\alpha_2 < 1$ then $X_{w_1} \leq_{mit} Y_{w_2}$.

Theorem 2.4:

If $f(\cdot)$ is log-convex and $w(\cdot)$ is increasing on $[0,\infty)$, then $X \leq_{lr\downarrow} X_w$.

Proof:

For fixed $a > 0$, consider

$$\frac{f_w(x+a)}{f(x)} = \frac{w(x+a)\, f(x+a)}{E[\, w(X+a)]\, f(x)}$$

$$= \frac{1}{E[\, w(X+a)]} \cdot w(x+a) \cdot \frac{f(x+a)}{f(x)},$$

which is increasing function, since $f(\cdot)$ is log-convex and $w(\cdot)$ is increasing. Hence $X \le_{lr\downarrow} X_w$.

Theorem 2.5:

If $X \le_{lr\downarrow} Y$ *and* $w(\cdot)$ *is log-convex, then* $X_w \le_{lr\downarrow} Y_w$.

Proof:

For fixed $a > 0$, consider

$$\frac{g_w(x+a)}{f_w(x)} = \frac{w(x+a)\, g(x+a)}{E[\, w(Y+a)]} \cdot \frac{E[w(x)]}{w(x)\, f(x)}$$

$$= \frac{E[w(x)]}{E[\, w(Y+a)]} \cdot \frac{w(x+a)}{w(x)} \cdot \frac{g(x+a)}{f(x)},$$

which is an increasing function, since $w(\cdot)$ is log-convex and $X \le_{lr\downarrow} Y$. Hence $X_w \le_{lr\downarrow} Y_w$.

3. CONCLUSION

In the context of reliability and life testing problems, we obtain reliability properties of mean inactivity time under weighting. The conditions under which such results have been obtained are described in this paper. Further, the conditions of stochastic comparison of weighted distributions in terms of mean inactivity time and shifted likelihood ratio order has been obtained.

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