Knowledge Acquisition under Imprecision through Neighborhood Approximation Operators

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ABSTRACT
The notion of rough sets, introduced by Z. Pawlak in 1982, is to capture impreciseness and indiscernibility of objects. The basic assumption of rough set theory is that human knowledge about a universe depends upon their capability to classify its objects. Classifications (or partitions) of a universe and equivalence relations defined on it are known to be interchangeable notions. So, for mathematical reasons, equivalence relations were considered by Pawlak to define rough sets. But in practice, we can get non-equivalence relations, rather than equivalence relations for the study of approximations. In this paper, we find notion of neighborhood systems instead of equivalence relations, proposed by Lin (1988), Chu (1992) and Lin & Yao (1996), for the study of approximation and also we study some properties of 1-neighborhood systems.

General Terms
Expert Systems, Rough Set Theory

Keywords
Rough sets, classifications, Neighborhood systems, approximation operators, Definability, Dependency.

1. INTRODUCTION
Uncertainty is an important part and is found a lot in our daily life. Theories to handle uncertainty are very before. Probability theory in statistics, the Dempster-Shafer theory of evidence [12] or the theory of belief functions, the fuzzy set theory, the rough set theory and their combinations are the main tool to deal with uncertainty. Recently rough set theory attracts not only the researcher of Artificial Intelligence but also the researcher of medical science, industry and business management etc. It has been successfully implemented in knowledge based systems in medicine [3, 10] and industry [1, 8].

Z. Pawlak [9] introduced the notion of rough set theory in 1982. It is an excellent tool to capture indiscernibility of objects. Vagueness, Impreciseness, inexactness of a set (concept) are manipulated, by two exact sets, known as, lower approximation and upper approximation of the set. For the finite universe *U*, the lower approximation of a rough set comprises of those elements of the universe, which can be said to belong to it certainly with the available knowledge (information) and on the other hand the upper approximation comprises of those elements which are possible in the set with respect to the available information.

Moreover, let *U* be a non-empty finite set, called universe of discourse. Let *R* be an equivalence relation on *U*, called an indiscernibility relation, and *U/R* be the family of all equivalence classes of *R* on *U*. An ordered pair *A = (U, R)* is called an approximation space.

For any subset *X* of *U*, the lower approximation of *X* in *A* under the indiscernibility relation R be defined by

(i) $$\underline{R}(X) = \{x \in U \mid [x]_R \subseteq X\}$$

and an upper approximation of X in A be defined by

(ii) $$\overline{R}(X) = \{x \in U \mid [x]_R \cap X \neq \phi\}$$

where $[x]_R \in$ U/R that is, $[x]_R$ be an equivalence class of R containing x, $x \in U$.

Definition 1.1 : A set $X \subset U$ is called rough with respect to the knowledge *R* (the equivalence relation *R*) if and only if $\underline{R}(X) \neq \overline{R}(X)$.

The set X is called definable with respect to knowledge *R* if and only if $\underline{R}(X)=\overline{R}(X)$.

$\overline{R}(X)\sim\underline{R}(X)$ is called the border line region of *X* with respect to the knowledge *R* and is denoted by $BN_R(X)$

Thus $\left(\underline{R}(X),\ \overline{R}(X)\right)$ is a rough set for X under the available knowledge R (an equivalence relation R) for any subset X of the universe U.

Lower and upper approximations in *A* have the following properties :

1.1 $\underline{R}(X)\subseteq X\subseteq\overline{R}(X)$

1.2 $\underline{R}(U)=\overline{R}(U)=U$, $\underline{R}(\phi)=\overline{R}(\phi)=\phi$

1.3 $\overline{R}(X\cup Y)=\overline{R}(X)\cup\overline{R}(Y)$ and

$\underline{R}(X\cap Y)=\underline{R}(X)\cap\underline{R}(Y)$

1.4 $\overline{R}(X\cap Y)\subseteq\overline{R}(X)\cap\overline{R}(Y) and$

$\underline{R}(X\cup Y)\supseteq\underline{R}(X)\cup\underline{R}(Y)$

1.5 $\overline{R}(\sim X)=\ \sim\underline{R}(X)$, $\underline{R}(\sim X)=\ \sim\overline{R}(X)$

1.6 $\underline{R}(\underline{R}(X))=\overline{R}(\underline{R}(X))=\underline{R}(X)\ and$

$\overline{R}(\overline{R}(X))=\underline{R}(\overline{R}(X))=\overline{R}(X)$

For an element $x\in U$, we say that x is certainly in *X* under the equivalence relation *R* (knowledge *R*) if and only if $x\in\underline{R}(X)$ and that x is possibly in *X* under *R* if and only if $x\in\overline{R}(X)$.

From property 1.4, according to Pawlak ([11]), the knowledge included in a distributed knowledge base is less than in the integrated one, in other words, dividing the knowledge base into smaller units in general, causes loss of information. In this article we find the conditions for which, there is no loss of information.

In the next section we study some properties of neighborhood operators. The objective of this paper is to extend the results of J.W. Grzymala – Busse [4] and other properties in the neighborhood systems.

2. NEIGHBORHOOD OPERATORS

let *U* be the universe of discourse, U be non-empty and finite and x be an element in *U*.

A neighborhood of x, denoted by n (x), is a non-empty subset of *U* which may or may not contain x. A neighborhood system of an element x, denoted by NS(x), is the maximal family of neighborhoods of x. If x has no neighborhood, then NS(x) is an empty family, in this case, we simply say that x has no neighborhood.

In this paper we consider only 1-neighborhood systems, that is, each object $x\in U$ has exactly one neighborhood in *U*.

A neighborhood system of *U*, denoted by NS(U), is the collection of NS(x) for all $x\in U$. The system (U, NS (U)) is called neighborhood system space or simply neighborhood system.

A neighborhood operator $n:U\rightarrow 2^U$ assigns a unique neighborhood n(x) to each element $x\in U$. We find the following properties of 1- neighborhood operator n. (Yao [14]).

2.1 : A neighborhood operator n is serial if for all $x\in U$, these exists a $y\in U$ such that $y\in n(x)$, that is, for all $x\in U,\ n(x)\neq\phi$

2.2 : The neighborhood operator n is inverse serial if for all $x\in U$, there exists $y\in U$ such that $x\in n(y)$, $\bigcup_{x\in U} n(x)\ =\ U$

2.3 : The neighborhood operator n is reflexive if for all $x\in U$, $x\in n(x)$.

2.4 : The neighborhood operator n is symmetric if for all $x,y\in U$

$$x\in n(y)\ \Rightarrow y\in n(x)$$

2.5 : The neighborhood operator n is transitive if for all $x,y,z\in U$

$$y\in n(x)\,,z\in n(y)\Rightarrow z\in n(x)\,.$$

A reflexive neighborhood operator is both serial and inverse serial. The family of neighborhoods $\{n(x) \mid x \in U\}$ of an 'inverse serial neighbourhood operator' n forms a covering of the universe.

Let n denote an arbitrary neighbourhood operator and n(x) be the corresponding neighborhood of $x \in U$ Then we define a pair of approximation operators [6, 14], for any subset X of U.

(iii) $$\underline{R}_n(X) = \{x \in U \mid n(x) \subseteq X\}$$
$$= \{x \in U \mid \forall y[y \in n(x) \Rightarrow y \in X]\}$$

(iv) $$\overline{R}_n(X) = \{x \in U \mid n(x) \cap X \neq \phi\}$$
$$= \{x \in U \mid \exists y[y \in n(x),\ y \in X]\}$$

For an equivalence relation R on U, the equivalence class $[x]_R$ may be considered as a neighborhood of. $x \in U$ Let n denote an arbitrary 1-neighborhood operator and n(x) be the corresponding neighborhood of $x \in U$ By substituting $[x]_R$ instead of n(x) we get the approximation operators (i) & (ii). Thus the approximation operators (iii) and (iv) are the generalization of (i) & (ii). The system $(2^U, \cap, \cup, \sim, \underline{R}_n, \overline{R}_n)$ is called the rough set algebra. The subscript n indicates that the approximation operators are defined and based on a particular neighborhood operator n.

Thus under the information available on U with respect to the neighborhood operator n, $(\underline{R_n}(X),\ \overline{R_n}(X))$ is a rough set for X, $X \subset U$ The border line region of X be give by $BN_n(x) = \overline{R}_n(x) \sim \underline{R}_n(x)$

Theorem : 2.1 (Yao, [14]) For an arbitrary neighborhood operator n, the pair of approximation operators satisfies the properties :

(a) $\underline{R}_n(X) = \sim(\overline{R}_n(\sim X))$

(b) $\overline{R}_n(X) = \sim(\underline{R}_n(\sim X))$

(c) $\underline{R}_n(U) = U, \overline{R}_n(\phi) = \phi$

(d) $\underline{R}_n(X \cap Y) = \underline{R}_n(X) \cap \underline{R}_n(Y)$

(e) $\overline{R}_n(X \cup Y) = \overline{R}_n(X) \cup \overline{R}_n(Y)$

where *X* and *Y* are two subsets of *U*. These properties also imply the following properties of approximation operators.

(f) $\underline{R}_n(X \cup Y) \supseteq \underline{R}_n(X) \cup \underline{R}_n(Y)$

(g) $\overline{R}_n(X \cap Y) \subseteq \overline{R}_n(X) \cap \overline{R}_n(Y)$

(h) $X \subseteq Y \Rightarrow \underline{R}_n(X) \subseteq \underline{R}_n(Y)$

(i) $X \subseteq Y \Rightarrow \overline{R}_n(X) \subseteq \overline{R}_n(Y)$

(j) $\underline{R}_n(X) = \bigcup_{x \in X} \overline{R}_n(\sim \{x\})$

(k) $\overline{R}_n(X) = \bigcup_{x \in X} \overline{R}_n(\{x\})$

Properties (a) and (b) show that approximation operators $\underline{R}_n$ & $\overline{R}_n$ are dual to each other. Properties (h) and (i) state that approximation operators are monotonic with respect to set inclusion. Inclusions in the properties (f) and (g) are of great interest, according to Pawlak ([11]), dividing the knowledge base into smaller units causes loss of information. These can be proved for equalities.

Additional properties of approximation operators are given below.

Theorem 2.2: (Yao [14]) Suppose $n : U \to 2^U$ is 1-neighborhood operator. If the neighborhood operator is serial then

(l) $\underline{R}_n(\phi) = \phi$, $\overline{R}_n(U) = U$ and $\underline{R}_n(X) \subseteq \overline{R}_n(X)$

The operator n is inverse serial then.

(m) for all $x \in U$, $\underline{R}_n(\sim(\{x\}) \neq U$ and $\overline{R}_n(\{x\}) \neq \phi$

The operator n is reflexive then.

(n) $\underline{R}_n(X) \subseteq X \subseteq \overline{R}_n(X)$

Now we will prove the following theorem.

Theorem 2.3: Let $n : U \to 2^U$ be a serial and inverse serial 1-neighborhood operator and $U = \{x_1, x_2, x_3, \ldots, x_n\}$ be the finite universe. Then for any two subsets $X_1, X_2 \subset U$ we have

(o) $\overline{R}_n(X_1 \cap X_2) \subsetneq \overline{R}_n(X_1) \cap \overline{R}_n(X_2)$

if and only if these exists at least one neighborhood *n(x_j)*, for $x_j \in U$, such that

(p) $X_1 \cap n(x_j) \neq \phi, X_2 \cap n(x_j) \neq \phi$

and

(q) $(X_1 \cap X_2) \cap n(x_j) = \phi$ for $1 \leq j \leq n$

Proof. (Sufficient Part)

We have, for $x_j \in U$

$X_1 \cap n(x_j) \neq \phi \Rightarrow x_j \in \overline{R}_n(X_1)$ and

$X_2 \cap n(x_j) \neq \phi \Rightarrow x_j \in \overline{R}_n(X_2)$

so that $x_j \in \overline{R}_n(X_1) \cap \overline{R}_n(X_2)$

But from hypothesis $(X_1 \cap X_2) \cap n(x_j) \neq \phi.$ This implies $x_j \notin \overline{R}_n(X_1 \cap X_2)$

Hence $\overline{R}_n(X_1 \cap X_2) \underset{\neq}{\subset} \overline{R}_n(X_1) \cap (X_2)$

Conversely from hypothesis there exists one neighborhood $n(x_k)$ for $x_k \in U$ such that

$n(x_k) \subset \overline{R}_n(X_1) \cap \overline{R}_n(X_2)$,but
$n(x_k) \not\subset \overline{R}_n(X_1 \cap X_2)$

that is, $n(x_k) \subseteq \overline{R}_n(X_1)$ *and* $n(x_k) \subseteq \overline{R}_n(X_2)$, but $n(x_k) \cap (X_1 \cap X_2) \neq \phi$

This implies $X_1 \cap n(x_k) = \phi$, $X_2 \cap n(x_k) = \phi$ and $(X_1 \cap X_2) \cap n(x_k) = \phi.$

This completes the proof:

The following theorem comes immediately.

Theorem 2.4 : Let U = {x_1, x_2, …………, x_n} be an finite universe and let $n : U \rightarrow 2^U$ be a 1- neighborhood operator having serial and inverse serial property. Then for any two subsets X_1, X_2 of U,

$$\overline{R}_n(X_1 \cap X_2) = \overline{R}_n(X_1) \cap \overline{R}_n(X_2)$$

if and only if these exists no $x(x_j)$ such that the properties (P) and (q) hold both.

Example 1: Let U = {x_1, x_2, x_3, x_4} be a universe, and a neighbourhood operator n on U be given by n (x_1) = {x_1, x_2}, n(x_2) = {x_3}, n(x_3) = {x_2}, n(x_4) = {x_2, x_4}

Let X_1 = {x_1, x_2}, X_2 = {x_2, x3} then

$X_1 \cap n(x_1) \neq \phi$, $X_2 \cap n(x_1) \neq \phi$,
but $(X_1 \cap X_2) \cap n(x_1) \neq \phi.$

Now
$\overline{R}_n(X_1) = \{x_1, x_3, x_4\}$, $\overline{R}_n(X_2) = \{x_1, x_2, x_3, x_4\}$

& as $X_1 \cap X_2 = \{x_2\}$, $\overline{R}_n(X_1 \cap X_2) = \{x_1, x_3, x_4\}$ = $\overline{R}_n(X_1) \cap \overline{R}_n(X_2)$.

Also, taking another neighbourhood n(x_2),

$X_1 \cap n(x_2) = \phi$, $X_2 \cap n(x_2) \neq \phi$ and $(X_1 \cap X_2) \cap n(x_2) = \phi$ and hence

$\overline{R}_n(X_1 \cap X_2) = \overline{R}_n(X_1) \cap \overline{R}_n(X_2)$.

Next, Let us consider two subsets of U, Y_1 = {x_1, x_3}, Y_2 = {x_2};

Then $Y_1 \cap n(x_1) \neq \phi$, $Y_2 \cap n(x_1) \neq \phi$ but $(Y_1 \cap Y_2) \cap n(x_1) = \phi$

Now
$\overline{R}_n(Y_1) = \{x_1, x_2\}$ *and* $\overline{R}_n(Y_2) = \{x_1, x_3, x_4\}$,

So that

$\phi = \overline{R}_n(Y_1 \cap Y_2) \underset{\neq}{\subset} \overline{R}_n(Y_1) \cap \overline{R}_n(Y_2) = \{x_1\}$ and hence Theorem 2.3.

Corollary 2.1 Let $R : U \rightarrow U$ be an equivalence relation and {Y_1, Y_2, ...Y_n} be a classification of U under R. Then for any two subsets X_1, X_2 *of U.*

$\overline{R}(X_1 \cap X_2) \underset{\neq}{\subset} \overline{R}(X_1) \cap \overline{R}(X_2)$ if and only if there exist at least one Y_j such that

$X_1 \cap Y_j \neq \phi$, $X_2 \cap Y_j \neq \phi, 1 \leq j \leq n$ *and* $(X_1 \cap X_2) \cap Y_j = \phi$.
for $1 \leq j \leq n$.

Corollary 2.2: Let $R : U \to U$ be an equivalence relation and $\{Y_1, Y_2, \ldots Y_n\}$ be a classification of U under R. Then for any two subnets X_1, X_2 of U.

$\overline{R}(X_1 \cap X_2) = \overline{R}(X_1) \cap \overline{R}(X_2)$ if and only if there exists no Y_j such that

$$X_1 \cap Y_j \neq \phi, \quad X_2 \cap Y_j \neq \phi \quad \text{and}$$

$(X_1 \cap X_2) \cap Y_j = \phi$ hold ; for each j = 1, 2, …..n.

Proof : It can be proved directly by taking R instead of R_n in Theorem 2.3 and 2.4.

We note here that there is no loss of information even if we divide the knowledge base into the smaller units.

In similar manner we can prove the following theorems.

Theorem 2.5: Let $U = \{x_1, x_2 \ldots\ldots, x_p\}$ be a finite universe and let $n : U \to 2^U$ be a serial and inverse serial neighborhood operator.

Then for any two subsets X_1, X_2 of U,

(r) $\underline{R}_n(X_1) \cup \underline{R}_n(X_2) \underset{\neq}{\subset} \underline{R}_n(X_1 \cup X_2)$ if and only if there exists at least one neighborhood $n(x_j)$ for $x_j \in U$ such that

(s) $X_1 \cap n(x_j) \underset{\neq}{\subset} n(x_j), \quad X_2 \cap n(x_j) \underset{\neq}{\subset} n(x_j)$

and

(t) $X_1 \cup X_2 \supseteq n(x_j), \quad 1 \le j \le p.$

Proof : (Sufficient part)

From Hypothesis (s) and (t),

$n(x_j) \not\subset \underline{R}_n(X_1)$ *and* $n(x_j) \not\subset \underline{R}_n(X_2)$ but $n(x_j) \subset \underline{R}_n(X_1 \cup X_2), \; 1 \le j \le p$

This implies $\underline{R}_n(X_1) \cup \underline{R}_n(X_2) \underset{\neq}{\subset} \underline{R}_n(X_1 \cup X_2)$

(Necessary part)

Suppose that $\underline{R}_n(X_1) \cup \underline{R}_n(X_2) \underset{\neq}{\subset} \underline{R}_n(X_1 \cup X_2)$

That is, there exists one $n(x_j)$ for $x_j \in U$ such that $n(x_k) \subset \underline{R}_n(X_1 \cup X_2)$ but

$$n(x_k) \not\subset \underline{R}_n(X_1) \cup \underline{R}_n(X_2)$$

that is, $n(x_k) \subseteq X_1 \cup X_2$ *and* $n(x_k) \not\subset \underline{R}_n(X_1)$, $n(x_k) \not\subset \underline{R}_n(X_2)$

This implies $X_1 \cup X_2 \supseteq n(x_k)$ and $X_1 \cap n(x_k) \underset{\neq}{\subset} n(x_k)$ and $X_2 \cap n(x_k) \underset{\neq}{\subset} n(x_k)$

for k = 1, 2,……., p.

Hence the theorem.

Theorem 2.6: Let $U = \{x_1, x_2, \ldots\ldots x_n\}$ be a finite universe and $n : U \to 2^U$ be a serial and inverse serial neighbourhood operator. Then for any two subset X_1, X_2 of U.

$$\underline{R}_n(X_1) \cup \underline{R}_n(X_2) = \underline{R}_n(X_1 \cup X_2)$$

if and only if these exists no $n(x_j)$ for $x_j \in U$, such that the properties (s) and (t) hold.

Corollary 2.3 : Let $R : U \to U$ be an equivalence relation and $\{Y_1, Y_2, \ldots\ldots Y_n\}$ be a classification of U under R. Then for any two subsets X_1, X_2 of U.

$$\underline{R}(X_1) \cup \underline{R}(X_2) = \underline{R}(X_1 \cup X_2)$$

if and only if there exists no Y_j such that

$$X_1 \cap Y_j \underset{\neq}{\subset} Y_j, X_2 \cap Y_j \underset{\neq}{\subset} Y_j$$

and $X_1 \cup X_2 \supseteq Y_j$, j = 1,2,3…….n

In this case there is no loss of information in the distributed knowledge base. We find a theorem for general case as.

Theorem 2.7: Let $U = \{x_1, x_2, \ldots\ldots x_p\}$ be a finite universe and $n : U \to 2^U$ be a serial and inverse serial neighbourhood operator. Then for a finite number of subsets $X_1, X_2, \ldots\ldots, X_k$ of U.

$$\overline{R}_n\left(\bigcap_{i=1}^{k} X_i\right) = \bigcap_{i=1}^{k} \overline{R}_n(X_i)$$

if and only if there exists no n(x_j), for $x_j \in U$, $1 \le j \le p$ such that $X_i \cap n(x_j) = \phi$ and

$\left(\bigcap_{i=1}^{k} X_i\right) \cap n(x_j) = \phi$ for each i=1,2,3….. k, and j = 1, 2, ….., p hold.

Suppose we are given an information system S = (U,Q,V,d), where U is a nonempty finite universe, Q = C ∪D is a set of attributes, C is a non-empty finite set called conditions of S and D is also a non-empty finite set called decision of S and C ∩ D=ϕ, $V = \bigcup_{q \in Q} V_q$ is a non-empty finite set called values of attributes, Vq is the set of values of attribute q, called domain of q and d is a function of $U \times Q$ onto V, called description function of *S* such that $d(x,q) \in V_q$ for all $x \in U$ and $q \in Q$.

Let $P \subset Q$, *P* is nonempty , the two elements *x, y* of *U* are indiscernible by *P* in *S* if and only if $d(x,a) = d(y,a)$ for each $a \in P$.

3. APPROXIMATION OF CLASSIFICATION

Classifications of universes play important roles in basic rough set theory. We define below a classification formally.

Definition 3.1 : Let $F = \{X_1, X_2, \ldots\ldots X_k\}$ be a classification of *U*, that is $X_i \cap X_j = \phi$ for $i \ne j$ and $\bigcup_{i=1}^{k} X_i = U$. Let R be an equivalence relation over *U*. Then $\overline{R}F and \underline{R}F$ denote respectively the R-upper and R-lower approximations of the classification *F* and are defined as

$$\overline{R}(F) = \{\overline{R}(X_1), \overline{R}(X_2), \ldots\ldots, \overline{R}(X_k)\} \quad \text{and}$$
$$\underline{R}(F) = \{\underline{R}(X_1), \underline{R}(X_2)\underline{R}(X_3)\ldots\ldots\ldots ., \underline{R}(X_k)\}$$

Properties of approximation of classifications established by Grzymala-Busse (1988 [4]) establish that the two concepts, approximation of sets and approximation of classifications are two different issues and that the equivalence classes of approximate classifications can not be arbitrary sets. These results of Busse are irreversible. It was observed by Pawlak [11] that, from the results of Busse "If we have positive example of each category in the approximate classification then we must have also negative examples of each category". In this article, we further analyze this aspect.

For the approximation operation $\overline{R}_n$ and $\underline{R}_n$, we have

$$\overline{R}_n(F) = \{\overline{R}_n(X_1), \overline{R}_n(X2), \ldots\ldots\ldots \ldots.., \overline{R}_n(X_k)\}$$

and

$$\underline{R}_n(F) = \{\underline{R}_n(X_1), \underline{R}_n(X_2), \ldots\ldots\ldots \ldots\ldots\ldots ., \underline{R}_n(X_k)\}$$

We use the following notation for the following theorems :

Let N_k = {1,2,3,….., k}. for any subset $I \subset N_k$, I^c is the complement of *I* in N_k.

Theorem 3.1 : Let $F = \{X_1, X_2, \ldots\ldots X_k\}$ be a classification of a finite universe *U*= {$x_1, x_2, \ldots\ldots x_p$} and let $n : U \to 2^U$ be a serial and inverse serial 1- neighborhood operator. Then

$$\underline{R}_n\left(\bigcup_{i \in I} X_i\right) \ne \phi \text{ if and only if } \underline{R}_n\left(\bigcup_{i \in I^c} X_i\right) \ne U$$

In particular, for any set $B \subset U$,

$\overline{R}_n(B) \ne \phi$ if and only if $\overline{R}_n(B^c) \ne U$; where B^c is the complement of *B* in *U*.

Also we note here that, for any subset B of U, $\underline{R}_n(B) = Y \ne \phi$ if and only if $\overline{R}_n(B^c) = Y^c = U \sim Y.$

Proof: (Necessary)Suppose that $\underline{R}_n\left(\bigcup_{i \in I} X_i\right) \ne \phi$, then there exists one $x_j \in U$ such that $n(x_j) \subset \bigcup_{i \in I} X_i$ so that $n(x_j) \cap \left(\bigcup_{i \in I^c} X_i\right) = \phi$, that is, $n(x_j) \not\subset \left(\bigcup_{j \in I^c} X_j\right)$.

Hence $\overline{R}_n\left(\bigcup_{j \in I^c} X_j\right) \ne U$.

(Sufficiency)

Let us suppose $\overline{R}_n\left(\bigcup_{j \in I^c} X_j\right) \ne U$

This implies, $\bigcup_{j \in I^c}\left(\overline{R}_n X_j\right) \ne U.$

then there exists one $x_j \in U$ such that

$n(x_j) \cap \left(\bigcup_{j \in I^c} X_j\right) = \phi$

That is $n(x_j) \subset \bigcup_{i \in I} X_i$.

Hence $\overline{R}_n \left(\bigcup_{i \in I} X_i \right) \neq \phi$. Hence the theorem.

Corollary 3.1 : [13] Let $F = \{X_1, X_2, \ldots\ldots X_k\}$ be a classification of U and R be an equivalence relation on *U* and $I \subset N_k$. Then

$$\underline{R}\left(\bigcup_{i \in I} X_i \right) \neq \phi \text{ if and only if } \bigcup_{j \in I^c} \overline{R}X_j \neq U.$$

Proof : This can be proved directly from Theorem 3.1. We prove this corollary for clarity.

Let us suppose that

$$\bigcup_{j \in I^c} \overline{R}X_j \neq U, \text{ so that } \overline{R}\left(\bigcup_{j \in I^c} X_j \right) \neq U$$

So there exists $[x]_R$ for some $x \in U$ such that $[x]_R \cap \left(\bigcup_{j \in I^c} X_j \right) = \phi$

That is, $[x]_R \subset \bigcup_{j \in I} X_i$, *and hence* $\underline{R}\left(\bigcup_{j \in I} X_i \right) \neq \phi$.

Conversely, suppose $\underline{R}\left(\bigcup_{j \in I} X_i \right) \neq \phi$.

Then there exists $x \in U$ such that $[x]_R \subseteq \bigcup_{j \in I} X_i$. *Thus* $[x]_R \cap X_j = \phi$ for $j \in I^c$. *So,* $x \notin \overline{R}X_j$ *for all* $j \in I^c$.

Hence $\bigcup_{j \in I^c} \overline{R}X_j \neq U$.

Corollary 3.2 : (Proposition 2.5 of Pawlak [11], Busse [4])

Let $F = \{X_1, X_2, \ldots\ldots X_k\}$ be a classification of *U* and *R* be an equivalence relation on *U*. If

$$\underline{R}X_i \neq \phi \text{ for } i \in N_k$$
$$\text{then for each } j \in N_k \ (j \neq i),\ \overline{R}X_j \neq U.$$

Corollary 3.3 : (Proposition 2.7 of Pawlak [11], Busse [4])

Let *F* and *R* are defined on *U* as above. If $\underline{R}X_i \neq \phi$ *holds then* $\overline{R}X_i \neq U$ for all $i \in N_k$

Now we will prove another theorem for neighborhood operator.

Theorem 3.2 : *Let* $F = \{X_1, X_2, \ldots\ldots, X_k\}$ be a classification of a finite universe $U = \{x_1, x_2, \ldots\ldots x_p\}$ and let $n : U \to 2^U$ be a serial and inverse serial neighborhood operator. Then for $I \subset N_k$,

$$\underline{R}_n \left(\bigcup_{i \in I} X_i \right) = \phi \text{ if and only if } \overline{R}_n \left(\bigcup_{j \in I^c} X_j \right) = U.$$

Proof: The proof follows from the following equivalences.

$$\underline{R}_n \left(\bigcup_{i \in I} X_i \right) = \phi$$

$$\Leftrightarrow \text{for all } x_j \in U, \text{ we have } n(x_j) \not\subset \bigcup_{i \in I} X_i$$

$$\Leftrightarrow \text{for all } x_j \in U, \quad n(x_j) \cap \left(\bigcup_{j \in I^c} X_j \right) \neq \phi$$

$$\Leftrightarrow \overline{R}_n \left(\bigcup_{j \in I^c} X_j \right) = U.$$

Corollary 3.4 : [13] Let $F = \{X_1, X_2, \ldots\ldots, X_k\}$ be a classification of an universe U and R be an equivalence redaction on *U*. for any $I \subset N_k$

$$\underline{R}\left(\bigcup_{i \in I} X_i \right) = \phi \text{ if and only if } \overline{R}\left(\bigcup_{j \in I^c} X_j \right) = U.$$

Corollary 3.5 : (Proposition 2.6 or Pawlak [11], Busse [4]).

Let $F = \{X_1, X_2, \ldots\ldots X_k\}$ be a classification of *U* and *R* be an equivalence relation on *U*. If there exists $i \in N_k$ such that $\overline{R}X_i = U$ then for each $j \in N_k$, $j \neq i$, $\underline{R}X_j = \phi$ (The apposite is not true).

Corollary 3.6 : (Proposition 2.8 of Pawlak [11], Busse [4])

Let $F = \{X_1, X_2, \ldots\ldots X_k\}$ be a classification of *U* and *R* be an equivalence relation on U. if $\overline{R}X_i = U$ for all $i \in N_k$ than $\underline{R}X_i = \phi$ for all $i \in N_k$. (The opposite is not true).

4. DEPENDENCY

Let *U* be a non-empty, finite Universe, $n, k : U \to 2^U$ be two 1- neighborhood operators. Then union, denotes $n \cup k$, be a neighborhood operator, $n \cup k : U \to 2^U$ be defined by $(n \cup k)(x_j) = n(x_j) \cup k(x_j)$ for each $x_j \in U$. Similarly a neighborhood operator, intersection, $n \cap k : U \to 2^U$ be defined by $(n \cap k)(x_j) = n(x_j) \cap k(x_j)$ for each $x_j \in U$.

Definition 4.1 : Let *U* be finite universe, $n, k : U \to 2^U$ be two 1-neighborhood operators and R_n , R_k be their corresponding approximation operators. The approximation operator R_n depends upon the approximation operator R_k, denoted by $R_k \Rightarrow R_n$, if and only if $k(x_j) \subseteq n(x_j)$ for every element $x_j \in U$.

We note here that $k(x_j) \subseteq n(x_j)$ for each $x_j \in U$ if and only if $\underline{R}_k(X) \supseteq \underline{R}_n(X)$ for any set $X \subset U$ and $\overline{R}_k(X) \subseteq \overline{R}_n(X)$. This is equivalent to

$BN_k(X) = \overline{R}_k(X) - \underline{R}_k(X) \subseteq \overline{R}_n(X) - \underline{R}_n(X) = BN_n(X)$ for $X \subset U$.

Note: *K = (U, **R**)* be a knowledge base when ***R*** be the family of all equivalence relations defined on *U*. Let ***P*** be a family of equivalence relations defined on *U* and ***Q*** be another family of equivalence relations on *U* and ***P*,*Q*** $\subset$ ***R***. According to Powlak ([11]), Knowledge ***Q*** depends upon knowledge ***P*,** denotes ***P*** $\Rightarrow$ ***Q*** if and only if *IND (**P**)* $\subset$ *IND (**Q**)*, which is equivalent to, for any subset *X* $\subset$ *U*, the borderline region of *X* under the equivalence relation *IND (**P**)* is contained in the borderline region of X under the relation *IND (**Q**)* that is,

$$BN_{IND(P)}(X) \subseteq BN_{IND(Q)}(X) \text{ for } X \subseteq U .$$

Taking this point of view, we get the Definition 4.1 the dependency on the neighborhood operator.

Definition 4.2 : Let *U* be a finite universe, $n, k : U \to 2^U$ be two 1- neighborhood operators. The approximation operators R_n and R_k are equivalent, denoted as $R_n \equiv R_k$ if and only if $R_k \Rightarrow R_n$ and $R_n \Rightarrow R_k$ also R_n and R_k are independent, denoted as $R_n \neq R_k$ if and only if neither $R_k \Rightarrow R_n$ and $R_n \Rightarrow R_k$ hold.

Proposition 4.1: Let *U* be a finite universe, $n, k, p : U \to 2^U$ be the 1-neighborhood operators with serial and inverse serial property, and R_n, R_k, R_p be their corresponding approximation operators Then.

(i) $R_k \Rightarrow R_n$ and $R_n \Rightarrow R_p$ implies $R_k \Rightarrow R_p$

(ii) $R_k \Rightarrow R_{k \cup n}$, $R_n \Rightarrow R_{k \cup n}$

(iii) $R_{k \cap n} \Rightarrow R_k$ and $R_{k \cap n} \Rightarrow R_n$ provided $k \cap n$ is an inverse serial operator.

(iv) $R_k \Rightarrow R_n$, $R_p \Rightarrow R_n$ implies $R_{k \cup p} \Rightarrow R_n$

(v) $R_k \Rightarrow R_n$, $R_k \Rightarrow R_p$ implies $R_k \Rightarrow R_{n \cap p}$ provided $n \cap p$ is an inverse serial operators.

(vi) $R_k \Rightarrow R_n$ if and only if $R_k \equiv R_{k \cup n}$.

Proof: For $R_k \Rightarrow R_n$ we get $k(x_j) \subseteq n(x_j)$ for all $x_j \in U$. and for $R_n \Rightarrow R_p$ we have $n(x_j) \subseteq p(x_j)$ for each $x_j \in U$.

Hence $k(x_j) \subseteq n(x_j) \subseteq p(x_j)$ for each $x_j \in U$. that is, $R_k \Rightarrow R_p$, (i) is proved ; similarly others can be proved.

Definition 4.3: Let $U = \{x_1, x_2, x_3, \ldots x_n\}$ be a finite universe and $k, p : U \to 2^U$ be two serial and inverse serial neighborhood operators. Now we say that the approximation operator R_k depends on the approximation operator R_p in a degree d, denotes $R_p \Rightarrow_d R_k$, if and only if

$$d = \gamma_p(k) = \frac{card\ M}{card\ U} \text{ where}$$

$M = \{x_j \in U \mid p(x_j) \subseteq k(x_j)\}$

If d = 1, we say that R_k depends totally on R_p and if d = 0, we say that R_k is independent to R_p and at that time $M = \phi$

If o < d < 1, we say that R_k depends partially on R_p with degree d.

Example 2 : Let $U = \{x_1, x_2, x_3, x_4, x_5\}$ Let $k : U \rightarrow 2^U$ be an serial and inverse serial neighborhood operator.

$$k(x_1) = \{x_2\}, \quad k(x_2) = \{x_2, x_4\}, \quad k(x_3) = (x_2),$$
$$k(x_4) = \{x_1, x_2\}, \quad k(x_5) = \{x_3, x_5\}$$

Let $n : U \rightarrow 2^U$ be another serial of inverse serial neighborhood operator such that

$$n(x_1) = \{x_2, x_3\}, \; n(x_2) = \{x_2, x_4\}, \; n(x_3) = \{x_2, x_4\}$$
$$, n(x_4) = \{x_1, x_2, x_5\}, \; n(x_5) = \{x_2, x_3, x_5\}$$

Let $X = \{x_1, x_3, x_5\} \subset U$

Then $\underline{R}_k(X) = \{x_5\}, \; \overline{R}_k(X) = \{x_4, x_5\}$ and $\underline{R}_n(X) = \phi, \; \overline{R}_n(X) = \{x_1, x_4, x_5\}$

Now, $BN_k(x) = \overline{R}_k(X) - \underline{R}_k(X) = \{x_4\}$ and $BN_n(x) = \overline{R}_n(X) - \underline{R}_n(X) = \{x_1, x_4, x_5\}$

Thus as $BN_k(x) \subset BN_n(x), R_k \Rightarrow R_n$, Also, here $k(x_j) \subseteq n(x_j) \;\; for \;\; each \;\; x_j \in U.$

Let $p : U \rightarrow 2^U$ be a serial and inverse serial neighborhood operator such that

$$p(x_1) = \{x_3\}, \; p(x_2) = \{x_2, x_4\}, \; p(x_3) = \{x_4\},$$
$$p(x_4) = (x_1), \; p(x_5) = \{x_3, x_5\}$$

Now $k \cup p(x_1) = \{x_2, x_3\}, \; k \cup p(x_2) = \{x_2, x_4\},$

$$k \cup p(x_3) = \{x_2, x_4\}, \; k \cup p(x_4) = \{x_1, x_2\}, \; k \cup p(x_5) = \{x_3, x_5\}$$

Thus

$$R_p \Rightarrow R_n \;\; and \;\; R_k \Rightarrow R_n \;\; implies \;\; R_{kup} \Rightarrow R_n,$$

as. $k \cup p(x_j) \subseteq n(x_j) \; for \;\; each \;\; x_j \in U$

Next. $M = \{x_j \in U \mid p(x_j) \subseteq k(x_j)\} = \{x_2, x_4, x_5\}$

Then

$d = \gamma_p(k) = \dfrac{card\ M}{card\ U} = \dfrac{3}{5}$: that is, degree of dependency of R_k on R_p be 0.6.

5. CONCLUSION

In computing world a notion of partitioned rough set (Pawlak rough set) is too restrictive, for that, we propose a generalized notion, namely, neighborhood systems which may be an effective notion in expressing some complex uncertainty. In this article the class of 1-neighborhood system, that is, each element has exactly one neighborhood are studied and we find the condition for which there is no loss of information in a distributed knowledge base by dividing the knowledge base into smaller fragments. Also we extended the result of Busse [1988] to obtain properties of approximations of classifications which are necessary and sufficient type. Dependency through the neighborhood operator be defined and a proposition is established.

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