A Computer based Numerical Method for Singular Boundary Value Problems

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ABSTRACT
A numerical method is presented in this paper which employs cubic trigonometric B-spline to solve linear two point second order singular boundary value problems for ordinary differential equations. The given singular boundary value problem is modified at the point of singularity. Then method utilizing the values of cubic trigonometric B-spline and its derivatives at nodal points is applied. Selected numerical examples are solved using *MATLAB*, which demonstrate the applicability and competence of present method.

General Terms
Computational method, Boundary value problem

Keywords
Singular Boundary Value Problem, Cubic Trigonometric B-spline, nodal points, system of equations, maximum absolute error

1. INTRODUCTION
Singular boundary value problems for ordinary differential equations arise very frequently in several areas of science and engineering. For example, in analysis of heat conduction through a solid with heat generation, Thomas–Fermi model in atomic physics and in the study of generalized axially symmetric potentials after separation of variables has been employed. These problems also occur very frequently in the study of electro hydrodynamics and the theory of thermal explosions. These arise in Physiology as well e.g. in the study of various tumor growth problems, in the study of steady state oxygen diffusion in a spherical cell with Michaelis– Menten uptake kinetics and in the study of the distribution of heat sources in the human head.

Consider a class of singular two-point boundary value problems

$$x^{-\alpha}(x^{\alpha}u')' = f(x,u), \qquad x \in (0,1) \tag{1}$$

with $$u'(0) = 0,\ u(1) = B \tag{2}$$

Here $\alpha \in (0,1)$ (weakly singular) or it may take values 1 or 2 (strongly singular). If $\alpha = 1$, then given problem becomes a cylindrical problem and if $\alpha = 2$, then it becomes a spherical problem. B is a finite constant. It is well known that above problem has a unique solution, if $f(x,u)$ is continuous, $\frac{\partial f}{\partial u}$ exists and is continuous and $\frac{\partial f}{\partial u} \geq 0$. (See reference [1])

After some algebraic manipulations, linear form of above problem can be taken as

$$u''(x) + \frac{k}{x}u'(x) - r(x)u(x) + s(x) = 0, \qquad x \in (0,1) \tag{3}$$

Subject to boundary conditions (2).

It is convenient to introduce the differential operator $Lu(x) = -\left(u''(x) + \frac{k}{x}u'(x) - r(x)u(x)\right)$ and to write (3) as $Lu(x) = s(x)$. Considering $r(x) = a$, which is a constant; the following results are described in [1],

Theorem. *Suppose that* $s(x) \in C[0,1]$ *and the constant* a *satisfies the inequality*

$$a > -J_0^2 \ \textit{if}\ k = 1,\ a > -\pi^2 \ \textit{if}\ k = 2$$

Where $J_0 = 2.40483$ *is the smallest positive zero of the Bessel function. Then problem (3) with equation (2) has a unique solution u(x).*

The result is easily established when $a \geq 0$ using standard arguments and maximum principle.

Corollary. *Suppose that* $u(x) \in C^2[0,1],\ \ u'(0) = 0, u(1) \geq 0$ *and* $Lu(x) \geq 0,\ 0 < x < 1,$ *then* $u(x) \geq 0,\ 0 \leq x \leq 1.$

Various numerical methods have been developed for solution of singular boundary value problems. An account of these methods can be found in [2]. Among these methods spline based numerical methods provide an important tool which are reviewed in [3]. B-spline has been employed for solution of singular boundary value problems by several authors [1, 4, 5, 6]. In the present paper we make use of Trigonometric B-spline which is a non-polynomial B-spline containing trigonometric terms. The theory and properties of Trigonometric B-spline could be traced in [7, 8, 9, 10]. However, little is found in literature about numerical methods applying Trigonometric B-spline for solution of differential equations including singular boundary value problems and perturbation problems. Quadratic Trigonometric spline has been used by Nikolis [11] for solution of initial value problems in ordinary differential equations. Hamid *et al*. [12] devised a method for boundary value problem of order two using cubic trigonometric B-spline. The present paper depicts the numerical method based on cubic trigonometric B-spline for linear two point second order singular boundary value problem, which demonstrates more accurate numerical results than existing methods for the considered problems.

Present paper is organized as follows: section 2 contains definition and values of derivatives of Cubic Trigonometric B-spline. Section 3 deals with the derivation of the TB-spline method for the singular boundary value problem. Numerical examples are given in section 4. Finally, paper is concluded in section 5.

2. CUBIC TRIGONOMETRIC B-SPLINE

We subdivide the interval $[0,1]$ and choose piecewise uniform grid points represented by $\prod = x_0 < x_1 < .... < x_n$, such that $x_0 = 0, x_n = 1$ and h is the piecewise uniform spacing. Let $TS_3(\prod)$ be the space of cubic trigonometric spline functions over the partition $\prod$. We can define the cubic trigonometric B-spline basis functions $\{TB_i(x)\}$, for $i = -1, 0, 1, ..., n+1$, for $TS_3(\prod)$ after including two more points on each side of the partition $\prod$. Thus the partition $\prod$ becomes

$$\prod : x_{-2} < x_{-1} < x_0 < x_1 < .... < x_n < x_{n+1} < x_{n+2} \tag{4}$$

Now, the cubic trigonometric B-Spline basis function is defined as, (see references [10, 12])

$$TB_i(x) = \frac{1}{\mu(h)} \begin{cases} \sin^3\left(\frac{x - x_{i-2}}{2}\right), & if\ x \in [x_{i-2}, x_{i-1}] \\ \sin\left(\frac{x - x_{i-2}}{2}\right)\left[\sin\left(\frac{x - x_{i-2}}{2}\right)\sin\left(\frac{x_i - x}{2}\right) + \sin\left(\frac{x_{i+1} - x}{2}\right)\sin\left(\frac{x - x_{i-1}}{2}\right)\right] \\ \quad + \sin\left(\frac{x_{i+2} - x}{2}\right)\sin^2\left(\frac{x - x_{i-1}}{2}\right), & if\ x \in [x_{i-1}, x_i] \\ \sin\left(\frac{x - x_{i-2}}{2}\right)\sin^2\left(\frac{x_{i+1} - x}{2}\right) \\ \quad + \sin\left(\frac{x_{i+2} - x}{2}\right)\left[\sin\left(\frac{x - x_{i-1}}{2}\right)\sin\left(\frac{x_{i+1} - x}{2}\right) + \sin\left(\frac{x_{i+2} - x}{2}\right)\sin\left(\frac{x - x_i}{2}\right)\right], & if\ x \in [x_i, x_{i+1}] \\ \sin^3\left(\frac{x_{i+2} - x}{2}\right), & if\ x \in [x_{i+1}, x_{i+2}] \\ 0, & otherwise. \end{cases}$$

$$\mu(h) = \sin\left(\frac{h}{2}\right)\sin(h)\sin\left(\frac{3h}{2}\right) \tag{5}$$

It can easily be verified from equation (5) that each of the functions $TB_i(x)$ is twice continuously differentiable on the entire real line. We need values of up to second derivatives to solve second order boundary value problem. For, we have the followings from the above definition,

$$TB_i(x_k) = \begin{cases} \dfrac{2}{1 + 2\cos(h)}, & if\ i = k \\ \sin^2\left(\dfrac{h}{2}\right)\csc(h)\csc\left(\dfrac{3h}{2}\right), & if\ i - k = \pm 1 \\ 0, & if\ i - k = \pm 2 \end{cases} \tag{6}$$

$$TB_i'(x_k) = \begin{cases} 0, & if\ i = k \\ \pm\dfrac{3}{4}\csc\left(\dfrac{3h}{2}\right), & if\ i - k = \pm 1 \\ 0, & if\ i - k = \pm 2 \end{cases} \tag{7}$$

$$TB_i''(x_k) = \begin{cases} \dfrac{-3\cot^2\left(\frac{h}{2}\right)}{2 + 4\cos(h)}, & if\ i = k \\ \dfrac{3(1 + 3\cos(h))\csc^2\left(\frac{h}{2}\right)}{16\left(2\cos\left(\frac{h}{2}\right) + \cos\left(\frac{3h}{2}\right)\right)}, & if\ i - k = \pm 1 \\ 0, & if\ i - k = \pm 2 \end{cases} \tag{8}$$

These values for cubic trigonometric B-spline function at nodal points can be presented in tabular form as given in Table 1.

Table 1. Values of $TB_i(x), TB_i'(x)$ and $TB_i''(x)$ at nodes

	$TB_i(x)$	$TB_i'(x)$	$TB_i''(x)$
x_{i-2}	0	0	0
x_{i-1}	$\sin^2\left(\frac{h}{2}\right)\csc(h)\csc\left(\frac{3h}{2}\right)$	$\frac{3}{4}\csc\left(\frac{3h}{2}\right)$	$\frac{3(1+3\cos(h))\csc^2\left(\frac{h}{2}\right)}{16\left(2\cos\left(\frac{h}{2}\right)+\cos\left(\frac{3h}{2}\right)\right)}$
x_i	$\frac{2}{1+2\cos(h)}$	0	$\frac{-3\cot^2\left(\frac{h}{2}\right)}{2+4\cos(h)}$
x_{i+1}	$\sin^2\left(\frac{h}{2}\right)\csc(h)\csc\left(\frac{3h}{2}\right)$	$-\frac{3}{4}\csc\left(\frac{3h}{2}\right)$	$\frac{3(1+3\cos(h))\csc^2\left(\frac{h}{2}\right)}{16\left(2\cos\left(\frac{h}{2}\right)+\cos\left(\frac{3h}{2}\right)\right)}$
x_{i+2}	0	0	0

Suppose $TS_3(\prod) = span\{TB_{-1}(x), TB_0(x), ..., TB_n(x), TB_{n+1}(x)\}$

Where $\dim TS_3(\prod) = n+3$. Further, let $TS(x)$ be the trigonometric B-Spline interpolating the function $u(x)$ at the nodal points and $TS(x) \in TS_3(\prod)$, then we have

$$u(x) = TS(x) = c_{-1}TB_{-1}(x) + c_0 TB_0(x) + ... + c_n TB_n(x) + c_{n+1}TB_{n+1}(x)$$
$$= \sum_{j=-1}^{n+1} c_j TB_j(x) \quad (9)$$

where c_j's are unknown coefficients and $TB_j(x)$'s are third degree trigonometric B-Spline functions.

3. DESCRIPTION OF THE METHOD

In this section, we apply the cubic trigonometric B-spline method to solve the singular problem given by equation (3) written as

$$u''(x) + \frac{k}{x}u'(x) - r(x)u(x) = -s(x) \ : x \in (0,1) \quad (10)$$

With boundary conditions given by equation (2).

As discussed in previous section, let the trigonometric B-Spline solution of the problem is $u(x) = \sum_{j=-1}^{n+1} c_j TB_j(x)$, which must satisfy the given boundary value problem at the nodal points $x = x_i$. For, putting values in equation (10), we get

$$\sum_{j=-1}^{n+1} c_j TB_j''(x_i) + \sum_{j=-1}^{n+1} c_j \frac{k}{x_i} TB_j'(x_i) + -\sum_{j=-1}^{n+1} c_j r(x_i) TB_j(x_i) = -s(x_i);$$
$$i = 1,2,....,n \quad (11)$$

Equivalently,

$$c_{-1}\{TB_{-1}''(x_i) + \frac{k}{x_i}TB_{-1}'(x_i) - r(x_i)TB_{-1}(x_i)\} + c_0\{TB_0''(x_i) + \frac{k}{x_i}TB_0'(x_i) -$$
$$r(x_i)TB_0(x_i)\} + ... + c_n\{TB_n''(x_i) + \frac{k}{x_i}TB_n'(x_i) - r(x_i)TB_n(x_i)\}$$
$$+c_{n+1}\{TB_{n+1}''(x_i) + \frac{k}{x_i}TB_{n+1}'(x_i) - r(x_i)TB_{n+1}(x_i)\} = -s(x_i), i = 1,2,....,n$$
$$(12)$$

For any nodal point $x = x_i$ above equation reduces to,

$$c_{i-1}\{TB_{i-1}''(x_i) + \frac{k}{x_i}TB_{i-1}'(x_i) - r(x_i)TB_{i-1}(x_i)\} + c_i\{TB_i''(x_i) + \frac{k}{x_i}TB_i'(x_i) -$$
$$r(x_i)TB_i(x_i)\} + c_{i+1}\{TB_{i+1}''(x_i) + \frac{k}{x_i}TB_{i+1}'(x_i) - r(x_i)TB_{i+1}(x_i)\} = f(x_i)$$

$$(13)$$

At the singular point $x_0 = 0$, we modify given boundary value problem (10) as

$$(k+1)u''(x) - r(x)u(x) = -s(x) \quad \text{for } x = x_0 \quad (14)$$

Now, we apply the above mentioned TB-spline method at $x_0 = 0$ and obtain

$$(k+1)\sum_{j=-1,0,1} c_j TB_j''(x_0) - r(x_0)\sum_{j=-1,0,1} c_j TB_j(x_0) = -s(x_0) \quad (15)$$

And boundary conditions give

$$\sum_{j=-1}^{n+1} c_j TB_j'(x) = 0, \quad \text{for } x = x_0 \quad (16)$$

$$\Rightarrow \sum_{j=-1,0,1} c_j TB_j'(x_0) = 0 \quad (17)$$

and

$$\sum_{j=-1}^{n+1} c_j TB_j(x) = B, \quad \text{for } x = x_n \quad (18)$$

$$\Rightarrow \sum_{j=n-1,n,n+1} c_j TB_j(x_n) = B \quad (19)$$

Now, using the values of trigonometric spline functions and derivatives at the knots given by relations (6), (7) and (8) in equations (13), (15), (17) and (19), a system of $(n+3)$ linear equations in $(n+3)$ unknowns $c_{-1}, c_0, ..., c_{n+1}$ is obtained. This system can be written in matrix vector form as

$$AC = F \quad (20)$$

Where

$$C = \left[c_{-1}, c_0, ..., c_{n+1}\right]^T \quad (21)$$

$$F = \left[0, -s(x_0), ..., -s(x_n), B\right]^T \quad (22)$$

And coefficient matrix A is given by

$$A=\begin{bmatrix} \frac{3}{4}\csc\left(\frac{3h}{2}\right) & 0 & -\frac{3}{4}\csc\left(\frac{3h}{2}\right) & 0 & \ldots & 0 & 0 \\ \alpha(x_0) & \beta(x_0) & \gamma(x_0) & 0 & \ldots & 0 & 0 \\ 0 & \alpha(x_1) & \beta(x_1) & \gamma(x_1) & 0 & \ldots & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & \ldots & 0 & 0 & \alpha(x_n) & \beta(x_n) & \gamma(x_n) \\ . & . & . & . & \sin^2\left(\frac{h}{2}\right)\csc(h)\csc\left(\frac{3h}{2}\right) & \frac{2}{1+2\cos(h)} & \sin^2\left(\frac{h}{2}\right)\csc(h)\csc\left(\frac{3h}{2}\right) \end{bmatrix} \quad (23)$$

Where elements of A are given by,

$$\begin{aligned}\alpha(x_0) &= (k+1)\frac{3(1+3\cos(h))\csc^2\left(\frac{h}{2}\right)}{16\left(2\cos\left(\frac{h}{2}\right)+\cos\left(\frac{3h}{2}\right)\right)} \\ &\quad + r(x_0)\sin^2\left(\frac{h}{2}\right)\csc(h)\csc\left(\frac{3h}{2}\right) \\ &= \gamma(x_0)\end{aligned} \quad (24)$$

$$\beta(x_0)=(k+1)\frac{-3\cot^2\left(\frac{h}{2}\right)}{2+4\cos(h)}-r(x_0)\frac{2}{1+2\cos(h)} \quad (25)$$

$$\begin{aligned}\alpha(x_i) &= \frac{3(1+3\cos(h))\csc^2\left(\frac{h}{2}\right)}{16\left(2\cos\left(\frac{h}{2}\right)+\cos\left(\frac{3h}{2}\right)\right)}+\frac{k}{x_i}\frac{3}{4}\csc\left(\frac{3h}{2}\right) \\ &\quad - r(x_i)\sin^2\left(\frac{h}{2}\right)\csc(h)\csc\left(\frac{3h}{2}\right),\ i=1,2.......,n\end{aligned} \quad (26)$$

$$\beta(x_i)=\frac{-3\cot^2\left(\frac{h}{2}\right)}{2+4\cos(h)}-r(x_i)\frac{2}{1+2\cos(h)},\ i=1,2.......,n \quad (27)$$

$$\begin{aligned}\gamma(x_i) &= \frac{3(1+3\cos(h))\csc^2\left(\frac{h}{2}\right)}{16\left(2\cos\left(\frac{h}{2}\right)+\cos\left(\frac{3h}{2}\right)\right)}+\frac{k}{x_i}-\frac{3}{4}\csc\left(\frac{3h}{2}\right) \\ &\quad - r(x_i)\sin^2\left(\frac{h}{2}\right)\csc(h)\csc\left(\frac{3h}{2}\right),\ i=1,2.......,n\end{aligned} \quad (28)$$

On solving the above system, we get the values of c_j's and, in turn, the required TB-spline solution is obtained.

4. NUMERICAL EXAMPLES

Problem 1.

$$\begin{aligned}&u''(x)+\frac{2}{x}u'(x)=3\cos x - x\sin x,\quad x\in(0,1) \\ &u'(0)=0, u(1)=\cos 1+\sin 1\end{aligned} \quad (29)$$

The analytical solution is $u(x)=\cos x + x\sin x$.

Table 2 . Maximum absolute errors for problem 1

h	Maximum absolute error by collocation method [13]	Maximum absolute error by our method
1/8	1.09E-5	3.76E-09
1/16	1.08E-6	5.78E-10
1/32	7.89E-8	9.18E-11

Problem 2.

$$\begin{aligned}&u''(x)+\frac{1}{x}u''(x)=\frac{\pi}{2x}\left(\sin\frac{\pi x}{2}+\frac{\pi x}{2}\cos\frac{\pi x}{2}\right),\quad x\in(0,1) \\ &u'(0)=u(1)=0\end{aligned} \quad (30)$$

The analytical solution is $u(x)=-\cos\frac{\pi x}{2}$.

Table 3 . Maximum absolute errors for problem 2

h	Max absolute error by modified hierarchy basis method [14]	Maximum absolute error by our method
1/32	3.07E-2	2.88E-6
1/64	1.34E-2	4.96E-7
1/128	6.20E-3	9.30E-8

5. CONCLUSION

We conclude by taking note that the present study furnished a numerical treatment for singular linear two-point boundary value problems of order two by trigonometric B-spline method. The approach has been tested on some existing problems from the literature. The observed maximum absolute errors for problems for various values of *h* are presented in Table 2 and Table 3. It is evident from the numerical examples that the results give a better estimation to the solution than the stated existing numerical methods with the same number of knots.

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