Complement of an Extended Fuzzy Set

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ABSTRACT
It has been accepted that for a fuzzy set A and its complement A^c, neither $A \cap A^c$ is the null set, nor $A \cup A^c$ is the universal set. Whereas the operations of union and intersection of two crisp sets are indeed special cases of the corresponding operations of two fuzzy sets, they end up with peculiar results while defining $A \cap A^c$ and $A \cup A^c$. In this regard, H. K. Baruah proposed that in the current definition of the complement of a fuzzy set, fuzzy membership function and fuzzy membership value had been taken to be the same, which led to the conclusion that the fuzzy sets do not follow the set theoretic axioms of exclusion and contradiction. H. K. Baruah has put forward an extended definition of fuzzy set and redefined the complement of a fuzzy set accordingly. In this paper, we are trying to improve the notion of union and intersection of fuzzy sets proposed by Baruah and generalize the concept of complement of a fuzzy set when the fuzzy reference function is not zero. We support our definition of complement of an extended fuzzy set with examples and show that indeed our definition satisfies all those properties that complement of a set really does in classical sense.

Keywords
Fuzzy set, fuzzy membership function, fuzzy reference function, fuzzy membership value, complement of an extended fuzzy set.

1. INTRODUCTION
Fuzzy Set Theory was introduced by Lofti Zadeh [5] in 1965 and it was specifically designed to mathematically represent uncertainty and vagueness with formalized logical tools for dealing with the imprecision inherent in many real world problems. Fuzzy sets are sets with boundaries that are not precise. The membership in a fuzzy set is not a matter of affirmation or denial, but rather a matter of a degree. Zadeh's fuzzy set theory challenged not only probability theory as the sole agent for uncertainty, but the very foundations upon which probability theory is based: Aristotelian two valued logic. Out of several higher order fuzzy sets, Intuitionistic Fuzzy Sets (IFS) introduced by Atanassov [1, 2] is of great importance. Although IFS are defined with the help of membership functions, these are not necessarily fuzzy sets. Fuzzy sets, on the other hand, are Intuitionistic Fuzzy Sets. Research on the theory of fuzzy sets has been growing steadily since the inception of the theory in the mid 1960's.The body of concepts and results pertaining to the theory is now quite impressive. Research on a broad variety of applications has also been very active and has produced results that are perhaps even more impressive.

Fuzzy set theory proposed by Professor L. A. Zadeh [5] is assumed as a generalization of classical or crisp sets. The theory of fuzzy sets should actually have been a generalization of the classical theory of sets in the sense that the theory of sets should have been a special case of the theory of fuzzy sets. Unfortunately, this is not the case. It has been accepted that for a fuzzy set A and its complement A^c, neither $A \cap A^c$ is the null set, nor $A \cup A^c$ is the universal set. Whereas the operations of union and intersection of two crisp sets are indeed special cases of the corresponding operations of two fuzzy sets, they end up giving peculiar results while defining $A \cap A^c$ and $A \cup A^c$. In this regard H. K. Baruah, [3, 4] has forwarded an extended definition of fuzzy sets which enables us to define complement of fuzzy set in a way that give us $A \cap A^c$ = the null set and $A \cup A^c$ = the universal set. We agree with him as this new definition satisfies all the properties regarding complement of a fuzzy set.

In this article, we put forward a definition of complement of an extended fuzzy set where the fuzzy reference function is not always zero. The definition of complement of a fuzzy set proposed by Baruah [3, 4] can be seen as a particular case of what we are giving. We support our definition of complement of an extended fuzzy set with examples and show that indeed our definition satisfies all those properties that complement of a set really does in classical sense. We improve the notion of fuzzy union and intersection proposed by Baruah [3, 4] and prove DeMorgan Laws in Fuzzy Set Theory. We further put forward an extended definition of subset of a fuzzy set from our stand point. Finally we define union and intersection for an arbitrary collection of fuzzy sets over the same universe and prove DeMorgan Laws for an arbitrary collection of fuzzy sets over the same universe.

2. PRELIMINARIES
H. K. Baruah [3, 4] gave an extended definition of fuzzy set in the following manner. According to him, to define a fuzzy set, two functions namely fuzzy membership function and fuzzy reference function are necessary. Fuzzy membership value is the difference between fuzzy membership function and reference function. Fuzzy membership function and fuzzy membership value are two different things. In the Zadehian definition of complementation, these two things have been taken to be the same, and that is where the error lies.

Let $\mu_1(x)$ and $\mu_2(x)$ be two functions, $0 \leq \mu_2(x) \leq \mu_1(x) \leq 1$. For a fuzzy number denoted by $\{x, \mu_1(x), \mu_2(x); x \varepsilon U\}$, we would call $\mu_1(x)$ the fuzzy membership function, and $\mu_2(x)$ a reference function, such that $\{\mu_1(x) - \mu_2(x)\}$ is the fuzzy membership value for any x. In the definition of complement of a fuzzy set, the fuzzy membership value and the fuzzy membership function have to be different, in the sense that for a usual fuzzy set the membership value and the membership function are of course equivalent.

Let $A(\mu_1,\mu_2)=\{x,\mu_1(x),\mu_2(x);x\in U\}$ and

$B(\mu_3,\mu_4)=\{x,\mu_3(x),\mu_4(x);x\in U\}$ be two fuzzy sets defined over the same universe *U*. Then the operations intersection and union are defined as

$A(\mu_1,\mu_2)\cap B(\mu_3,\mu_4)$

$=\{x,\min(\mu_1(x),\mu_3(x)),\max(\mu_2(x),\mu_4(x));x\in U\}$

and

$A(\mu_1,\mu_2)\cup B(\mu_3,\mu_4)$

$=\{x,\max(\mu_1(x),\mu_3(x)),\min(\mu_2(x),\mu_4(x));x\in U\}$

Two fuzzy sets

$C=\{x,\mu_C(x):x\in U\}$ and

$D=\{x,\mu_D(x):x\in U\}$ in the usual definition would be expressed as

$C(\mu_C,0)=\{x,\mu_C(x),0;x\in U\}$ and
$D(\mu_D,0)=\{x,\mu_D(x),0;x\in U\}$

Accordingly, we have

$C(\mu_C,0)\cap D(\mu_D,0)$

$=\{x,\min(\mu_C(x),\mu_D(x)),\max(0,0);x\in U\}$

$=\{x,\min(\mu_C(x),\mu_D(x)),0;x\in U\}$
$=\{x,\mu_C(x)\wedge\mu_D(x);x\in U\}$

which in the usual definition is nothing but $C\cap D$. Similarly, we have

$C(\mu_C,0)\cup D(\mu_D,0)$

$=\{x,\max(\mu_C(x),\mu_D(x)),\min(0,0);x\in U\}$

$=\{x,\max(\mu_C(x),\mu_D(x)),0;x\in U\}$

$=\{x,\mu_C(x)\vee\mu_D(x);x\in U\}$

which in the usual definition is nothing but $C\cup D$. Thus we have seen that for union and intersection of two fuzzy sets, the extended definition leads to the union and intersection under the standard definition.

These new definitions lead to the conclusion that for usual fuzzy sets $A(\mu,0)=\{x,\mu(x),0;x\in U\}$ and $B(1,\mu)=\{x,1,\mu(x);x\in U\}$ defined over the same universe *U* we have

$A(\mu,0)\cap B(1,\mu)$

$=\{x,\min(\mu(x),1),\max(0,\mu(x));x\in U\}$

$=\{x,\mu(x),\mu(x);x\in U\}$,

which is nothing but the null set φ and

$A(\mu,0)\cup B(1,\mu)$

$=\{x,\max(\mu(x),1),\min(0,\mu(x));x\in U\}$

$=\{x,1,0;x\in U\}$,

which is nothing but the universal set *U*

This means if we define a fuzzy set

$(A(\mu,0))^c=\{x,1,\mu(x);x\in U\}$, it is nothing but the complement of $A(\mu,0)=\{x,\mu(x),0;x\in U\}$.

Thus it can be concluded that $A(\mu,0)\cap(A(\mu,0))^c=\phi$, the null set and $A(\mu,0)\cup(A(\mu,0))^c=U$, the universal set.

Example 1.

Let $U=\{a,b,c\}$ be the universal set. We take two fuzzy sets *A* and *B* as –

$A(\mu_1,\mu_2)=\{(a,0.1,0),(b,0.2,0.1),(c,0.4,0.2)\}$ and
$B(\mu_3,\mu_4)=\{(a,0.9,0.3),(b,0.5,0.3),(c,0.5,0.3)\}$.

Then $(A\cap B)(\mu_5,\mu_6)$

$=\{x,\mu_5(x),\mu_6(x);x\in U\}$

$=\{(a,0.1,0.3),(b,0.2,0.3),(c,0.4,0.3)\}$

We have seen that $\mu_5(a)<\mu_6(a)$, $\mu_5(b)<\mu_6(b)$ which is going against our assumption that $\mu_5(x)\geq\mu_6(x)\forall x\in U$

Again $(A\cup B)(\mu_7,\mu_8)$

$=\{x,\mu_7(x),\mu_8(x);x\in U\}$

$=\{(a,0.9,0),(b,0.5,0.1),(c,0.5,0.2)\}$, which is not true.

To avoid such degenerate cases, we improve these definitions of union and intersection as follows.

Definition 1.

Let $A(\mu_1,\mu_2)=\{x,\mu_1(x),\mu_2(x);x\in U\}$

and $B(\mu_3,\mu_4)=\{x,\mu_3(x),\mu_4(x);x\in U\}$

be two fuzzy sets defined over the same universe *U*. To avoid degenerate cases we assume that

$\min(\mu_1(x),\mu_3(x))\geq\max(\mu_2(x),\mu_4(x))\forall x\in U$.

Then the operation intersection is defined as

$A(\mu_1,\mu_2)\cap B(\mu_3,\mu_4)$

$=\{x,\min(\mu_1(x),\mu_3(x)),\max(\mu_2(x),\mu_4(x));x\in U\}$

If for some $x\in U$, $\min(\mu_1(x),\mu_3(x))<\max(\mu_2(x),\mu_4(x))$,

then our conclusion is that $A\cap B=\varphi$.

If for some $x \in U$, $\min(\mu_1(x), \mu_3(x)) = \max(\mu_2(x), \mu_4(x))$,

then also $A \cap B = \varphi$.

Further, we define the operation union, with $\min(\mu_1(x), \mu_3(x)) \geq \max(\mu_2(x), \mu_4(x)) \forall x \in U$ as

$$A(\mu_1, \mu_2) \cup B(\mu_3, \mu_4)$$

$$= \{x, \max(\mu_1(x), \mu_3(x)), \min(\mu_2(x), \mu_4(x)); x \in U\}$$

Also, our another conclusion is, that if for some $x \in U$, $\min(\mu_1(x), \mu_3(x)) < \max(\mu_2(x), \mu_4(x))$, then the union of the fuzzy sets *A* and *B* cannot be expressed as one single fuzzy set. The union, however, can be expressed in one single fuzzy set if $\min(\mu_1(x), \mu_3(x)) = \max(\mu_2(x), \mu_4(x))$. Above example makes this clear. For usual fuzzy sets with reference function 0, it is quite obvious to see that the above conditions for defining intersection and union hold good.

Definition 2.

Let $A(\mu_1, \mu_2) = \{x, \mu_1(x), \mu_2(x); x \in U\}$ be a fuzzy set defined over the universe U. Then the complement of the extended fuzzy set $A(\mu_1, \mu_2)$ is defined as

$$(A(\mu_1, \mu_2))^c$$

$$= \{x, \mu_1(x), \mu_2(x); x \in U\}^c$$

$$= \{x, \mu_2(x), 0; x \in U\} \cup \{x, 1, \mu_1(x); x \in U\}$$

Membership value of x in $(A(\mu_1, \mu_2))^c$ is given by $\mu_2(x) + (1 - \mu_1(x)) = 1 + \mu_2(x) - \mu_1(x)$

If $\mu_2(x) = 0$, then membership value of x is

$$1 + 0 - \mu_1(x) = 1 - \mu_1(x)$$

For $x \in U$, $\min(\mu_2(x), 1) < \max(0, \mu_1(x))$, so the union of these two fuzzy sets cannot be expressed as one single fuzzy set.

Remark 1.

If $\mu_1(x) = \mu_2(x), \forall x \in U$, this definition gives

$$(A(\mu_1, \mu_1))^c = \{x, \mu_1(x), \mu_1(x); x \in U\}^c$$

$$= \{x, \mu_1(x), 0; x \in U\} \cup \{x, 1, \mu_1(x); x \in U\}$$

$$= \{x, \max(\mu_1(x), 1), \min(0, \mu_1(x)); x \in U\}$$

$$= \{x, 1, 0; x \in U\}$$

$$= U$$

Thus $\varphi^c = U$.

Example 2.

Let $U = \{a, b, c\}$ be the universal set. We take a fuzzy null set

$$\varphi = \{(a, 0.1, 0.1), (b, 0.2, 0.2), (c, 0.4, 0.4)\}$$

$$\varphi^c = \{(a, 0.1, 0), (b, 0.2, 0), (c, 0.4, 0)\} \cup \{(a, 1, 0.1), (b, 1, 0.2), (c, 1, 0.4)\}$$

$$= \{(a, \max(0.1, 1), \min(0, 0.1)), (b, \max(0.2, 1), \min(0, 0.2)),$$

$$(c, \max(0.4, 1), \min(0, 0.4))\}$$

$$= \{(a, 1, 0), (b, 1, 0), (c, 1, 0)\}$$

$$= U$$

Remark 2.

Let us see what happens when we take fuzzy reference function = 0 in this definition.

$$(A(\mu_1, 0))^c = \{x, \mu_1(x), 0; x \in U\}^c$$

$$= \{x, 0, 0; x \in U\} \cup \{x, 1, \mu_1(x); x \in U\},$$

(Here $0 = \min(0, 1) < \max(0, \mu_1(x)) = \mu_1(x)$)

$$= \varphi \cup \{x, 1, \mu_1(x); x \in U\}$$

$$= \{x, 1, \mu_1(x); x \in U\}$$

Which is what Baruah [3, 4] has defined. Thus we have seen that our definition of complement of an extended fuzzy set yields Baruah's definition [3, 4] when we take fuzzy reference function = 0 $\forall x \in U$. Next we show that our definition of complement of an extended fuzzy set satisfies the set theoretic axioms of contradiction and exclusion. Thus we put forward the following two propositions.

Proposition 1.

For a fuzzy set $A(\mu_1, \mu_2)$, we have

1. $A(\mu_1, \mu_2) \cap (A(\mu_1, \mu_2))^c = \varphi$ (Contradiction)
2. $A(\mu_1, \mu_2) \cup (A(\mu_1, \mu_2))^c = U$ (Exclusion)

Proof.

1. $A(\mu_1, \mu_2) \cap (A(\mu_1, \mu_2))^c$

$$= \{x, \mu_1(x), \mu_2(x); x \in U\} \cap \{x, \mu_1(x), \mu_2(x); x \in U\}^c$$
$$= \{x, \mu_1(x), \mu_2(x); x \in U\} \cap \{\{x, \mu_2(x), 0; x \in U\} \cup \{x, 1, \mu_1(x); x \in U\}\}$$
$$= \{\{x, \mu_1(x), \mu_2(x); x \in U\} \cap \{x, \mu_2(x), 0; x \in U\}\}$$
$$\cup \{\{x, \mu_1(x), \mu_2(x); x \in U\} \cap \{x, 1, \mu_1(x); x \in U\}\}$$
$$= \{x, \min(\mu_1(x), \mu_2(x)), \max(\mu_2(x), 0); x \in U\}$$
$$\cup \{x, \min(\mu_1(x), 1), \max(\mu_2(x), \mu_1(x)); x \in U\}$$
$$= \{x, \mu_2(x), \mu_2(x); x \in U\} \cup \{x, \mu_1(x), \mu_1(x); x \in U\}$$
$$= \varphi \cup \varphi$$
$$= \varphi$$

Thus $A(\mu_1, \mu_2) \cap (A(\mu_1, \mu_2))^c = \varphi$ (Contradiction)

2. $A(\mu_1, \mu_2) \cup (A(\mu_1, \mu_2))^c$

$$= \{x, \mu_1(x), \mu_2(x); x \in U\} \cup \{x, \mu_1(x), \mu_2(x); x \in U\}^c$$

$= \{x, \mu_1(x), \mu_2(x); x \in U\}$
$\cup \{\{x, \mu_2(x), 0; x \in U\} \cup \{x, 1, \mu_1(x); x \in U\}\}$
$= \{\{x, \mu_1(x), \mu_2(x); x \in U\} \cup \{x, \mu_2(x), 0; x \in U\}\}$
$\cup \{x, 1, \mu_1(x); x \in U\}$
$= \{x, \max(\mu_1(x), \mu_2(x)), \min(\mu_2(x), 0); x \in U\} \cup \{x, 1, \mu_1(x); x \in U\}$
$= \{x, \mu_1(x), 0; x \in U\} \cup \{x, 1, \mu_1(x); x \in U\}$
$= \{x, \max(\mu_1(x), 1), \min(0, \mu_1(x)); x \in U\}$
$= \{x, 1, 0; x \in U\}$
$= U$

Thus $A(\mu_1, \mu_2) \cup (A(\mu_1, \mu_2))^c = U$ (Exclusion)

Proposition 2. (DeMorgan Laws)

Let $A(\mu_1, \mu_2) = \{x, \mu_1(x), \mu_2(x); x \in U\}$ and $B(\mu_3, \mu_4) = \{x, \mu_3(x), \mu_4(x); x \in U\}$ be two fuzzy sets defined over the same universe U. To avoid degenerate case we assume that $\min(\mu_1(x), \mu_3(x)) \geq \max(\mu_2(x), \mu_4(x)) \forall x \in U$. Then

1. $(A(\mu_1, \mu_2) \cup B(\mu_3, \mu_4))^c = (A(\mu_1, \mu_2))^c \cap (B(\mu_3, \mu_4))^c$
2. $(A(\mu_1, \mu_2) \cap B(\mu_3, \mu_4))^c = (A(\mu_1, \mu_2))^c \cup (B(\mu_3, \mu_4))^c$

Proof.

1. $(A(\mu_1, \mu_2) \cup B(\mu_3, \mu_4))^c$

$= \{\{x, \mu_1(x), \mu_2(x); x \in U\} \cup \{x, \mu_3(x), \mu_4(x); x \in U\}\}^c$
$= \{x, \max(\mu_1(x), \mu_3(x)), \min(\mu_2(x), \mu_4(x)); x \in U\}^c$
$= \{x, \min(\mu_2(x), \mu_4(x)), 0; x \in U\} \cup \{x, 1, \max(\mu_1(x), \mu_3(x)); x \in U\}$

Again

$(A(\mu_1, \mu_2))^c \cap (B(\mu_3, \mu_4))^c$

$= \{x, \mu_1(x), \mu_2(x); x \in U\}^c \cap \{x, \mu_3(x), \mu_4(x); x \in U\}^c$
$= \{\{x, \mu_2(x), 0; x \in U\} \cup \{x, 1, \mu_1(x); x \in U\}\}$
$\cap \{\{x, \mu_4(x), 0; x \in U\} \cup \{x, 1, \mu_3(x); x \in U\}\}$
$= [\{\{x, \mu_2(x), 0; x \in U\} \cup \{x, 1, \mu_1(x); x \in U\}\} \cap \{x, \mu_4(x), 0; x \in U\}]$
$\cup [\{\{x, \mu_2(x), 0; x \in U\} \cup \{x, 1, \mu_1(x); x \in U\}\} \cap \{x, 1, \mu_3(x); x \in U\}]$
$= [\{\{x, \mu_2(x), 0; x \in U\} \cap \{x, \mu_4(x), 0; x \in U\}\}$
$\cup \{\{x, 1, \mu_1(x); x \in U\} \cap \{x, 1, \mu_3(x); x \in U\}\}]$
$\cup [\{\{x, 1, \mu_1(x); x \in U\} \cap \{x, \mu_4(x), 0; x \in U\}\}$
$\cup \{\{x, \mu_2(x), 0; x \in U\} \cap \{x, 1, \mu_3(x); x \in U\}\}]$
$= [\{x, \min(\mu_2(x), \mu_4(x)), 0; x \in U\} \cup \varphi]$
$\cup [\varphi \cup \{x, 1, \max(\mu_1(x), \mu_3(x)); x \in U\}]$
$= \{x, \min(\mu_2(x), \mu_4(x)), 0; x \in U\}$
$\cup \{x, 1, \max(\mu_1(x), \mu_3(x)); x \in U\}$

Thus $(A(\mu_1, \mu_2) \cup B(\mu_3, \mu_4))^c = (A(\mu_1, \mu_2))^c \cap (B(\mu_3, \mu_4))^c$

We have assumed that

$\min(\mu_1(x), \mu_3(x)) \geq \max(\mu_2(x), \mu_4(x)) \forall x \in U$, so

$\mu_4(x) = \min(1, \mu_4(x)) \leq \max(\mu_1(x), 0) = \mu_1(x) \forall x \in U$

and as such

$\{\{x, 1, \mu_1(x); x \in U\}\} \cap \{x, \mu_4(x), 0; x \in U\} = \varphi$.

Similarly

$\mu_2(x) = \min(\mu_2(x), 1) \leq \max(0, \mu_3(x)) = \mu_3(x) \forall x \in U$

and hence

$\{\{x, \mu_2(x), 0; x \in U\}\} \cap \{x, 1, \mu_3(x); x \in U\} = \varphi$

2. $(A(\mu_1, \mu_2) \cap B(\mu_3, \mu_4))^c$

$= \{\{x, \mu_1(x), \mu_2(x); x \in U\} \cap \{x, \mu_3(x), \mu_4(x); x \in U\}\}^c$
$= \{x, \min(\mu_1(x), \mu_3(x)), \max(\mu_2(x), \mu_4(x)); x \in U\}^c$
$= \{x, \max(\mu_2(x), \mu_4(x)), 0; x \in U\} \cup$
$\{x, 1, \min(\mu_1(x), \mu_3(x)); x \in U\}$

Again

$(A(\mu_1, \mu_2))^c \cup (B(\mu_3, \mu_4))^c$

$= \{x, \mu_1(x), \mu_2(x); x \in U\}^c \cup \{x, \mu_3(x), \mu_4(x); x \in U\}^c$
$= \{\{x, \mu_2(x), 0; x \in U\} \cup \{x, 1, \mu_1(x); x \in U\}\}$
$\cup \{\{x, \mu_4(x), 0; x \in U\} \cup \{x, 1, \mu_3(x); x \in U\}\}$
$= \{\{x, \mu_2(x), 0; x \in U\} \cup \{x, \mu_4(x), 0; x \in U\}\}$
$\cup \{\{x, 1, \mu_1(x); x \in U\} \cup \{x, 1, \mu_3(x); x \in U\}\}$
$= \{x, \max(\mu_2(x), \mu_4(x)), 0; x \in U\}$
$\cup \{x, 1, \min(\mu_1(x), \mu_3(x)); x \in U\}$

Thus $(A(\mu_1, \mu_2) \cap B(\mu_3, \mu_4))^c = (A(\mu_1, \mu_2))^c \cup (B(\mu_3, \mu_4))^c$

DeMorgan laws for usual fuzzy sets with reference function 0 can be obtained from the above DeMorgan laws by taking

$\mu_2(x) = \mu_4(x) = 0 \forall x \in U$.

Proposition 3.

For a fuzzy set $A(\mu_1, \mu_2)$, we have

$$\left((A(\mu_1, \mu_2))^c\right)^c = A(\mu_1, \mu_2)$$

Proof.

$\left((A(\mu_1,\mu_2))^c\right)^c$

$=\left\{\{x,\mu_1(x),\mu_2(x);x\in U\}^c\right\}^c$

$=\{\{x,\mu_2(x),0;x\in U\}\cup\{x,1,\mu_1(x);x\in U\}\}^c$

$=\{x,\mu_2(x),0;x\in U\}^c\cap\{x,1,\mu_1(x);x\in U\}^c$

$=\{x,1,\mu_2(x);x\in U\}^c\cap\{x,\mu_1(x),0;x\in U\}^c$

$=\{x,\min(1,\mu_1(x)),\max(\mu_2(x),0);x\in U\}$

$=\{x,\mu_1(x),\mu_2(x);x\in U\}$

$=(A(\mu_1,\mu_2))$

Thus $\left((A(\mu_1,\mu_2))^c\right)^c = A(\mu_1,\mu_2)$ (Involution)

Example 3.

Let $U=\{a,b,c\}$ be the universal set. We take a fuzzy set

$A \ =\{(a,0.1,0),(b,0.2,0.1),(c,0.4,0.2)\}$

$A^c=\{(a,0,0),(b,0.1,0),(c,0.2,0)\}\cup\{(a,1,0.1),(b,1,0.2),(c,1,0.4)\}$

Thus $\left(A^c\right)^c$

$=(\{(a,0,0),(b,0.1,0),(c,0.2,0)\}\cup\{(a,1,0.1),(b,1,0.2),(c,1,0.4)\})^c$

$=\{(a,0,0),(b,0.1,0),(c,0.2,0)\}^c\cap\{(a,1,0.1),(b,1,0.2),(c,1,0.4)\}^c$

$=\{(a,1,0),(b,1,0.1),(c,1,0.2)\}\cap\{(a,0.1,0),(b,0.2,0),(c,0.4.0)\}$

$=\{(a,0.1,0),(b,0.2,0.1),(c,0.4,0.2)\}$

$=A$

We now proceed to define subset of a fuzzy set from our stand point.

Definition 3.

Let $A(\mu_1,\mu_2)=\{x,\mu_1(x),\mu_2(x);x\in U\}$ and

$B(\mu_3,\mu_4)=\{x,\mu_3(x),\mu_4(x);x\in U\}$ be two fuzzy sets defined over the same universe *U*. The fuzzy set $A(\mu_1,\mu_2)$ is a subset of the fuzzy set $B(\mu_3,\mu_4)$

if $\forall x\in U$, $\mu_1(x)\le\mu_3(x)$ and $\mu_4(x)\le\mu_2(x)$.

Two fuzzy sets

$C=\{x,\mu_C(x):x\in U\}$ and $D=\{x,\mu_D(x):x\in U\}$ in the usual definition would be expressed as $C(\mu_C,0)=\{x,\mu_C(x),0;x\in U\}$ and $D(\mu_D,0)=\{x,\mu_D(x),0;x\in U\}$.

Accordingly, we have $C(\mu_C,0)\subseteq D(\mu_D,0)$

if $\forall x\in U$, $\mu_C(x)\le\mu_D(x)$, Which can be obtained by putting $\mu_2(x)=\mu_4(x)=0$ in our new definition.

Proposition 4.

For fuzzy sets $A(\mu_1,\mu_2),B(\mu_3,\mu_4),C(\mu_5,\mu_6)$ over the same universe *U*, the following propositions are valid. To avoid degenerate cases we assume that

$\min(\mu_1(x),\mu_3(x))\ge\max(\mu_2(x),\mu_4(x))\forall x\in U$.

1. $A(\mu_1,\mu_2)\subseteq B(\mu_3,\mu_4), B(\mu_3,\mu_4)\subseteq C(\mu_5,\mu_6)$

   $\Rightarrow A(\mu_1,\mu_2)\subseteq C(\mu_5,\mu_6)$

2. $A(\mu_1,\mu_2)\cap B(\mu_3,\mu_4)\subseteq A(\mu_1,\mu_2)$,

   $A(\mu_1,\mu_2)\cap B(\mu_3,\mu_4)\subseteq B(\mu_3,\mu_4)$

3. $A(\mu_1,\mu_2)\subseteq A(\mu_1,\mu_2)\cup B(\mu_3,\mu_4)$,

   $B(\mu_3,\mu_4)\subseteq A(\mu_1,\mu_2)\cup B(\mu_3,\mu_4)$

4. $A(\mu_1,\mu_2)\subseteq B(\mu_3,\mu_4)$

   $\Rightarrow A(\mu_1,\mu_2)\cap B(\mu_3,\mu_4)=A(\mu_1,\mu_2)$

5. $A(\mu_1,\mu_2)\subseteq B(\mu_3,\mu_4)$

   $\Rightarrow A(\mu_1,\mu_2)\cup B(\mu_3,\mu_4)=B(\mu_3,\mu_4)$

Proof.

Let $A(\mu_1,\mu_2)=\{x,\mu_1(x),\mu_2(x);x\in U\}$,

$B(\mu_3,\mu_4)=\{x,\mu_3(x),\mu_4(x);x\in U\}$ and

$C(\mu_5,\mu_6)=\{x,\mu_5(x),\mu_6(x);x\in U\}$

be three fuzzy sets over the same universe *U*.

1. $A(\mu_1,\mu_2)\subseteq B(\mu_3,\mu_4)$

   $\Rightarrow\forall x\in U,\mu_1(x)\le\mu_3(x),\mu_4(x)\le\mu_2(x)$

   $B(\mu_3,\mu_4)\subseteq C(\mu_5,\mu_6)$

   $\Rightarrow\forall x\in U,\mu_3(x)\le\mu_5(x),\mu_6(x)\le\mu_4(x)$

Thus $\forall x\in U,\mu_1(x)\le\mu_5(x),\mu_6(x)\le\mu_2(x)$

and hence

$A(\mu_1,\mu_2)\subseteq C(\mu_5,\mu_6)$

2. $A(\mu_1,\mu_2)\cap B(\mu_3,\mu_4)$

   $=\{x,\min(\mu_1(x),\mu_3(x)),\max(\mu_2(x),\mu_4(x));x\in U\}$

It is clear that

$\forall x\in U,\min(\mu_1(x),\mu_3(x))\le\mu_1(x),\mu_2(x)\le\max(\mu_2(x),\mu_4(x))$

Thus $A(\mu_1,\mu_2)\cap B(\mu_3,\mu_4)\subseteq A(\mu_1,\mu_2)$

The second result can be similarly found out.

3. $A(\mu_1,\mu_2)\cup B(\mu_3,\mu_4)$

$= \{x, \max(\mu_1(x),\mu_3(x)), \min(\mu_2(x),\mu_4(x)); x\in U\}$

It is clear that

$\forall x\in U, \mu_1(x)\le \max(\mu_1(x),\mu_3(x)), \min(\mu_2(x),\mu_4(x))\le \mu_2(x)$

Thus $A(\mu_1,\mu_2)\subseteq A(\mu_1,\mu_2)\cup B(\mu_3,\mu_4)$

The second result can be similarly found out.

4. $A(\mu_1,\mu_2)\subseteq B(\mu_3,\mu_4)$

$\Rightarrow \forall x\in U, \mu_1(x)\le \mu_3(x), \mu_4(x)\le \mu_2(x)$

Now, $A(\mu_1,\mu_2)\cap B(\mu_3,\mu_4)$

$= \{x, \min(\mu_1(x),\mu_3(x)), \max(\mu_2(x),\mu_4(x)); x\in U\}$

$= \{x, \mu_1(x), \mu_2(x); x\in U\}$

$= A(\mu_1,\mu_2)$

5. $A(\mu_1,\mu_2)\subseteq B(\mu_3,\mu_4)$

$\Rightarrow \forall x\in U, \mu_1(x)\le \mu_3(x), \mu_4(x)\le \mu_2(x)$

Now, $A(\mu_1,\mu_2)\cup B(\mu_3,\mu_4)$

$= \{x, \max(\mu_1(x),\mu_3(x)), \min(\mu_2(x),\mu_4(x)); x\in U\}$

$= \{x, \mu_3(x), \mu_4(x); x\in U\}$

$= B(\mu_3,\mu_4)$

Taking $\mu_2(x)=\mu_4(x)=\mu_6(x)=0$, we obtain the same results for usual fuzzy sets.

We now proceed to define arbitrary fuzzy union and intersection using the extended definition of fuzzy sets given by Baruah [3, 4].

Definition 4.

Let $\Im=\{A_i(\mu_{i1},\mu_{i2}) \mid i\in I\}$ be a family of fuzzy sets over the same universe U. To avoid degenerate cases we assume that $\min(\mu_{i1}(x))\ge \max(\mu_{i2}(x))\forall x\in U$. Then the union of fuzzy sets in $\Im$ is a fuzzy set given by

$\underset{i}{\cup} A_i(\mu_{i1},\mu_{i2}) = \{x, \max(\mu_{i1}(x)), \min(\mu_{i2}(x)); x\in U\}$. And the intersection of fuzzy sets in $\Im$ is a fuzzy set given by

$\underset{i}{\cap} A_i(\mu_{i1},\mu_{i2}) = \{x, \min(\mu_{i1}(x)), \max(\mu_{i2}(x)); x\in U\}$

Example 4.

Let $U=\{a,b,c\}$ and

$A_1(\mu_{11},\mu_{12}) = \{(a,0.5,0.1),(b,0.6,0),(c,1,0.3)\}$,

$A_2(\mu_{21},\mu_{22}) = \{(a,0.6,0),(b,0.8,0.2),(c,0.9,0.1)\}$

and $A_3(\mu_{31},\mu_{32}) = \{(a,1,0.2),(b,0.9,0.1),(c,0.4,0.2)\}$ be three fuzzy sets over U. Then

$A_1(\mu_{11},\mu_{12})\cup A_2(\mu_{21},\mu_{22})\cup A_3(\mu_{31},\mu_{32})$

$= \{(a, \max(0.5,0.6,1), \min(0.1,0,0.2)),$

$(b, \max(0.6,0.8,0.9), \min(0,0.2,0.1)),$

$(c, \max(1,0.9,0.4), \min(0.3,0.1,0.2))\}$

$= \{(a,1,0),(b,0.9,0),(c,1,0.1)\}$

and

$A_1(\mu_{11},\mu_{12})\cap A_2(\mu_{21},\mu_{22})\cap A_3(\mu_{31},\mu_{32})$

$= \{(a, \min(0.5,0.6,1), \max(0.1,0,0.2)),$

$(b, \min(0.6,0.8,0.9), \max(0,0.2,0.1)),$

$(c, \min(1,0.9,0.4), \max(0.3,0.1,0.2))\}$

$= \{(a,0.5,0.2),(b,0.6,0.2),(c,0.4,0.3)\}$

Proposition 5.

1. $A_i(\mu_{i1},\mu_{i2})\subseteq \underset{i}{\cup} A_i(\mu_{i1},\mu_{i2})\forall i\in I$

2. $\underset{i}{\cap} A_i(\mu_{i1},\mu_{i2})\subseteq A_i(\mu_{i1},\mu_{i2})\forall i\in I$

Proof.

1. $\underset{i}{\cup} A_i(\mu_{i1},\mu_{i2})$

$= \{x, \max(\mu_{i1}(x)), \min(\mu_{i2}(x)); x\in U\}$

It is quite obvious that

$\mu_{i1}(x)\le \max(\mu_{i1}(x))$ and $\min(\mu_{i2}(x))\le \mu_{i2}(x)\ \forall i$ and $\forall x\in U$

Thus by our extended definition of subset of a fuzzy set, $A_i(\mu_{i1},\mu_{i2})\subseteq \underset{i}{\cup} A_i(\mu_{i1},\mu_{i2})\forall i\in I$.

2. $\underset{i}{\cap} A_i(\mu_{i1},\mu_{i2})$

$= \{x, \min(\mu_{i1}(x)), \max(\mu_{i2}(x)); x\in U\}$

It is quite obvious that

$\min(\mu_{i1}(x))\le \mu_{i1}(x)$ and $\mu_{i2}(x)\le \max(\mu_{i2}(x))\ \forall i$ and $\forall x\in U$.

Thus by our extended definition of subset of a fuzzy set, $\underset{i}{\cap} A_i(\mu_{i1},\mu_{i2})\subseteq A_i(\mu_{i1},\mu_{i2})\forall i\in I$.

Proposition 6. (DeMorgan Laws)

Let $\Im=\{A_i(\mu_{i1},\mu_{i2}) \mid i\in I\}$ be a family of fuzzy sets over the same universe U. To avoid degenerate cases we assume that $\min(\mu_{i1}(x))\ge \max(\mu_{i2}(x))\forall x\in U$. Then

1. $\left\{\underset{i}{\cup} A_i(\mu_{i1},\mu_{i2})\right\}^c = \underset{i}{\cap}\{A_i(\mu_{i1},\mu_{i2})\}^c$

2. $\left\{\underset{i}{\cap} A_i(\mu_{i1},\mu_{i2})\right\}^c = \underset{i}{\cup}\{A_i(\mu_{i1},\mu_{i2})\}^c$

Proof.

1. $\left\{\bigcup_i A_i(\mu_{i1},\mu_{i2})\right\}^c$

$=\{\{x,\max(\mu_{i1}(x)),\min(\mu_{i2}(x));x\in U\}\}^c$

$=\{x,\min(\mu_{i2}(x)),0;x\in U\}\cup\{x,1,\max(\mu_{i1}(x));x\in U\}$

$\bigcap_i\{A_i(\mu_{i1},\mu_{i2})\}^c$

$=\bigcap_i\{\{x,\mu_{i2}(x),0;x\in U\}\cup\{x,1,\mu_{i1}(x);x\in U\}\}$

$=\left\{\bigcap_i\{x,\mu_{i2}(x),0;x\in U\}\right\}\cup\left\{\bigcap_i\{x,1,\mu_{i1}(x);x\in U\}\right\}$

$=\{x,\min(\mu_{i2}(x)),0;x\in U\}\cup\{x,1,\max(\mu_{i1}(x));x\in U\}$ Thus

$$\left\{\bigcup_i A_i(\mu_{i1},\mu_{i2})\right\}^c=\bigcap_i\{A_i(\mu_{i1},\mu_{i2})\}^c$$

In a similar way, the result (2) can be established.

3. CONCLUSION

We have seen that if we use the extended definition of fuzzy set and complement of an extended fuzzy set, we arrive at the conclusion that the fuzzy sets, too, follow the set theoretic axioms of exclusion and contradiction, even when the fuzzy reference function is not zero. We have defined arbitrary fuzzy union and intersection and finally proved DeMorgan Laws for an arbitrary collection of fuzzy sets over the same universe *U*. We hope that our findings will help enhancing this study on fuzzy sets.

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5. REFERENCES

[1] Atanassov K, *Intuitionistic Fuzzy Sets*, Fuzzy Sets and Systems, vol. 20, pp. 87-96, 1986.

[2] Atanassov K, *Intuitionistic Fuzzy Sets–Theory and Applications*, Physica–Verlag, A Springer–Verlag Company, New-York (1999)

[3] Baruah H K, "*The Theory of Fuzzy Sets: Beliefs and Realities*", International Journal of Energy, Information and Communications, Vol. 2, Issue 2, pp. 1-22, May 2011.

[4] Baruah H K, "*Towards Forming A Field Of Fuzzy Sets*", International Journal of Energy, Information and Communications, Vol. 2, Issue 1, pp. 16-20, February 2011.

[5] Zadeh L A, "*Fuzzy Sets*", Information and Control, 8, pp. 338- 353, 1965.