# Fuzzy Goal Programming Approach to Quadratic Bi-Level Multi-Objective Programming Problem

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## ABSTRACT
This paper deals with fuzzy goal programming approach to quadratic bi-level multi-objective programming problem involving a single decision maker with multiple objectives at the upper level and a single decision maker with multiple objectives at the lower level. The objective functions of each level decision maker are quadratic in nature and the system constraints are linear functions. In the model formulation of the problem, we first determine the individual best solution of the quadratic objective functions subject to the system constraints and construct the quadratic membership functions of the objective functions of both levels. The quadratic membership functions are then transformed into equivalent linear membership functions by first order Taylor series at the individual best solution point. A possible relaxation of each level decision is considered by providing preference bounds on the decision variables for avoiding decision deadlock. Fuzzy goal programming approach is then used to achieve maximum degree of each of the membership goals by minimizing negative deviational variables. To demonstrate the efficiency of the proposed approach, an illustrative numerical example is provided.

## General Terms
Quadratic bi-level multi-objective programming.

## Keywords
Fuzzy goal programming, Quadratic programming, Quadratic bi-level programming, Quadratic bi-level multi-objective programming.

## 1. INTRODUCTION
A quadratic bi-level multi-objective programming problem (QBLMOPP) involves a single decision maker viz. upper level decision maker (ULDM) with multiple objectives at the upper level and a single decision maker viz. lower level decision maker (LLDM) with multiple objectives at the lower level. The objective functions of each level decision maker (DM) are quadratic in nature and the system constraints are linear functions. Here, ULDM and LLDM independently control a set of decision variables.

Our primary objectives of the study are (i) to transform the quadratic membership functions into equivalent linear membership functions at the individual best solution point by first order Taylor series approximation and (ii) to introduce an alternative fuzzy goal programming (FGP) approach for solving QBLMOPP.

Rest of the paper is organized as follows. Section 2 provides a brief literature review. Section 3 presents QBLMOPP formulation. Section 4 discusses fuzzy programming formulation of QBLMOPP. Subsection 4.1 describes transformation of quadratic membership functions into equivalent linear membership functions by first order Taylor polynomial series. Subsection 4.2 describes preference bounds of both level DMs. In subsection 4.3, formulation of FGP model for solving QBLMOPP is presented. Section 5 provides FGP algorithm for solving QBLMOPP. Section 6 is devoted to solve the model for a numerical example and to show the efficiency of the proposed approach. Section 7 presents the concluding remarks and future research directions.

## 2. LITERATURE REVIEW
The formal formulation of bi-level programming problem (BLPP) was studied by Candler and Townsley [1] and Fortuny-Amat and McCarl [2]. Anandalingam [3] discussed multi-level programming problem (MLPP) as well as bi-level decentralized programming problem based on Stackelberg solution concept in 1988. Lai [4] applied the concept of fuzzy set theory to MLPP for the first time. Shih et al. [5], Shih and Lee [6] extended Lai's concept by introducing non-compensatory max-min aggregation operator and compensatory fuzzy operator respectively for MLPP. Sakawa et al. [7] presented interactive fuzzy programming for MLPP in 1998. Pramanik and Roy [8] discussed FGP approach for solving MLPP and they also extend the concept for solving decentralized bi-level programming problem.

Edmund and Bard [9] dealt with nonlinear bi-level mathematical problems in 1991. Savard and Gauvin [10] proposed steepest decent direction for the nonlinear bi-level programming. Vicente et. al. [11] discussed descent approaches for quadratic bi-level programming problem (QBLPP) in 1994. Thirwani and Arora [12] proposed an algorithm to QBLPP for integer variables. Pal and Moitra [13] proposed FGP procedure to QBLPP.

In this study, we formulate quadratic membership functions of the objective functions of both level DMs. The quadratic membership functions are then transformed into equivalent linear membership functions at the individual best solution point by first order Taylor series approximation. A possible relaxation of decision of ULDM and LLDM are considered by providing preference bounds on the decision variables under their control in the decision-making situation for avoiding decision deadlock. Then FGP approach due to Pramanik and Roy [8] and Pramanik and Dey [14, 15] is used for achieving highest degree of each of the membership goals by minimizing negative deviational variables. To demonstrate the efficiency of the proposed FGP approach, a numerical example is solved.

## 3. FORMULATION OF QBLMOPP

We consider QBLMOPP of maximization – type of objective functions at each level. Let us suppose that the ULDM controls the decision vector $\bar{x}_1 = (x_{11}, x_{12}, ..., x_{1N_1})$ and the LLDM controls the decision vector $\bar{x}_2 = (x_{21}, x_{22}, ..., x_{2N_2})$ in the decision-making situation. Mathematically, QBLMOPP can be stated as:

ULDM:

$$\max_{\bar{x}_1} Z_{1i}(\bar{x}) = \{\bar{C}_{1i}\bar{x} + \frac{1}{2}\bar{x}^T\bar{D}_{1i}\bar{x}\}\ (i = 1, 2, \ldots, m_1) \tag{1}$$

LLDM:

$$\max_{\bar{x}_2} Z_{2j}(\bar{x}) = \{\bar{C}_{2j}\bar{x} + \frac{1}{2}\bar{x}^T\bar{D}_{2j}\bar{x}\}\ (j = 1, 2, \ldots, m_2) \tag{2}$$

subject to

$$\bar{x} \in S = \{(\bar{x}_1, \bar{x}_2) \mid \bar{A}_1\bar{x}_1 + \bar{A}_2\bar{x}_2 \le \bar{b}, \bar{x}_1 \ge \bar{0}, \bar{x}_2 \ge \bar{0}\} \tag{3}$$

Here, $\bar{x} = \bar{x}_1 \cup \bar{x}_2$ is the set of decision vector, $N_1 + N_2 = N$ = total number of decision variables of the system and M is the total number of constraints. $\bar{C}_{1i}$ (i = 1, 2, …, $m_1$), $\bar{C}_{2j}$ (j = 1, 2, …, $m_2$) and $\bar{b}$ are constant vectors. $\bar{A}_1$ and $\bar{A}_2$ are constant matrices. $\bar{D}_{1i}$ (i = 1, 2, …, $m_1$), $\bar{D}_{2j}$ (j = 1, 2, …, $m_2$) are constant symmetric matrices. The symbol 'T' represents transposition. We assume that the objective functions $Z_{1i}(\bar{x})$ (i = 1, 2, …, $m_1$) and $Z_{2j}(\bar{x})$ (j = 1, 2, …, $m_2$) are concave and the system constraints are convex. We also assume S ($\neq \Phi$) to be bounded.

## 4. FUZZY PROGRAMMING FORMULATION OF QBLMOPP

We now formulate the fuzzy programming model of QBLMOPP by transforming the objective functions $Z_{1i}(\bar{x})$ (i = 1, 2, …, $m_1$), $Z_{2j}(\bar{x})$ (j = 1, 2, …, $m_2$) into fuzzy goals by means of assigning an imprecise aspiration level to each of them.

Let, $Z_{1i}^B = Z_{1i}\left(\bar{x}_{1i}^B\right) = \max_{\bar{x} \in S} Z_{1i}(\bar{x})$ (i = 1, 2, …, $m_1$) and $Z_{2j}^B = Z_{2j}\left(\bar{x}_{2j}^B\right) = \max_{\bar{x} \in S} Z_{2j}(\bar{x})$ (j = 1, 2, …, $m_2$) be the optimal solutions of the objective functions of ULDM and LLDM respectively when calculated in isolation subject to the system constraints.

Then the fuzzy goals appear in the form:

$Z_{1i}(\bar{x}) \gtrsim Z_{1i}^B$ (i = 1, 2, …, $m_1$) and $Z_{2j}(\bar{x}) \gtrsim Z_{2j}^B$ (j = 1, 2, …, $m_2$).

Using the individual best solutions, we formulate a payoff matrix as follows:

$$\begin{bmatrix} & Z_{11}(\bar{x}) & \ldots & Z_{1m_1}(\bar{x}) & \ldots & Z_{21}(\bar{x}) & \ldots & Z_{2m_2}(\bar{x}) \\ \bar{x}_{11}^B & Z_{11}(\bar{x}_{11}^B) & \ldots & Z_{1m_1}(\bar{x}_{11}^B) & \ldots & Z_{21}(\bar{x}_{11}^B) & \ldots & Z_{2m_2}(\bar{x}_{11}^B) \\ \ldots & \ldots & \ldots & \ldots & & \ldots & \ldots & \ldots \\ \bar{x}_{1m_1}^B & Z_{11}(\bar{x}_{1m_1}^B) & \ldots & Z_{1m_1}(\bar{x}_{1m_1}^B) & \ldots & Z_{21}(\bar{x}_{1m_1}^B) & \ldots & Z_{2m_2}(\bar{x}_{1m_1}^B) \\ \ldots & \ldots & \ldots & \ldots & & \ldots & \ldots & \ldots \\ \bar{x}_{21}^B & Z_{11}(\bar{x}_{21}^B) & \ldots & Z_{1m_1}(\bar{x}_{21}^B) & \ldots & Z_{21}(\bar{x}_{21}^B) & \ldots & Z_{2m_2}(\bar{x}_{21}^B) \\ \ldots & \ldots & \ldots & \ldots & & \ldots & \ldots & \ldots \\ \bar{x}_{2m_2}^B & Z_{11}(\bar{x}_{2m_2}^B) & \ldots & Z_{1m_1}(\bar{x}_{2m_2}^B) & \ldots & Z_{21}(\bar{x}_{2m_2}^B) & \ldots & Z_{2m_2}(\bar{x}_{2m_2}^B) \end{bmatrix} \tag{4}$$

The maximum value of each column gives the upper tolerance limit or aspired level of achievement for the objective functions $Z_{1i}(\bar{x})$ (i = 1, 2, …, $m_1$) and $Z_{2j}(\bar{x})$ (j = 1, 2, …, $m_2$). The minimum value of each column gives lower tolerance limit or lowest acceptable level of achievement for the objective function i.e. $Z_{1i}^W = \min\{Z_{1i}(\bar{x}_{1i})\}$ (i = 1, 2, …, $m_1$) and $Z_{2j}^W = \min\{Z_{2j}(\bar{x}_{2j})\}$ (j = 1, 2, …, $m_2$).

The objective values, which are equal to or larger than $Z_{1i}^B(\bar{x})$ (i = 1, 2, …, $m_1$) should be absolutely satisfactory to ULDM. Similarly, the objective values, which are equal to or larger than $Z_{2j}^B(\bar{x})$ (j = 1, 2, …, $m_2$) should be absolutely satisfactory to LLDM. If the individual best solutions are identical, then a satisfactory optimal solution of the system is reached. However, this situation arises rarely because the objectives of ULDM and LLDM are conflicting in general.

The quadratic membership function $\mu_{1i}(\bar{x})$ corresponding to the objective function $Z_{1i}(\bar{x})$ (i = 1, 2, …, $m_1$) of the ULDM can be formulated as:

$$\mu_{1i}(\bar{x}) = \begin{cases} 1, & \text{if } Z_{1i}(\bar{x}) \ge Z_{1i}^B \\ \dfrac{Z_{1i}(\bar{x}) - Z_{1i}^W}{Z_{1i}^B - Z_{1i}^W}, & \text{if } Z_{1i}^W \le Z_{1i}(\bar{x}) \le Z_{1i}^B \\ 0, & \text{if } Z_{1i}(\bar{x}) \le Z_{1i}^W \end{cases}\ (i = 1, 2, \ldots, m_1) \tag{5}$$

Here, $Z_{1i}^B$ and $Z_{1i}^W$ (i = 1, 2, …, $m_1$) are respectively the upper and lower tolerance limits of the fuzzy objective goal for ULDM.

The quadratic membership function $\mu_{2j}(\bar{x})$ corresponding to the objective function $Z_{2j}(\bar{x})$ (j = 1, 2, …, $m_2$) of the LLDM can be written as:

$$\mu_{2j}(\bar{x}) = \begin{cases} 1, & \text{if } Z_{2j}(\bar{x}) \geq Z_{2j}^B \\ \dfrac{Z_{2j}(\bar{x}) - Z_{2j}^W}{Z_{1i}^B - Z_{1i}^W}, & \text{if } Z_{2j}^W \leq Z_{2j}(\bar{x}) \leq Z_{2j}^B \\ 0, & \text{if } Z_{2j}(\bar{x}) \leq Z_{2j}^W \end{cases} \quad (j = 1, 2, \ldots, m_2) \qquad (6)$$

Here, $Z_{2j}^B$ and $Z_{2j}^W$ (j = 1, 2, …, $m_2$) are respectively the upper and lower tolerance limit of the fuzzy objective goal for LLDM.

Now the problem reduces to

$$\max \mu_{1i}(\bar{x}), (i = 1, 2, \ldots, m_1) \qquad (7)$$

$$\max \mu_{2j}(\bar{x}), (j = 1, 2, \ldots, m_2) \qquad (8)$$

subject to

$$\bar{x} \in S = \{(\bar{x}_1, \bar{x}_2) \mid \bar{A}_1 \bar{x}_1 + \bar{A}_2 \bar{x}_2 \leq \bar{b}, \bar{x}_1 \geq \bar{0}, \bar{x}_2 \geq \bar{0}\}.$$

## 4.1 Linearization of the quadratic membership functions by first order Taylor series

Let, $\bar{x}_{1i}^* = (x_1^{1i*}, x_2^{1i*}, \ldots, x_{N_1}^{1i*}, x_{N_1+1}^{1i*}, \ldots, x_N^{1i*})$ be the individual best solution of the quadratic membership function $\mu_{1i}(\bar{x})$ (i = 1, 2, …, $m_1$) subject to the constraints for ULDM. Also let, $\bar{x}_{2j}^* = (x_1^{2j*}, x_2^{2j*}, \ldots, x_{N_2}^{2j*}, x_{N_2+1}^{2j*}, \ldots, x_N^{2j*})$ be the individual best solution of the membership function $\mu_{2j}(\bar{x})$ (j = 1, 2, …, $m_2$) subject to the constraints for LLDM. Next, we transform the quadratic membership functions $\mu_{1i}(\bar{x})$ (i = 1, 2, …, $m_1$) and $\mu_{2j}(\bar{x})$ (j = 1, 2, …, $m_2$) into equivalent linear membership functions at the individual best solution point by first order Taylor series as follows:

$$\mu_{1i}(\bar{x}) \cong \mu_{1i}(\bar{x}_{1i}^*) + (x_1 - x_1^{1i*})\frac{\partial}{\partial x_1}\mu_{1i}(\bar{x}_{1i}^*) + (x_2 - x_2^{1i*})\frac{\partial}{\partial x_2}\mu_{1i}(\bar{x}_{1i}^*) + \ldots + (x_{N_1} - x_{N_1}^{1i*})\frac{\partial}{\partial x_{N_1}}\mu_{1i}(\bar{x}_{1i}^*) + (x_{N_1+1} - x_{N_1+1}^{1i*})\frac{\partial}{\partial x_{N_1+1}}\mu_{1i}(\bar{x}_{1i}^*) + \ldots + (x_N - x_N^{1i*})\frac{\partial}{\partial x_N}\mu_{1i}(\bar{x}_{1i}^*) = \xi_{1i}(\bar{x}), (i = 1, 2, \ldots, m_1) \qquad (9)$$

$$\mu_{2j}(\bar{x}) \cong \mu_{2j}(\bar{x}_{2j}^*) + (x_1 - x_1^{2j*})\frac{\partial}{\partial x_1}\mu_{2j}(\bar{x}_{2j}^*) + (x_2 - x_2^{2j*})\frac{\partial}{\partial x_2}\mu_{2j}(\bar{x}_{2j}^*) + \ldots + (x_{N_2} - x_{N_2}^{2j*})\frac{\partial}{\partial x_{N_2}}\mu_{2j}(\bar{x}_{2j}^*) + (x_{N_2+1} - x_{N_2+1}^{2j*})\frac{\partial}{\partial x_{N_2+1}}\mu_{2j}(\bar{x}_{2j}^*) + \ldots + (x_N - x_N^{2j*})\frac{\partial}{\partial x_N}\mu_{2j}(\bar{x}_{2j}^*) = \xi_{2j}(\bar{x}) (j = 1, 2, \ldots, m_2) \qquad (10)$$

## 4.2 Characterization of preference bounds on the decision variables

In the decision-making situation, each level DM desires to maximize his/her own objective function over a common feasible region. However, since the individual best solutions of ULDM and LLDM are distinct, the direct compromise optimal solution does not arise. Therefore, cooperation between ULDM and LLDM is essential to reach a compromise optimal solution. In this context, each level DM tries to get maximum benefit by considering the benefit of other DM also. Therefore, we consider the relaxation on decision of ULDM and LLDM simultaneously to reach a compromise optimal solution by providing their preference upper and lower bounds on the decision variables.

Let, $\ell_{1i}$ and $u_{1i}$ be the lower and upper bounds on the decision variable $x_{1i}$ (i = 1, 2, …, $N_1$) provided by the ULDM such that $\ell_{1i} \leq x_{1i} \leq u_{1i}$ (i = 1, 2, …, $N_1$). Also let, $\ell_{2j}$ and $u_{2j}$ be the lower and upper bounds on the decision variable $x_{2j}$ (j = 1, 2, …, $N_2$) provided by the LLDM so that $\ell_{2j} \leq x_{2j} \leq u_{2j}$ (j = 1, 2, …, $N_2$).

## 4.3 FGP model of QBLMOPP

The QBLMOPP represented by (7) and (8) reduces to the following problem

$$\max \xi_{1i}(\bar{x}), (i = 1, 2, \ldots, m_1) \qquad (11)$$

$$\max \xi_{2j}(\bar{x}), (j = 1, 2, \ldots, m_2) \qquad (12)$$

subject to

$$\bar{x} \in S = \{(\bar{x}_1, \bar{x}_2) \mid \bar{A}_1 \bar{x}_1 + \bar{A}_2 \bar{x}_2 \leq \bar{b}, \bar{x}_1 \geq \bar{0}, \bar{x}_2 \geq \bar{0}\},$$

$$\ell_{1i} \leq x_{1i} \leq u_{1i}, (i = 1, 2, \ldots, N_1)$$

$$\ell_{2j} \leq x_{2j} \leq u_{2j} (j = 1, 2, \ldots, N_2).$$

The maximum value of a membership function is unity (one), so for the defined membership functions in (11) & (12), the flexible membership goals having the aspiration level unity can be presented as:

$$\xi_{1i}(\bar{x}) + d_{1i}^- - d_{1i}^+ = 1, (i = 1, 2, \ldots, m_1) \qquad (13)$$

$$\xi_{2j}(\bar{x}) + d_{2j}^- - d_{2j}^+ = 1 (j = 1, 2, \ldots, m_1) \qquad (14)$$

Here, $d_{1i}^- \geq 0$ (i = 1, 2, …, $m_1$), $d_{2j}^- \geq 0$ (j = 1, 2, …, $m_2$) represent the negative deviational variables and $d_{1i}^+ \geq 0$ (i = 1, 2, …, $m_1$),

$d_{2j}^{+} \geq 0$ (j = 1, 2, …, $m_2$) represent the positive deviational variables such that $d_{1i}^{-} \times d_{1i}^{+} = 0$ (i = 1, 2, …, $m_1$) and $d_{2j}^{-} \times d_{2j}^{+} = 0$ (j = 1, 2, …, $m_2$). Following Pramanik and Roy [8] and Pramanik and Dey [14, 15], (13) & (14) can be written as:

$$\xi_{1i}(\bar{x}) + d_{1i}^{-} \geq 1, \ (i = 1, 2, \ldots, m_1) \quad (15)$$

$$\xi_{2j}(\bar{x}) + d_{2j}^{-} \geq 1 \ (j = 1, 2, \ldots, m_2) \quad (16)$$

Then proposed FGP model can be formulated as:

$$\min \left(\sum_{i=1}^{m_1} d_{1i}^{-} + \sum_{j=1}^{m_2} d_{2j}^{-}\right) \quad (17)$$

subject to

$\xi_{1i}(\bar{x}) + d_{1i}^{-} \geq 1$, (i = 1, 2, …, $m_1$)

$\xi_{2j}(\bar{x}) + d_{2j}^{-} \geq 1$, (j = 1, 2, …, $m_2$)

$\bar{x} \in S = \{(\bar{x}_1, \bar{x}_2) \mid \bar{A}_1 \bar{x}_1 + \bar{A}_2 \bar{x}_2 \leq \bar{b}, \bar{x}_1 \geq \bar{0}, \bar{x}_2 \geq \bar{0}\}$

$\ell_{1i} \leq x_{1i} \leq u_{1i}$, (i = 1, 2, …, $N_1$)

$\ell_{2j} \leq x_{2j} \leq u_{2j}$, (j = 1, 2, …, $N_2$)

$d_{1i}^{-} \geq 0$, (i = 1, 2, …, $m_1$)

$d_{2j}^{-} \geq 0$ (j = 1, 2, …, $m_2$).

i.e.
$$\min \left(\sum_{i=1}^{m_1} d_{1i}^{-} + \sum_{j=1}^{m_2} d_{2j}^{-}\right) \quad (18)$$

$$\mu_{1i}(\bar{x}_{1i}^{*}) + (x_1 - x_1^{1i*}) \frac{\partial}{\partial x_1} \mu_{1i}(\bar{x}_{1i}^{*}) + (x_2 - x_2^{1i*}) \frac{\partial}{\partial x_2} \mu_{1i}(\bar{x}_{1i}^{*}) + \ldots + (x_{N_1} - x_{N_1}^{1i*}) \frac{\partial}{\partial x_{N_1}} \mu_{1i}(\bar{x}_{1i}^{*}) + (x_{N_1+1} - x_{N_1+1}^{1i*}) \frac{\partial}{\partial x_{N_1+1}} \mu_{1i}(\bar{x}_{1i}^{*}) + \ldots + (x_N - x_N^{1i*}) \frac{\partial}{\partial x_N} \mu_{1i}(\bar{x}_{1i}^{*}) + d_{1i}^{-} \geq 1,$$
(i = 1, 2, …, $m_1$)

$$\mu_{2j}(\bar{x}_{2j}^{*}) + (x_1 - x_1^{2j*}) \frac{\partial}{\partial x_1} \mu_{2j}(\bar{x}_{2j}^{*}) + (x_2 - x_2^{2j*}) \frac{\partial}{\partial x_2} \mu_{2j}(\bar{x}_{2j}^{*}) + \ldots + (x_{N_2} - x_{N_2}^{2j*}) \frac{\partial}{\partial x_{N_2}} \mu_{2j}(\bar{x}_{2j}^{*}) + (x_{N_2+1} - x_{N_2+1}^{2j*}) \frac{\partial}{\partial x_{N_2+1}} \mu_{2j}(\bar{x}_{2j}^{*}) + \ldots + (x_N - x_N^{2j*}) \frac{\partial}{\partial x_N} \mu_{2j}(\bar{x}_{2j}^{*}) + d_{2j}^{-} \geq 1,$$
(j = 1, 2, …, $m_2$)

$\bar{x} \in S = \{(\bar{x}_1, \bar{x}_2) \mid \bar{A}_1 \bar{x}_1 + \bar{A}_2 \bar{x}_2 \leq \bar{b}, \bar{x}_1 \geq \bar{0}, \bar{x}_2 \geq \bar{0}\}$,

$\ell_{1i} \leq x_{1i} \leq u_{1i}$, (i = 1, 2, …, $N_1$)

$\ell_{2j} \leq x_{2j} \leq u_{2j}$, (j = 1, 2, …, $N_2$)

$d_{1i}^{-} \geq 0$, (i = 1, 2, …, $m_1$)

$d_{2j}^{-} \geq 0$ (j = 1, 2, …, $m_2$).

# 5. FGP ALGORITHM FOR QBLMOPP

From the discussion of the previous section, the proposed FGP algorithm for solving QBLMOPP can be outlined as given below:

**Step 1:** Find the individual best solution of each quadratic objective function for both ULDM and LLDM subject to the system constraints.

**Step 2:** Formulate the payoff matrix as given by (4). Then define upper and lower tolerance limits of each objective function for both ULDM and LLDM.

**Step 3:** Construct quadratic membership function $\mu_{1i}(\bar{x})$ (i = 1, 2, …, $m_1$) corresponding to the objective function $Z_{1i}(\bar{x})$ (i = 1, 2, …, $m_1$) of ULDM. Similarly, construct quadratic membership function $\mu_{2j}(\bar{x})$ (j = 1, 2, …, $m_2$) corresponding to the objective function $Z_{2j}(\bar{x})$ (j = 1, 2, …, $m_2$) of LLDM.

**Step 4:** Find the individual best solution of the quadratic membership functions $\mu_{1i}(\bar{x})$ (i = 1, 2, …, $m_1$) and $\mu_{2j}(\bar{x})$ (j = 1, 2, …, $m_2$) subject to the system constraints.

**Step 5:** Transform the quadratic membership functions $\mu_{1i}(\bar{x})$ (i = 1, 2, …, $m_1$) and $\mu_{2j}(\bar{x})$ (j = 1, 2, …, $m_2$) into equivalent linear membership functions $\xi_{1i}(\bar{x})$ (i = 1, 2, …, $m_1$) and $\xi_{2j}(\bar{x})$ (j = 1, 2, …, $m_2$) respectively at the individual best solution point by first order Taylor series approximation as given by (9) and (10).

**Step 6:** Determine the preference bounds on the decision variables provided by the DMs under their control such that $\ell_{1i} \leq x_{1i} \leq u_{1i}$ (i = 1, 2, … $N_1$) and $\ell_{2j} \leq x_{2j} \leq u_{2j}$ (j = 1, 2, … $N_2$).

**Step 7:** Formulate the FGP model (18) for QBLMOPP.

**Step 8:** Solve the FGP model. If the solution is acceptable to ULDM and LLDM, then compromise optimal solution is reached. Otherwise, both the level DMs provide another set of preference upper and lower bounds on the decision variables to reach a compromise optimal solution i.e. go to step 6 until the compromise optimal solution is reached.

**Step 9:** End.

## 6. NUMERICAL EXAMPLE

To illustrate the proposed FGP approach for solving QBLMOPP, we consider the following numerical example:

ULDM:

$$\max_{x_1} \begin{pmatrix} Z_{11}(x) = (6x_1 + 3x_2 - x_1^2 - x_2^2), \\ Z_{12}(x) = (7x_1 + 4x_2 - x_1^2 - 5x_2^2), \\ Z_{13}(x) = (5x_1 + 3x_2 - x_1^2 - 4x_2^2) \end{pmatrix}$$

LLDM:

$$\max_{x_2} \begin{pmatrix} Z_{21}(x) = (2x_1 + 6x_2 - 3x_1^2 - x_2^2), \\ Z_{22}(x) = (3x_1 + 7x_2 - x_1^2 - x_2^2) \end{pmatrix}$$

subject to

$x_1 + x_2 \le 3$,

$4x_1 + x_2 \le 9$,

$x_1 \ge 0,\ x_2 \ge 0$.

The individual best solution of the objective functions subject to the constraints are $Z_{11}^B = 10$ at (2, 1); $Z_{12}^B = 11.25$ at (2.167, 0.333); $Z_{13}^B = 6.696$ at (2.161, 0.354); $Z_{21}^B = 9.25$ at (0.25, 2.75); $Z_{22}^B = 12.5$ at (0.5, 2.5).

Then, the fuzzy goals appear as: $Z_{11}(\bar{x}) \gtrsim 10$, $Z_{12}(\bar{x}) \gtrsim 11.25$, $Z_{13}(\bar{x}) \gtrsim 6.696$, $Z_{21}(\bar{x}) \gtrsim 9.25$, $Z_{22}(\bar{x}) \gtrsim 12.5$.

$$\text{Payoff matrix} = \begin{bmatrix} 10 & 9 & 5 & -3 & 2.125 \\ 9.194 & 11.25 & 6.694 & -7.866 & 4.025 \\ 9.233 & 11.246 & 6.696 & -7.689 & 4.166 \\ 2.125 & -25.125 & -20.812 & 9.25 & 12.375 \\ 4 & -18 & -15.25 & 9 & 12.5 \end{bmatrix}$$

Here, $Z_{11}^B = 10$, $Z_{11}^W = 2.125$, $Z_{12}^B = 11.25$, $Z_{12}^W = -25.125$, $Z_{13}^B = 6.696$, $Z_{13}^W = -20.812$, $Z_{21}^B = 9.25$, $Z_{21}^W = -7.866$, $Z_{22}^B = 12.5$, $Z_{22}^W = 4.025$.

The quadratic membership functions of ULDM are

$$\mu_{11}(\bar{x}) = \frac{Z_{11}(\bar{x}) - 2.125}{10 - 2.125} = \frac{6x_1 + 3x_2 - x_1^2 - x_2^2 - 2.125}{10 - 2.125},$$

$$\mu_{12}(\bar{x}) = \frac{Z_{12}(\bar{x}) + 25.125}{11.25 + 25.125} = \frac{7x_1 + 4x_2 - x_1^2 - 5x_2^2 + 25.125}{11.25 + 25.125},$$

$$\mu_{13}(\bar{x}) = \frac{Z_{13}(\bar{x}) + 20.812}{6.696 + 20.812} = \frac{5x_1 + 3x_2 - x_1^2 - 4x_2^2 + 20.812}{6.696 + 20.812}$$

The quadratic membership functions of LLDM are

$$\mu_{21}(\bar{x}) = \frac{Z_{21}(\bar{x}) + 7.866}{9.25 + 7.866} = \frac{2x_1 + 6x_2 - 3x_1^2 - x_2^2 + 7.866}{9.25 + 7.866},$$

$$\mu_{22}(\bar{x}) = \frac{Z_{22}(\bar{x}) - 4.025}{12.5 - 4.025} = \frac{3x_1 + 7x_2 - x_1^2 - x_2^2 - 4.025}{12.5 - 4.025}.$$

The membership functions $\mu_{11}(\bar{x})$, $\mu_{12}(\bar{x})$ and $\mu_{13}(\bar{x})$ for ULDM are maximal at the points (2, 1), (2.167, 0.333) and (2.161, 0.354) respectively. The membership functions $\mu_{21}(\bar{x})$, $\mu_{22}(\bar{x})$ for LLDM are maximal at the points (0.25, 2.75) and (0.5, 2.5) respectively.

Then, the quadratic membership functions are transformed into linear at the individual best solution point by first order Taylor polynomial series as follows:

$\mu_{11}(\bar{x}) \cong \mu_{11}(2, 1) + (x_1 - 2)\frac{\partial}{\partial x_1}\mu_{11}(2, 1) + (x_2 - 1)\frac{\partial}{\partial x_2}\mu_{11}(2, 1) = 1 + (x_1 - 2) \times 0.254 + (x_2 - 1) \times 0.127 = \xi_{11}(\bar{x})$,

$\mu_{12}(\bar{x}) \cong \mu_{12}(2.167, 0.333) + (x_1 - 2.167)\frac{\partial}{\partial x_1}\mu_{12}(2.167, 0.333) + (x_2 - 0.333)\frac{\partial}{\partial x_2}\mu_{12}(2.167, 0.333) = 1 + (x_1 - 2.167) \times 0.073 + (x_2 - 0.333) \times 0.018 = \xi_{12}(\bar{x})$,

$\mu_{13}(\bar{x}) \cong \mu_{13}(2.161, 0.354) + (x_1 - 2.161)\frac{\partial}{\partial x_1}\mu_{11}(2.161, 0.354) + (x_2 - 0.354)\frac{\partial}{\partial x_2}\mu_{11}(2.161, 0.354) = 1 + (x_1 - 2.161) \times 0.025 + (x_2 - 0.354) \times 0.006 = \xi_{13}(\bar{x})$,

$\mu_{21}(\bar{x}) \cong \mu_{21}(0.25, 2.75) + (x_1 - 0.25)\frac{\partial}{\partial x_1}\mu_{21}(0.25, 2.75) + (x_2 - 2.75)\frac{\partial}{\partial x_2}\mu_{21}(0.25, 2.75) = 1 + (x_1 - 0.25) \times 0.029 + (x_2 - 2.75) \times 0.029 = \xi_{21}(\bar{x})$,

$\mu_{22}(\bar{x}) \cong \mu_{22}(0.5, 2.5) + (x_1 - 0.5)\frac{\partial}{\partial x_1}\mu_{21}(0.5, 2.5) + (x_2 - 2.5)\frac{\partial}{\partial x_2}\mu_{21}(0.5, 2.5) = 1 + (x_1 - 0.5) \times 0.236 + (x_2 - 2.5) \times 0.236 = \xi_{22}(\bar{x})$.

Let the preference bounds provided by the respective DMs be

$1 \le x_1 \le 1.9$,

$1 \le x_2 \le 2$

Then, the proposed FGP model for solving QBLMOPP is formulated as follows:

$$\min \left( \sum_{i=1}^{3} d_{1i}^{-} + \sum_{j=1}^{2} d_{2j}^{-} \right)$$

subject to

$1 + (x_1 - 2) \times 0.254 + (x_2 - 1) \times 0.127 + d_{11}^{-} \geq 1,$

$1 + (x_1 - 2.167) \times 0.073 + (x_2 - 0.333) \times 0.018 + d_{12}^{-} \geq 1,$

$1 + (x_1 - 2.161) \times 0.025 + (x_2 - 0.354) \times 0.006 + d_{13}^{-} \geq 1,$

$1 + (x_1 - 0.25) \times 0.029 + (x_2 - 2.75) \times 0.029 + d_{21}^{-} \geq 1,$

$1 + (x_1 - 0.5) \times 0.236 + (x_2 - 2.5) \times 0.236 + d_{22}^{-} \geq 1,$

$x_1 + x_2 \leq 3,$

$4x_1 + x_2 \leq 9,$

$1 \leq x_1 \leq 2,$

$1 \leq x_2 \leq 1.9,$

$x_1 \geq 0, x_2 \geq 0,$

$d_{1i}^{-} \geq 0, (i = 1, 2, 3)$

$d_{2j}^{-} \geq 0, (j = 1, 2).$

By solving the FGP model, we find the optimal solution $x_1 = 1.9$, $x_2 = 1.1$.

The objective values are $Z_{11} = 9.88$, $Z_{12} = 8.04$, $Z_{13} = 4.35$, $Z_{21} = -1.64$, $Z_{22} = 8.54$.

The resulting membership values are $\mu_{11} = 0.9894$, $\mu_{12} = 0.9117$, $\mu_{13} = 0.9147$, $\mu_{21} = 0.3637$, $\mu_{22} = 0.5375$.

Note: All solutions of the problem are obtained by using Lingo 6.0.

## 7. CONCLUSION

This paper introduced an alternative technique for solving QBLMOPP. Proposed concept can be extended to multi-level as well as decentralized multi-level multi-objective quadratic programming problems. We hope that the proposed approach can contribute to future study in the field of practical implementation to real world hierarchical decision-making problems involving quadratic objectives especially in quadratic assignment problem, portfolio problems etc.

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