The Total Irregularity of some Composite Graphs

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ABSTRACT

The *total irregularity* of a simple undirected graph $G = (V, E)$ is defined as $\mathrm{irr}_t(G) = \frac{1}{2}\sum_{u,v\in V(G)} |d_G(u) - d_G(v)|$, where $d_G(u)$ is the degree of the vertex u. In this paper we investigate the change of the total irregularity of graphs under various subdivision operations. Also, we present exact expressions and upper bounds on the total irregularity of different composite graphs such as the double graph, the extended double cover of a graph, the generalized thorn graph, several variants of subdivision corona graphs, and the hierarchical product graphs.

Keywords

irregularity of a graph, total irregularity of a graph, graph invariants, composite graphs

1. INTRODUCTION

In this paper, all graphs are simple and connected. Let $G = (V, E)$ be a graph with n vertices and m edges. We denote the degree of a vertex u by $d_G(u)$ and the maximum and minimum degree of the graph G by Δ and δ, respectively. The imbalance of an edge $e = (u, v) \in E(G)$ is defined as $imb(e) = |d_G(u) - d_G(v)|$. In [4], Albertson defined the irregularity of G as

$$\mathrm{irr}(G) = \sum_{e\in E(G)} \mathrm{imb}(e) = \sum_{(u,v)\in E(G)} |d_G(u) - d_G(v)| \quad (1)$$

and found upper bounds of irregularity for bipartite graphs, triangle-free graphs and a sharp upper bound of irregularity for trees. Hansen and Mélot [15] characterized the graphs with n vertices and m edges with maximal irregularity.

Recently in [1] a new measure of irregularity of a graph, so-called the *total irregularity*, that depends also on one single parameter (the pairwise difference of vertex degrees) was introduced. It was defined as

$$\mathrm{irr}_t(G) = \frac{1}{2}\sum_{u,v\in V(G)} |d_G(u) - d_G(v)|\,. \quad (2)$$

In this paper, we assume that for a graph G with the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ the order $d(v_1) \geq d(v_2) \geq \ldots \geq d(v_n)$ holds. Then,

$$\mathrm{irr}_t(G) = \sum_{i<j}(d(v_i) - d(v_j)) = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(d(v_i) - d(v_j)). \quad (3)$$

The motivation to introduce the total irregularity of a graph, as modification of the irregularity of graph, is twofold [1]. First, in contrast to $\mathrm{irr}(G)$, $\mathrm{irr}_t(G)$ can be computed directly from the sequence of the vertex degrees (degree sequence) of G. Second, the most irregular graphs with respect to irr are graphs that have only two degrees. On the contrary the most irregular graphs with respect to irr_t, as it is shown in [1], are graphs with maximal number of different vertex degrees, which is much closer to what one can expect from "very" irregular graphs. Also in [1], sharp upper bounds on the total irregularity of a graph were given. Recently, irregularity measures irr and irr_t were compared in [13], where it was shown that for a connected graph G with n vertices $\mathrm{irr}_t(G) \leq n^2\,\mathrm{irr}(G)/4$. Moreover, if G is a tree, then it was shown that $\mathrm{irr}_t(G) \leq (n-2)\,\mathrm{irr}(G)$. A lot of researches on irregularity of graphs have been carried out recently [2, 3, 7]. In this work we investigate the change of the total irregularity under different graph operations. The rest of the paper is structured as follows. In Sections 2 exact expressions of the total irregularity of graphs under several graph subdivision operations are presented. Exact expressions of the total irregularity for double graphs and extended double cover graphs are given in Section 3. The sharp upper bound on the total irregularity of thorn graphs is presented in Section 4, while in Sections 5 and 6 exact expressions and upper bounds on the total irregularity of several subdvision corona graphs and hierarchical product graphs are given.

2. TOTAL IRREGULARITY OF SOME SUBDIVISION GRAPHS

DEFINITION 1. *The line graph of G, denoted by $L(G)$ is the graph whose vertices are the edges of G and two vertices of $L(G)$ are adjacent if and only if the corresponding edges are adjacent in G.*

Next, we presented definitions of several subdivision graphs whose the total irregularities will be studied in this section.

DEFINITION 2. *The subdivision graph of a graph G denoted by $S(G)$ is obtained from G by replacing each edge of G by a path of length two.*

DEFINITION 3. *The triangle parallel graph of a graph G is denoted by $R(G)$ and is obtained from G by replacing each edge of G by a triangle.*

DEFINITION 4. *The line superposition graph $Q(G)$ of a graph G is obtained from G by inserting a new vertex into each edge of*

G, and joining with edges each pair of new vertices on adjacent edges of G.

DEFINITION 5. *The total graph $T(G)$ of a graph G has a vertex for each edge and vertex of G, and two vertces in $T(G)$ are adjacent if and only if their corresponding elements are either adjacent or incident in G.*

DEFINITION 6. *A sequence $D = [d_1, d_2, \ldots, d_n]$ is graphical if there is a graph whose vertex degrees are d_i, $i = 1, \ldots, n$. If in addition $d_1 \geq d_2 \geq \cdots \geq d_n$, then D is a degree sequence. Furthermore, the notation $D(G) = [x_1^{n_1}, x_2^{n_2}, \cdots, x_t^{n_t}]$ means that the degree sequence is comprised of n_i vertices of degree x_i, where $i = 1, 2, \cdots, t$.*

First, we determine the total irregularity of the subdivision graph $S(G)$ and the line graph $L(G)$ of G in terms of the number of vertices, edges, and the total irregularity of G.

THEOREM 7. *Let G be simple undirected connected graphs with $|V(G)| = n$, $|E(G)| = m$ and r pendent vertices. Then,*

$$\mathrm{irr}_t(S(G)) = \mathrm{irr}_t(G) + 2m(m + r - n).$$

PROOF. Since $S(G)$ is obtained from G by replacing each edge by a path of length two, it follows that $|V(S(G))| = n + m$ and $|E(S(G))| = 2m$. Also, $d_{S(G)}(v) = d_G(v)$, if $v \in V(G)$, otherwise $d_{S(G)}(v) = 2$. Consider the degree sequence of $V(G) = \{v_1, v_2, \ldots, v_n\}$ with $d(v_1) \geq d(v_2) \geq \ldots \geq d(v_{n-r}) > \underbrace{1 = \ldots = 1}_{r-\text{times}}$. Then, $V(S(G)) = \{v_1, v_2, \ldots, v_n, \underbrace{2, \ldots, 2}_{m-\text{times}}\}$ with $d(v_1) \geq d(v_2) \geq \ldots \geq d(v_{n-r}) \geq \underbrace{2 = \ldots = 2}_{m-\text{times}} > \underbrace{1 = \ldots = 1}_{r-\text{times}}$, and the total irregularity of G is

$$\begin{aligned}
\mathrm{irr}_t(G) &= \frac{1}{2} \sum_{u,v \in V(G)} |d_G(u) - d_G(v)| \\
&= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (d_G(v_i) - d_G(v_j)) \\
&= \sum_{i=1}^{n-r} \sum_{j=i+1}^{n-r} (d_G(v_i) - d_G(v_j)) + \sum_{i=1}^{n-r} \sum_{j=n-r+1}^{n} (d_G(v_i) - 1) \\
&= \sum_{i=1}^{n-r} \sum_{j=i+1}^{n-r} (d_G(v_i) - d_G(v_j)) + \sum_{i=1}^{n-r} (d_G(v_i) - 1)(n - n + r) \\
&= \sum_{i=1}^{n-r} \sum_{j=i+1}^{n-r} (d_G(v_i) - d_G(v_j)) + r \sum_{i=1}^{n-r} d_G(v_i) - r(n - r) \\
&= \sum_{i=1}^{n-r} \sum_{j=i+1}^{n-r} (d_G(v_i) - d_G(v_j)) + r\,(2m - r - n + r) \\
&= \sum_{i=1}^{n-r} \sum_{j=i+1}^{n-r} (d_G(v_i) - d_G(v_j)) + r\,(2m - n)\,, \qquad (4)
\end{aligned}$$

and the total irregularity of $S(G)$ is

$$\begin{aligned}
\mathrm{irr}_t(S(G)) &= \frac{1}{2} \sum_{u,v \in V(S(G))} \left| d_{S(G)}(u) - d_{S(G)}(v) \right| \\
&= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (d_{S(G)}(v_i) - d_{S(G)}(v_j)) \\
&= \sum_{i=1}^{n-r} \sum_{j=i+1}^{n-r} (d_{S(G)}(v_i) - d_{S(G)}(v_j)) + \\
&\quad \sum_{i=1}^{n-r} \sum_{j=n-r+1}^{n-r+m} (d_{S(G)}(v_i) - 2) + \\
&\quad \sum_{i=1}^{n-r} \sum_{j=n-r+m+1}^{n+m} (d_{S(G)}(v_i) - 1) + \sum_{i=n-r+1}^{n-r+m} \sum_{j=n-r+m+1}^{n+m} 1 \\
&= \sum_{i=1}^{n-r} \sum_{j=i+1}^{n-r} (d_G(v_i) - d_G(v_j)) + \\
&\quad \sum_{i=1}^{n-r} [d_G(v_i)(m + r) - 2m - r)] + mr \\
&= \sum_{i=1}^{n-r} \sum_{j=i+1}^{n-r} (d_G(v_i) - d_G(v_j)) + (2m - r)(m + r) - \\
&\quad (2m + r)(n - r) + mr \\
&= \sum_{i=1}^{n-r} \sum_{j=i+1}^{n-r} (d_G(v_i) - d_G(v_j)) + r(2m - n) + 2m(m + r - n)
\end{aligned} \qquad (5)$$

From (4) and (5), we obtain
$\mathrm{irr}_t(S(G)) = \mathrm{irr}_t(G) + 2m(m + r - n)$. □

Next, we determine the total irregularity of the line graph. Let $G = (E, V)$ be a graph with $|V(G)| = n$, $|E(G)| = m$. Then $L(G)$ satisfies $|V(L(G))| = m$, $|E(L(G))| = \sum_{v \in V(G)} d_G(v)^2 - m$. It follows that $d_{L(G)} e_i = d_G(u_i) + d_G(v_i) - 2$ for all $e_i = u_i v_i \in L(G)$.

PROPOSITION 1. *Let $L(G)$ be the line graph of G . Then, $\mathrm{irr}_t(L(G)) \leq 2\,\mathrm{irr}_t(G)$. The bound is sharp.*

PROOF. Since $d_{L(G)} e_i = d_G(u_i) + d_G(v_i) - 2$. Then

$$\begin{aligned}
\mathrm{irr}_t(L(G)) &= \frac{1}{2} \sum_{e_i, e_j \in V(L(G))} \left| d_{L(G)}(e_i) - d_{L(G)}(e_j) \right| \\
&= \frac{1}{2} \sum_{u_i u_j, v_i v_j \in E(G)} |(d_G(u_i) + d_G(v_i) - 2) - \\
&\quad (d_G(u_j) + d_G(v_j) - 2)| \\
&= \frac{1}{2} \sum_{u_i v_i, u_j v_j \in E(G)} |(d_G(u_i) - d_G(v_i)) + (d_G(u_j) - d_G(v_j))| \\
&\leq \frac{1}{2} \sum_{u_i, v_i \in V(L(G))} |d_G(u_i) - d_G(v_i)| + \\
&\quad \frac{1}{2} \sum_{u_j, v_j \in V(L(G))} |d_G(u_j) - d_G(v_j)| \\
&= 2\,\mathrm{irr}_t(G)
\end{aligned}$$

To show the bound is sharp, consider the complete graph K_n. The line graph $L(K_n)$ is the $(2n-4)-$regular graph of order $\binom{n}{2}$. Then $\mathrm{D(K_n)} = [(\mathrm{n} - 1)^{\mathrm{n}}]$ and $\mathrm{D(L(K_n))} = [(2\mathrm{n} - 4)^{\frac{\mathrm{n(n-1)}}{2}}]$. Hence, $\mathrm{irr}_t(K_n) = \mathrm{irr}_t(L(K_n)) = 0$. □

Next, we determine the total irregularity of the triangle parallel graph.

THEOREM 8. *The total irregularity of the triangle parallel graph $R(G)$ of a graph G is*

$$\mathrm{irr}_t(R(G)) = 2\,\mathrm{irr}_t(G) + 2m(2m-n).$$

PROOF. From the definition of the triangle parallel graph $R(G)$ of G, one can conclude that $|V(R(G))| = n+m$ and $|E(R(G))| = 3m$. Also, it follows that $d_{R(G)}(v) = 2d_G(v)$, if $v \in V(G)$, and $d_{R(G)}(v) = 2$, if $v \in V(R(G))\backslash V(G)$. Hence, from (3) the total irregularity of $R(G)$ is

$$\begin{aligned}
\mathrm{irr}_t(R(G)) &= \sum_{i=1}^{n+m-1}\sum_{j=i+1}^{n+m}(d_{R(G)}(v_i) - d_{R(G)}(v_j)) \\
&= \sum_{i=1}^{n}\left(\sum_{j=i+1}^{n}(d_{R(G)}(v_i) - d_{S(G)}(v_j)) + \sum_{j=n+1}^{n+m}(d_{S(G)}(v_i) - 2)\right) \\
&= \sum_{i=1}^{n}\sum_{j=i+1}^{n}(2\,d_G(v_i) - 2\,d_G(v_j)) + \sum_{i=1}^{n}\sum_{j=n+1}^{n+m}(2d_G(v_i) - 2) \\
&= 2\sum_{i=1}^{n}\sum_{j=i+1}^{n}(d_G(v_i) - d_G(v_j)) + 2m\sum_{i=1}^{n}(d_G(v_i) - 1) \\
&= 2\,\mathrm{irr}_t(G) + 2m(2m-n).
\end{aligned}$$

□

Now, we determine the total irregularity of the line superposition graph $Q(G)$ in terms of the total irregularities of the graph G and its line graph $L(G)$.

THEOREM 9. *The total irregularity of the line superposition graph $Q(G)$ of G is*

$$\begin{aligned}
\mathrm{irr}_t(Q(G)) \;=\; & \mathrm{irr}_t(G) + \mathrm{irr}_t(L(G)) + \\
& \frac{1}{2}\sum_{\substack{t\in V(L(G))\\ z\in V(G)}}\left|d_{L(G)}(t) - d_G(z) + 2\right|
\end{aligned}$$

PROOF. From the construction of $Q(G)$, we obtain that $|V(Q(G))| = n+m$ and $|E(Q(G))| = 2m + |E(L(G))|$. Also, $d_{Q(G)}(v) = d_G(v)$, if $v \in V(G)$, and $d_{R(G)}(v) = d_{L(G)}(v) + 2$, if $v \in V(R(G))\backslash V(G)$. Hence, from (2), the total irregularity of $Q(G)$ is

$$\begin{aligned}
\mathrm{irr}_t(Q(G)) &= \frac{1}{2}\sum_{u,v\in V(Q(G))}\left|d_{Q(G)}(u) - d_{Q(G)}(v)\right| \\
&= \frac{1}{2}\sum_{u,v\in V(Q(G))\backslash V(G)}\left|d_{Q(G)}(u) - d_{Q(G)}(v)\right| + \\
&\quad \frac{1}{2}\sum_{p,q\in V(G)}\left|d_{Q(G)}(p) - d_{Q(G)}(q)\right| + \\
&\quad \frac{1}{2}\sum_{\substack{t\in V(Q(G))\backslash V(G)\\ z\in V(G)}}\left|d_{Q(G)}(t) - d_{Q(G)}(z)\right| \\
&= \frac{1}{2}\sum_{u,v\in V(L(G))}\left|d_{L(G)}(u) + 2) - (d_{L(G)}(v) + 2)\right| + \\
&\quad \frac{1}{2}\sum_{p,q\in V(G)}|d_G(p) - d_G(q)| + \\
&\quad \frac{1}{2}\sum_{\substack{t\in V(L(G))\\ z\in V(G)}}\left|d_{L(G)}(t) + 2 - d_G(z)\right| \\
&= \mathrm{irr}_t(L(G)) + \mathrm{irr}_t(G) + \\
&\quad \frac{1}{2}\sum_{\substack{t\in V(L(G))\\ z\in V(G)}}\left|d_{L(G)}(t) - d_G(z) + 2\right|.
\end{aligned}$$

□

Finally, we present the total irregularity of the total graph $T(G)$ in terms of the total irregularity of graph G and its line graph $L(G)$.

THEOREM 10. *The total irregularity of the total graph $T(G)$ of G is*

$$\begin{aligned}
\mathrm{irr}_t(T(G)) \;=\; & 2\,\mathrm{irr}_t(G) + \mathrm{irr}_t(L(G)) + \\
& \frac{1}{2}\sum_{\substack{u\in V(G)\\ v\in V(T(G))\backslash V(G)}}\left|2\,d_G(u) - d_{L(G)}(v) - 2\right|
\end{aligned}$$

PROOF. From the definition of total graph, it follows that $V(T(G)) = V(G) \cup E(G)$. Also from the fact that two vertices of $T(G)$ are adjacent if and only if the corresponding elements of G are either adjacent or incident, one can obtain that $|V(T(G))| = n+m$ and $|E(T(G))| = 3m + |E(L(G))|$. Also, it follows that $d_{T(G)}(v) = 2d_G(v)$, if $v \in V(G)$ and $d_{T(G)}(v) = d_{L(G)}(v) + 2$, if $v \in V(T(G))\backslash V(G)$. Therefore, the total irregularity of the total graph $T(G)$ is

$$\begin{aligned}
\mathrm{irr}_t(T(G)) &= \frac{1}{2}\sum_{u,v\in V(T(G))}\left|d_{T(G)}(u) - d_{T(G)}(v)\right| \\
&= \frac{1}{2}\sum_{u,v\in V(G)}\left|d_{T(G)}(u) - d_{T(G)}(v)\right| + \\
&\quad \frac{1}{2}\sum_{u,v\in V(T(G))\backslash V(G)}\left|d_{T(G)}(u) - d_{T(G)}(v)\right| + \\
&\quad \frac{1}{2}\sum_{\substack{u\in V(G)\\ v\in V(T(G))\backslash V(G)}}\left|d_{T(G)}(u) - d_{T(G)}(v)\right| \\
&= \frac{1}{2}\sum_{u,v\in V(G)}|2\,d_G(u) - 2\,d_G(v)| + \\
&\quad \frac{1}{2}\sum_{u,v\in V(L(G))}\left|d_{L(G)}(u) + 2 - d_{L(G)}(v) - 2\right| + \\
&\quad \frac{1}{2}\sum_{\substack{u\in V(G)\\ v\in V(T(G))\backslash V(G)}}\left|2\,d_G(u) - d_{L(G)}(v) - 2\right| \\
&= 2\,\mathrm{irr}_t(G) + \mathrm{irr}_t(L(G)) + \\
&\quad \frac{1}{2}\sum_{\substack{u\in V(G)\\ v\in V(T(G))\backslash V(G)}}\left|2\,d_G(u) - d_{L(G)}(v) - 2\right|.
\end{aligned}$$

□

EXAMPLE 1. *Let $K_{a,b}$ be the complete bipartite graph where $a, b \geq 2$ and P_n be a path on n vertices ($n \geq 2$). The following results are obtained directly from Theorems 7, 8, 9 and 10.*

(1) $\mathrm{irr}_t(S(K_{a,b})) = ab(2ab - a - b)$

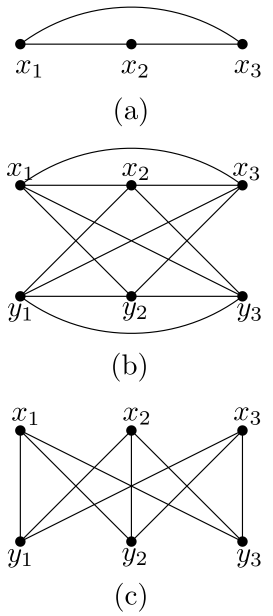

Fig. 1. (a) Cycle graph C_3, (b) Double graph C_3^*, (c) Extended double cover C_3^{**}.

(2) $\mathrm{irr}_t(R(P_n)) = 2(n+1)(n-2)$
(3) $\mathrm{irr}_t(Q(P_n)) = 2(n+3)(n-2)$
(4) $\mathrm{irr}_t(T(P_n)) = 2(6n-13)$

3. TOTAL IRREGULARITY OF DOUBLE GRAPH AND EXTENDED DOUBLE COVER

In this section, we give exact expressions of the total irregularity of double graph and extended double cover [5, 12, 17].

DEFINITION 11. *The double graph G^* of a given graph G is constructed by making two copies of G (including the initial edge set of each) and adding edges u_1v_2 and v_1u_2 for every edge uv of G.*

DEFINITION 12. *The extended double cover of G, denoted by G^{**} is the bipartite graph with bipartition (X,Y) where $X = \{x_1, x_2, ..., x_n\}$ and $Y = \{y_1, y_2, ..., y_n\}$ in which x_i and y_j are adjacent if and only if either v_i and v_j are adjacent in G or $i = j$.*

For example, the extended double cover graph of the complete graph is the complete bipartite graph. This construction of the extended double cover graph was introduced by Alon [5]. The double graph C_3^* and the extended double cover graph C_3^{**} of the cycle C_3 are illustrated respectively in Figure 1

THEOREM 13. *The total irregularity of the double graph G^* and the total irregularity of the extended double cover G^{**} of the graph G are*

$$\mathrm{irr}_t(G^*) = 8\,\mathrm{irr}_t(G), \quad \mathrm{irr}_t(G^{**}) = 4\,\mathrm{irr}_t(G).$$

PROOF. From the definition of double graph, we one can obtain that $d_{G^*}(x_i) = d_{G^*}(y_i) = 2d_G(v_i)$, where $v_i \in V(G)$ and $x_i, y_i \in V(G^*)$ are corresponding clone vertices of v_i. Thus the total irregularity of double graph G^* is

$$\begin{aligned}
\mathrm{irr}_t(G^*) &= \frac{1}{2}\sum_{u,v\in V(G^*)} |d_{G^*}(u) - d_{G^*}(v)| \\
&= \frac{1}{2}\sum_{x_i,x_j\in X} |2\,d_G(x_i) - 2\,d_G(x_j)| + \\
&\quad \frac{1}{2}\sum_{y_i,y_j\in Y} |2\,d_G(y_i) - 2\,d_G(y_j)| + \\
&\quad \sum_{x_i,\in X, y_j\in Y} |2\,d_G(x_i) - 2\,d_G(y_j)| \\
&= 8\,\mathrm{irr}_t(G)
\end{aligned}$$

By definition of the extended double cover G^{**}, one can conclude that G^{**} consists of $2n$ vertices, $(n+2m)$ edges and $d_{G^{**}}(x_i) = d_{G^{**}}(y_i) = d_G(v_i) + 1$ for $i = 1, 2, ..., n$. Here, $v_i \in V(G)$ and $x_i, y_i \in V(G^{**})$ are corresponding clone vertices of v_i. Thus the total irregularity of extended double cover G^{**} is

$$\begin{aligned}
\mathrm{irr}_t(G^{**}) &= \frac{1}{2}\sum_{u,v\in V(G^{**})} |d_{G^{**}}(u) - d_{G^{**}}(v)| \\
&= \frac{1}{2}\sum_{x_i,x_j\in X} |d_G(x_i) + 1 - d_G(x_j) - 1| + \\
&\quad \frac{1}{2}\sum_{y_i,y_j\in Y} |d_G(y_i) + 1 - d_G(y_j) - 1| + \\
&\quad \sum_{x_i,\in X, y_j\in Y} |d_G(x_i) + 1 - d_G(y_j) - 1| \\
&= 4\,\mathrm{irr}_t(G).
\end{aligned}$$

□

4. TOTAL IRREGULARITY OF THORN GRAPH

An edge $e = (u,v)$ of a graph G is called a *thorn* if either $d(u) = 1$ or $d(v) = 1$. The concept of thorn graph was first introduced by Gutman [14] by joining a number of thorns to each vertex of any given graph G. A variety of topological indices of thorn graphs have been recently studied [8, 10, 9, 11, 16, 18, 21, 22, 23]. Let $V(G) = \{v_1, v_2, ..., v_n\}$ and $V(G^T) = V(G) \cup V_1 \cup V_2 \cup \ldots \cup V_n$ be the vertex sets of G and its thorn graph G^T, respectively. The vertex set $V_i, i = 1, \ldots, n$, is the set of vertices of degree 1 attached to the vertex v_i in G^T. It holds that $V_i \cap V_j = \emptyset, i \neq j$. Let denote the vertices of the set V_i by v_{ij}, for $i = 1, 2, \ldots, n$ and $j = 1, 2, ..., p_i$. Thus, $|V(G^T)| = n + z$ where, $z = \sum_{i=1}^n p_i$ and the degree of the vertices v_i in G^T are $d_{G^T}(v_i) = d_G(v_i) + p_i$, for $i = 1, 2, \ldots, n$. In the following, we first present the total irregularity of the general thorn graph G^T and then consider some particular cases.

THEOREM 14. *The total irregularity of the thorn graph G^T of the graph G with $|V(G)| = n$, $|E(G)| = m$ and $\sum_{i=1}^n p_i = z$ is*

$$\mathrm{irr}_t(G^T) \leq \mathrm{irr}_t(G) + z(z + 2m - n) + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} |p_i - p_j|$$

PROOF. The total irregularity of the thorn graph G^T is

$$\begin{aligned}
\mathrm{irr}_t(G^T) &= \sum_{i=1}^{n+z-1}\sum_{j=i+1}^{n+z} |d_{G^T}(v_i) - d_{G^T}(v_j)| \\
&= \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} |d_G(v_i) + p_i - d_G(v_j) - p_j| +
\end{aligned}$$

$$\sum_{i=1}^{n}\sum_{j=n+1}^{n+z}|d_G(v_i)+p_i-1|$$
$$\leq \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}|d_G(v_i)-d_G(v_j)|+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}|p_i-p_j|+\sum_{i=1}^{n}(d_G(v_i)+p_i-1)z$$
$$= \operatorname{irr}_t(G)+z(z+2m-n)+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}|p_i-p_j|. \quad (6)$$

Moreover, the bound is sharp and can be obtained for infinitely many graphs. □

The following corollaries are direct consequences of the previous theorem.

COROLLARY 4.1. *Let G^T be a thorn graph with parameters $p_i = x$ for all i, then*

$$\operatorname{irr}_t(G^T) = \operatorname{irr}_t(G) + n\,x\,(2\,m + n\,(x-1)).$$

COROLLARY 4.2. *If the parameter $p_i(\geq 1)$ is equal to the degree of the corresponding vertex v_i, for all i, then*

$$\operatorname{irr}_t(G^T) = 2\operatorname{irr}_t(G) + 2\,m(4m-n).$$

COROLLARY 4.3. *If λ is an integer such that $\lambda > \Delta$ and if G^T is the thorn graph with parameters $p_i = \lambda - d_G(v_i)$, then*

$$irr_t(G^T) = n(\lambda-1)(n\lambda - 2m)$$

.

COROLLARY 4.4. *If the number of thorns, i.e., pendant edges attached to any vertex of the parent graph is a linear function of the degree of the corresponding vertex v_i, i.e., $p_i = a\,d_G(v_i) + b$, where a and b are positive numbers, the total irregularity of the thorn graph is*

$$\operatorname{irr}_t(G^T) = (a+1)\operatorname{irr}_t(G)+(2am+bn)\,(\,2\,(a+1)m+(b-1)n)$$

.

COROLLARY 4.5. *Let $C_{n,x}$ be the thorny ring (having n ring as parent and $(x-2)$ thorns at each vertex) then $\operatorname{irr}_t(C_{n,x}) = n^2(x-1)(x-2)$.*

COROLLARY 4.6. *Let $P_{n,x}$ be the thorn path (caterpillar) obtained from P_n by attaching x pendent vertices at each vertex of P_n, then we have $\operatorname{irr}_t(P_{n,x}) = n^2x^2 + (nx+2)(n-2)$.*

COROLLARY 4.7. *Let $S_{n,x}$ be the thorn star obtained from $S_n(\cong K_{1,n})$ by joining x thorns at each vertex of the parent graph S_n. Then, $\operatorname{irr}_t(S_{n,x}) = (n+1)^2x^2+(n+1)(n-1)x+n(n-1)$.*

5. TOTAL IRREGULARITY OF THE SUBDIVISION CORONA GRAPHS

Next we consider the total irregularity of the *subdivision-vertex corona* and the *subdivision-edge corona* graphs , introduced by Lu and Miao [20] and the total irregularity of the *subdivision-vertex neighborhood corona* and the *subdivision-edge neighborhood corona* graphs, introduced by Liu and Lub [19]. Let $\mathcal{S}(G)$ be the subdivision graph of G and let $I(G)$ be the set of inserted vertices to $\mathcal{S}(G)$.

DEFINITION 15. *The subdivision-vertex corona of two vertex-disjoint graphs G_1 and G_2, denoted by $G_1 \odot G_2$, is the graph obtained from $\mathcal{S}(G_1)$ and $|V(G_1)|$ copies of G_2, all vertex-disjoint, by joining the ith vertex of $V(G_1)$ to every vertex in the ith copy of G_2.*

DEFINITION 16. *The subdivision-edge corona of two vertex-disjoint graphs G_1 and G_2, denoted by $G_1 \ominus G_2$, is the graph obtained from $\mathcal{S}(G_1)$ and $|I(G_1)|$ copies of G_2, all vertex-disjoint, by joining the ith vertex of $I(G_1)$ to every vertex in the ith copy of G_2.*

DEFINITION 17. *The subdivision-vertex neighborhood corona of G_1 and G_2, denoted by $G_1 \boxdot G_2$, is the graph obtained from the subdivision graph $\mathcal{S}(G_1)$ and $|V(G_1)|$ copies of G_2, all vertex disjoint, and joining the neighbours of the ith vertex of $V(G_1)$ to every vertex in the ith copy of G_2.*

DEFINITION 18. *The subdivision-edge neighborhood corona of G_1 and G_2, denoted by $G_1 \boxminus G_2$, is the graph obtained from $\mathcal{S}(G_1)$ and $|I(G_1)|$ copies of G_2, all vertex disjoint, and joining the neighbours of the ith vertex of $I(G_1)$ to every vertex in the ith copy of G_2.*

Note that if G_1 and G_2 are two graphs on disjoint sets of n_1 and n_2 vertices, m_1 and m_2 edges, respectively, then $G_1 \odot G_2$ has $n_1+n_1n_2+m_1$ vertices, $2m_1+n_1m_2+n_1n_2$ edges, $G_1 \ominus G_2$, has $n_1+m_1+m_1n_2$ vertices, $2m_1+m_1m_2+m_1n_2$ edges, $G_1 \boxdot G_2$ has $n_1+m_1+n_1n_2$ vertices, $2m_1+n_1m_2+2m_1n_2$ edges, and $G_1 \boxminus G_2$ has $n_1+m_1+m_1n_2$ vertices, $2m_1+m_1n_2+2m_1n_2$ edges.

EXAMPLE 2. *An illustration of the above presented product corona graphs of the path and the cycle on n vertices is given in Figure 2.*

THEOREM 19. *The total irregularity of the subdivision corona graphs are*

(I) $\operatorname{irr}_t(G_1 \odot G_2) = \operatorname{irr}_t(G_1) + n_1\operatorname{irr}_t(G_2) + n_1^2(n_2^2 - n_2 - 2m_2) + 2n_1m_1(n_2+m_2-1) + 2m_1^2$,

(II) $\operatorname{irr}_t(G_1 \ominus G_2) = \operatorname{irr}_t(G_1) + m_1^2\operatorname{irr}_t(G_2) + m_1^2(n_2^2+n_2-2m_2) + m_1\sum_{i=1}^{n_1}|d_{G_1}(v_i)-n_2-2| + \sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n_1+m_1n_2}|d_{G_1}(v_i)-d_{G_2}(v_j)-1|$,

(III) $\operatorname{irr}_t(G_1 \boxdot G_2) \leq (n_2^2+n_2+1)\operatorname{irr}_t(G_1) + n_1^2\operatorname{irr}_t(G_2) + 2n_1^2m_2 + m_1\sum_{i=1}^{n_1}|d_{G_1}(v_i)-2n_2-2| + \sum_{i=1}^{n_1}\sum_{l=1}^{n_2}|d_{G_1}(v_i)+d_{G_2}(v_l)-2n_2-2|$,

(IV) $\operatorname{irr}_t(G_1 \boxminus G_2) = (n_2+1)\operatorname{irr}_t(G_1) + m_1^2\operatorname{irr}_t(G_2) + 2m_1^2(n_2^2+2n_2+m_2+1) - 2n_1m_1(n_2+m_2+1)$.

PROOF. (I) The vertex set of the subdivision-vertex corona $G_1 \odot G_2$ can be partitioned into the subsets

(1) $V_1 = \{v_i \in V(G_1 \odot G_2) : d_{G_1\odot G_2}(v_i) = d_{G_1}(v_i)+n_2$, $i = 1,2,\ldots,n_1\}$,
(2) $V_2 = \{v_i \in V(G_1 \odot G_2) : d_{G_1\odot G_2}(v_i) = d_{G_2}(v_i)+1$, $i = 1,2,\ldots,n_1n_2\}$, and
(3) $V_3 = \{v_i \in V(G_1 \odot G_2) : d_{G_1\odot G_2}(v_i) = 2$, $i = 1,2,\ldots,m_1\}$.

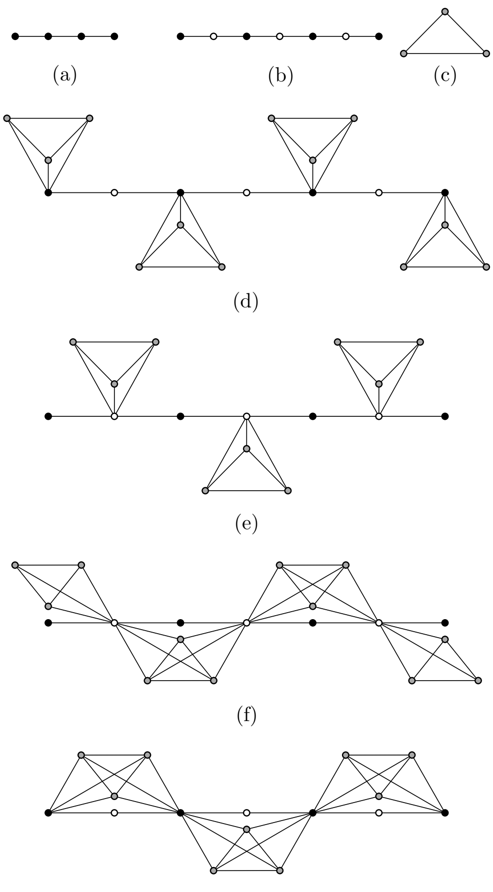

Fig. 2. (a) Path P_4, (b) Subdivision $\mathcal{S}(P_4)$, (c) Cycle C_3, (d) Subdivision-vertex corona $P_4 \odot C_3$, (e) Subdivision-edge corona $P_4 \ominus C_3$, (f) Subdivision-vertex neighbourhood corona $P_4 \boxdot C_3$, and (g) Subdivision-edge neighbourhood corona $P_4 \boxminus C_3$ graphs.

Thus, the total irregularity of $G_1 \odot G_2$ is

$$\mathrm{irr}_t(G_1 \odot G_2) = \sum_{\substack{i=1,\\ j=i+1}}^{\substack{n_1+n_1n_2+m_1-1,\\ n_1+n_1n_2+m_1}} |d_{G_1\odot G_2}(v_i) - d_{G_1\odot G_2}(v_j)|$$

$$= \sum_{i=1}^{n_1}\sum_{j=i+1}^{n_1} |d_{G_1}(v_i) + n_2 - d_{G_1}(v_j) - n_2| + \sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n_1+n_1n_2} |d_{G_1}(v_i) + n_2 - d_{G_2}(v_j) - 1| + \sum_{i=1}^{n_1}\sum_{j=n_1+n_1n_2+1}^{n_1+n_1n_2+m1} |d_{G_1}(v_i) + n_2 - 2| + \sum_{i=n_1+1}^{n_1+n_1n_2}\sum_{j=i+1}^{n_1+n_1n_2} |d_{G_2}(v_i) + 1 - d_{G_2}(v_j) - 1| + \sum_{i=n_1+1}^{n_1+n_1n_2}\sum_{j=n_1+n_1n_2+1}^{n_1+n_1n_2+m1} |d_{G_2}(v_i) + 1 - 2|$$

$$= \mathrm{irr}_t(G_1) + \sum_{i=1}^{n_1} (n_1n_2(d_{G_1}(v_i) + n_2 - 1) - 2n_1m_2 + m_1(d_{G_1}(v_i) + n_2 - 2)) + n_1\mathrm{irr}_t(G_2) + m_1 \sum_{i=n_1+1}^{n_1+n_1n_2} (d_{G_2}(v_i) - 1)$$

$$= \mathrm{irr}_t(G_1) + n_1^2n_2(n_2-1) - 2n_1^2m_2 + 2n_1n_2m_1 + 2m_1^2 + n_1m_1(n_2-2) + n_1\,\mathrm{irr}_t(G_2) + m_1n_1(2m_2 - n_2)$$

$$= \mathrm{irr}_t(G_1) + n_1\,\mathrm{irr}_t(G_2) + n_1^2(n_2^2 - n_2 - 2m_2) + 2n_1m_1(n_2 + m_2 - 1) + 2m_1^2.$$

(II) Let $V(G_1 \ominus G_2) = V_1 \cup V_2 \cup V_3$, where

(1) $V_1 = \{v_i \in V(G_1 \ominus G_2) : d_{G_1\ominus G_2}(v_i) = d_{G_1}(v_i), i = 1, 2, \ldots, n_1\}$,
(2) $V_2 = \{v_i \in V(G_1 \ominus G_2) : d_{G_1\ominus G_2}(v_i) = d_{G_2}(v_i) + 1, i = 1, 2, \ldots, m_1n_2\}$, and
(3) $V_3 = \{v_i \in V(G_1 \ominus G_2) : d_{G_1\ominus G_2}(v_i) = n_2 + 2, i = 1, 2, \ldots, m_1\}$.

Then, the total irregularity of $G_1 \ominus G_2$ is

$$\mathrm{irr}_t(G_1 \ominus G_2) = \sum_{\substack{i=1,\\ j=i+1}}^{\substack{n_1+m_1n_2+m_1-1,\\ n_1+m_1n_2+m_1}} |d_{G_1\ominus G_2}(v_i) - d_{G_1\ominus G_2}(v_j)|$$

$$= \sum_{i=1}^{n_1}\sum_{j=i+1}^{n_1} |d_{G_1}(v_i) - d_{G_1}(v_j)| + \sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n_1+m_1n_2} |d_{G_1}(v_i) - d_{G_2}(v_j) - 1| + \sum_{i=1}^{n_1}\sum_{j=n_1+m_1n_2+1}^{n_1+m_1n_2+m1} |d_{G_1}(v_i) - n_2 - 2| + \sum_{i=n_1+1}^{n_1+m_1n_2}\sum_{j=i+1}^{n_1+m_1n_2} |d_{G_2}(v_i) + 1 - d_{G_2}(v_j) - 1| + \sum_{i=n_1+1}^{n_1+m_1n_2}\sum_{j=n_1+m_1n_2+1}^{n_1+m_1n_2+m1} |d_{G_2}(v_i) + 1 - n_2 - 2|$$

$$= \mathrm{irr}_t(G_1) + \sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n_1+m_1n_2} |d_{G_1}(v_i) - d_{G_2}(v_j) - 1| + m_1 \sum_{i=1}^{n_1} |d_{G_1}(v_i) - n_2 - 2| + m_1^2\mathrm{irr}_t(G_2) + m_1^2(n_2^2 + n_2 - 2m_2)$$

$$= \mathrm{irr}_t(G_1) + m_1^2\mathrm{irr}_t(G_2) + m_1^2(n_2^2 + n_2 - 2m_2) + m_1 \sum_{i=1}^{n_1} |d_{G_1}(v_i) - n_2 - 2| + \sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n_1+m_1n_2} |d_{G_1}(v_i) - d_{G_2}(v_j) - 1|.$$

(III) Also here, one can partition the vertex set of $G_1 \boxdot G_2$ into

(1) $V_1 = \{v_i \in V(G_1 \boxdot G_2) : d_{G_1\boxdot G_2}(v_i) = d_{G_1}(v_i), i = 1, 2, \ldots, n_1\}$,
(2) $V_2 = \{v_i \in V(G_1 \boxdot G_2) : d_{G_1\boxdot G_2}(v_i) = d_{G_1}(v_i) + d_{G_2}(v_j), 1 \le i \le n_1, 1 \le j \le n_2\}$,

(3) $V_3 = \{v_i \in V(G_1 \boxdot G_2) : d_{G_1 \boxdot G_2}(v_i) = 2n_2 + 2, i = 1, 2, \ldots, m_1\}$.

Then,the total irregularity of $G_1 \boxdot G_2$ is

$$\begin{aligned}
&\operatorname{irr}_t(G_1 \boxdot G_2) = \\
&\quad \sum_{i=1}^{n_1+n_1n_2+m_1-1} \sum_{j=i+1}^{n_1+n_1n_2+m_1} |d_{G_1\boxdot G_2}(v_i) - d_{G_1\boxdot G_2}(v_j)| \\
&= \sum_{i=1}^{n_1} \sum_{j=i+1}^{n_1} |d_{G_1}(v_i) - d_{G_1}(v_j)| + \\
&\quad \sum_{i=1}^{n_1} \sum_{j=n_1+1}^{n_1+n_1n_2} |d_{G_1}(v_i) - d_{G_1\boxdot G_2}(v_j)| + \\
&\quad \sum_{i=1}^{n_1} \sum_{j=n_1+n_1n_2+1}^{n_1+n_1n_2+m1} |d_{G_1}(v_i) - 2n_2 - 2| + \\
&\quad \sum_{i=n_1+1}^{n_1+n_1n_2} \sum_{j=i+1}^{n_1+n_1n_2} |d_{G_1\boxdot G_2}(v_i) - d_{G_1\boxdot G_2}(v_j)| + \\
&\quad \sum_{i=n_1+1}^{n_1+n_1n_2} \sum_{j=n_1+n_1n_2+1}^{n_1+n_1n_2+m1} |d_{G_1\boxdot G_2}(v_i) - 2n_2 - 2|. \qquad (7)
\end{aligned}$$

Further, we have

$$\begin{aligned}
&\sum_{i=1}^{n_1} \sum_{j=n_1+1}^{n_1+n_1n_2} |d_{G_1\boxdot G_2}(v_j) - d_{G_1}(v_i)| \\
&\quad = \sum_{i=1}^{n_1} \sum_{k=1}^{n_1} \sum_{t=1}^{n_2} |d_{G_1}(v_k) + d_{G_2}(v_t) - d_{G_1}(v_i)| \\
&\quad \le \sum_{i=1}^{n_1} \sum_{k=1}^{n_1} \sum_{t=1}^{n_2} (|d_{G_1}(v_k) - d_{G_1}(v_i)| + d_{G_2}(v_t)) \\
&\quad = n_2 \operatorname{irr}_t(G_1) + 2n_1^2 m_2, \qquad (8)
\end{aligned}$$

$$\begin{aligned}
&\sum_{s=n_1+1}^{n_1+n_1n_2} \sum_{t=s+1}^{n_1+n_1n_2} |d_{G_1\boxdot G_2}(v_s) - d_{G_1\boxdot G_2}(v_t)| \\
&= \sum_{i,j=1}^{n_1,n_2} \sum_{k,l=1}^{n_1,n_2} |d_{G_1}(v_i) + d_{G_2}(v_j) - d_{G_1}(v_k) - d_{G_2}(v_l)| \\
&\le \sum_{i,j=1}^{n_1,n_2} \sum_{k,l=1}^{n_1,n_2} (|d_{G_1}(v_i) - d_{G_1}(v_k)| + |d_{G_2}(v_j) - d_{G_2}(v_l)|) \\
&= n_2^2 \operatorname{irr}_t(G_1) + n_1^2 \operatorname{irr}_t(G_2) \qquad (9)
\end{aligned}$$

$$\begin{aligned}
&\sum_{i=1}^{n_1} \sum_{j=n_1+n_1n_2+1}^{n_1+n_1n_2+m1} |d_{G_1}(v_i) - 2n_2 - 2| = \\
&\quad m_1 \sum_{i=1}^{n_1} |d_{G_1}(v_i) - 2n_2 - 2|, \quad \text{and} \qquad (10)
\end{aligned}$$

$$\begin{aligned}
&\sum_{i=n_1+1}^{n_1+n_1n_2} \sum_{j=n_1+n_1n_2+1}^{n_1+n_1n_2+m1} |d_{G_1\boxdot G_2}(v_i) - 2n_2 - 2| = \\
&\quad m_1 \sum_{k=1}^{n_1} \sum_{l=1}^{n_2} |d_{G_1}(v_k) + d_{G_2}(v_l) - 2n_2 - 2|. \qquad (11)
\end{aligned}$$

From (7), (8), (9), (10) and (11), we obtain

$$\begin{aligned}
&\operatorname{irr}_t(G_1 \boxdot G_2) \le (n_2^2 + n_2 + 1) \operatorname{irr}_t(G_1) + \\
&\quad n_1^2 \operatorname{irr}_t(G_2) + 2n_1^2 m_2 + m_1 \sum_{i=1}^{n_1} |d_{G_1}(v_i) - 2n_2 - 2| + \\
&\quad \sum_{i=1}^{n_1} \sum_{l=1}^{n_2} |d_{G_1}(v_i) + d_{G_2}(v_l) - 2n_2 - 2|.
\end{aligned}$$

(IV) Here again, we partition $V(G_1 \boxminus G_2)$ into three subsets

(1) $V_1 = \{v_i \in V(G_1 \boxminus G_2) : d_{G_1\boxminus G_2}(v_i) = (n_2 + 1)\, d_{G_1}(v_i), i = 1, 2, \ldots, n_1\}$,
(2) $V_2 = \{v_i \in V(G_1 \boxminus G_2) : d_{G_1\boxminus G_2}(v_i) = d_{G_2}(v_i) + 2, i = 1, 2, \ldots, m_1 n_2\}$,
(3) $V_3 = \{v_i \in V(G_1 \boxminus G_2) : d_{G_1\boxminus G_2}(v_i) = 2, i = 1, 2, \ldots, m_1\}$,

and then, the total irregularity of $G_1 \boxminus G_2$ is

$$\begin{aligned}
&\operatorname{irr}_t(G_1 \boxminus G_2) = \\
&\quad \sum_{i=1}^{n_1+m_1n_2+m_1-1} \sum_{j=i+1}^{n_1+m_1n_2+m_1} |d_{G_1\boxminus G_2}(v_i) - d_{G_1\boxminus G_2}(v_j)| \\
&= \sum_{i=1}^{n_1} \sum_{j=i+1}^{n_1} |(n_2+1)\, d_{G_1}(v_i) - (n_2+1)\, d_{G_1}(v_j)| + \\
&\quad \sum_{i=1}^{n_1} \sum_{j=n_1+1}^{n_1+m_1n_2} |(n_2+1)\, d_{G_1}(v_i) - d_{G_2}(v_j) - 2| + \\
&\quad \sum_{i=1}^{n_1} \sum_{j=n_1+m_1n_2+1}^{n_1+m_1n_2+m1} |(n_2+1)\, d_{G_1}(v_i) - 2| + \\
&\quad \sum_{i=n_1+1}^{n_1+m_1n_2} \sum_{j=i+1}^{n_1+m_1n_2} |d_{G_2}(v_i) + 2 - d_{G_2}(v_j) - 2| + \\
&\quad \sum_{i=n_1+1}^{n_1+m_1n_2} \sum_{j=n_1+m_1n_2+1}^{n_1+m_1n_2+m1} |d_{G_2}(v_i) + 2 - 2| \\
&= (n_2+1) \operatorname{irr}_t(G_1) + \\
&\quad \sum_{i=1}^{n_1} (m_1 n_2((n_2+1) d_{G_1}(v_i) - 2) - 2m_1 m_2) + \\
&\quad \sum_{i=1}^{n_1} m_1((n_2+1) d_{G_1}(v_i) - 2) + m_1^2 \operatorname{irr}_t(G_2) + 2m_1^2 m_2 \\
&= (n_2+1) \operatorname{irr}_t(G_1) + m_1^2 \operatorname{irr}_t(G_2) + \\
&\quad 2m_1^2(n_2^2 + 2n_2 + m_2 + 1) - 2n_1 m_1 (n_2 + m_2 + 1).
\end{aligned}$$

$\square$

The following results are direct consequences of Theorem 19.

COROLLARY 5.1. *Let C_n be a cycle on n vertices, P_n be a path on n vertices and G be a simple graph with $p = |V(G)|$ vertices and $q = |E(G)|$ edges. Then*

(1) $\operatorname{irr}_t(C_n \odot G) = n^2 \operatorname{irr}_t(G) + n^2 p(p+1)$
(2) $\operatorname{irr}_t(G \odot C_n) = \operatorname{irr}_t(G) + n(n-3)p^2 + 2(2n-1)pq + 2q^2$
(3) $\operatorname{irr}_t(P_n \odot G) = n^2 \operatorname{irr}_t(G) + n^2 p^2 + n(n-2)p - 2nq - 2$
(4) $\operatorname{irr}_t(G \odot P_n) = \operatorname{irr}_t(G) + (n-2)(n+1)p^2 + 4(n-1)pq + 2q^2$

COROLLARY 5.2. *Let C_n be a cycle on n vertices, P_n be a path on n vertices and G be a simple graph with $p = |V(G)|$ vertices and $q = |E(G)|$ edges. Then*

(1) $\operatorname{irr}_t(C_n \boxminus G) = n^2 \operatorname{irr}_t(G) + 2n^2 p(p+1)$

(2) $\operatorname{irr}_t(G \boxminus C_n) = (n+1) \operatorname{irr}_t(G) + 2(n^2+3n+1)q^2 - 2(2n+1)pq$

(3) $\operatorname{irr}_t(P_n \boxminus G) = (n-1)^2 \operatorname{irr}_t(G) + 2(n-1)^2 p^2 + 2n(n-2)p - 2(n-1)q - 2$

(4) $\operatorname{irr}_t(G \boxminus P_n) = (n+1) \operatorname{irr}_t(G) + 2(n^2+4n-2)q^2 - 4npq.$

6. TOTAL IRREGULARITY OF HIERARCHICAL PRODUCT GRAPHS

The hierarchical product of graphs, introduced in [6], is defined as follows. Let $G_i = (V_i, E_i)$ be graphs with vertex sets $V_i, i = 1, 2$, having a distinguished or root vertex, labeled 0. The hierarchical product $G_1 \sqcap G_2$ is the graph with vertices $x_1x_2, x_i \in V_i$, and edges $\{x_1x_2, y_1y_2\}$ where either $x_1 = y_1$ and $x_2 \sim y_2$ in G_2, or $x_2 = 0$ and $x_1 \sim y_1$ in G_1. Thus, $G_1 \sqcap G_2$ has $|V_1|\,|V_2|$ vertices and $|E_1| + |V_1|\,|E_2|$ edges and $d_{G_1 \sqcap G_2}(x_1x_2) = d_{G_1}(x_1) + d_{G_2}(x_2)$ if $x_2 = 0$ otherwise, $d_{G_1 \sqcap G_2}(x_1x_2) = d_{G_2}(x_2)$.

THEOREM 20. *Let $G_i = (V_i, E_i), i = 1, 2$ be graphs with $|V_1| = n_1$, $|V_2| = n_2$, $|E_1| = m_1$ and $|E_2| = m_2$. The total irregularity of the hierarchical product $G_1 \sqcap G_2$ is*

$$\operatorname{irr}_t(G_1 \sqcap G_2) \leq \operatorname{irr}_t(G_1) + 2\, n_1\, n_2\, m_1 + n_1^2 \operatorname{irr}_t(G_2).$$

PROOF. Let $v_1 \in V(G_2)$ be the root of G_2 and the degree sequences of G_1, G_2 are:

$$\begin{aligned} \mathrm{D(G_1)} &= [d_{G_1}(u_1),\ d_{G_1}(u_2),\ \cdots,\ d_{G_1}(u_{n_1-1}),\ d_{G_1}(u_{n_1})], \\ \mathrm{D(G_2)} &= [d_{G_2}(v_1),\ d_{G_2}(v_2),\ \cdots,\ d_{G_2}(v_{n_2-1}), d_{G_2}(v_{n_2})]. \end{aligned}$$

By the definition of hierarchical product graphs, the degree sequence of $G_1 \sqcap G_2$ is

$$\begin{aligned} \mathrm{D(G_1 \sqcap G_2)} = {} & [d_{G_2}(v_1) + d_{G_1}(u_1),\ d_{G_2}(v_1) + d_{G_1}(u_2), \cdots, \\ & d_{G_2}(v_1) + d_{G_1}(u_{n_1-1}), d_{G_2}(v_1) + d_{G_1}(u_{n_1}), \\ & d_{G_2}^{n_1}(v_2), d_{G_2}^{n_1}(v_3), \cdots, d_{G_2}^{n_1}(v_{n_2-1}), d_{G_2}^{n_1}(v_{n_2})]. \end{aligned}$$

Hence, the total irregularity of $G_1 \sqcap G_2$ is

$$\begin{aligned} & \operatorname{irr}_t(G_1 \sqcap G_2) = \\ & \quad \frac{1}{2} \sum_{u_iv_i, u_jv_j \in V(G_1 \sqcap G_2)} |d_{G_1 \sqcap G_2}(u_iv_i) - d_{G_1 \sqcap G_2}(u_jv_j)| \\ & = \frac{1}{2} \sum_{u_i, u_j \in V(G_1), v_1 \in V(G_2)} |(d_{G_2}(v_1) + d_{G_1}(u_i)) - \\ & \quad (d_{G_2}(v_1) + d_{G_1}(u_j))| + \frac{n_1}{2} \\ & \quad \sum_{u_i \in V(G_1), v_1, v_j \in V(G_2)} |d_{G_2}(v_1) + d_{G_1}(u_i) - d_{G_2}(v_j)| \\ & \quad + \frac{n_1^2}{2} \sum_{v_i, v_j \in V(G_2), v_i \neq v_1, v_j \neq v_1} |d_{G_2}(v_i) - d_{G_2}(v_j)| \\ & \leq \operatorname{irr}_t(G_1) + \frac{n_1}{2} 2 \sum_{u_i \in V(G_1), v_1, v_j \in V(G_2)} d_{G_1}(u_i) + \\ & \quad \frac{n_1}{2} \sum_{u_i \in V(G_1), v_1, v_j \in V(G_2)} |d_{G_2}(v_1) - d_{G_2}(v_j)| + \end{aligned}$$

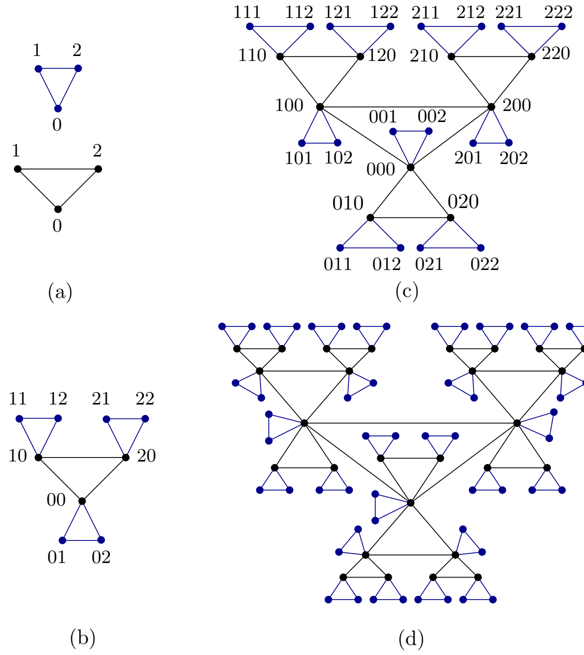

Fig. 3. Hierarchical product of graphs (a) K_3, (b) $K_3^2 = K_3 \sqcap K_3$, (c) $K_3^3 = K_3^2 \sqcap K_3$ and (d) $K_3^4 = K_3^3 \sqcap K_3$.

$$\begin{aligned} & \quad \frac{n_1^2}{2} \sum_{v_i, v_j \in V(G_2)} |d_{G_2}(v_i) - d_{G_2}(v_j)| - \\ & \quad \frac{n_1^2}{2} \sum_{v_1, v_j \in V(G_2)} |d_{G_2}(v_1) - d_{G_2}(v_j)|) \\ & = \operatorname{irr}_t(G_1) + n_1(2n_2m_1) + \\ & \quad \frac{n_1^2}{2} \sum_{v_1, v_j \in V(G_2)} |d_{G_2}(v_1) - d_{G_2}(v_j)| + \\ & \quad n_1^2 \operatorname{irr}_t(G_2) - \frac{n_1^2}{2} \sum_{v_1, v_j \in V(G_2)} |d_{G_2}(v_1) - d_{G_2}(v_j)|) \\ & = \operatorname{irr}_t(G_1) + 2\, n_1\, n_2\, m_1 + n_1^2 \operatorname{irr}_t(G_2). \end{aligned}$$

□

The total irregularity of $K_n^r = K_n^{r-1} \sqcap K_n$ for $r \geq 1$ is given in Proposition 2. The hierarchical products K_3^2, K_3^3, K_3^4 are illustrated in Figure 3.

PROPOSITION 2. *The total irregularity of the hierarchical product K_n^r is*

$$\operatorname{irr}_t(K_n^r) = \frac{n^2(n^r - 1)(n^{r-1} - 1)}{n+1}.$$

PROOF. The degree sequences of the hierarchical products $K_n^1, \cdots, K_n^4$ according to the definition of the hierarchical product are

$$\begin{aligned} & \mathrm{D(K_n^1)} = [(n-1)^n], \\ & \mathrm{D(K_n^2)} = [(2(n-1))^n, (n-1)^{n(n-1)}], \\ & \mathrm{D(K_n^3)} = [(3(n-1))^n, (2(n-1))^{n(n-1)}, (n-1)^{n^2(n-1)}], \end{aligned}$$

$$\mathrm{D}(\mathrm{K}_\mathrm{n}^4) = [(4(n-1))^n, (3(n-1))^{n(n-1)}, (2(n-1))^{n^2(n-1)}, (n-1)^{n^3(n-1)}].$$

The degree sequence of the hierarchical products K_n^r for positive integer r is

$$\mathrm{D}(\mathrm{K}_\mathrm{n}^\mathrm{r}) = [(r(n-1))^n, ((r-1)(n-1))^{(n-1)n}, ((r-2)(n-1))^{(n-1)n^2}, ((r-3)(n-1))^{(n-1)n^3}, \cdots, (3(n-1))^{(n-1)n^{(r-3)}}, (2(n-1))^{(n-1)n^{(r-2)}}, (n-1)^{(n-1)n^{(r-1)}}].$$

Then

$$\begin{aligned}\mathrm{irr}_t(K_n^r) &= n(n-1)^2\left(n+2n^2+3n^3+\cdots+(r-3)n^{r-3}+(r-2)n^{r-2}+(r-1)n^{r-1}\right)+n^2(n-1)^3\left(n+2n^2+3n^3+\cdots+(r-4)n^{r-4}+(r-3)n^{r-3}+(r-2)n^{r-2}\right)+n^4(n-1)^3\left(n+2n^2+3n^3+\cdots+(r-5)n^{r-5}+(r-4)n^{r-4}+(r-3)n^{r-3}\right)+\cdots+n^{2r-6}(n-1)^3\left(n+2n^2\right)+n^{2r-4}(n-1)^3 n\\
&= n(n-1)^2\sum_{i=1}^{r-1} i\,n^i + (n-1)^3\left(n^2\sum_{i=1}^{r-2} i\,n^i + n^4\sum_{i=1}^{r-3} i\,n^i+\cdots+n^{2(r-2)}\sum_{i=1}^{r-(r-1)} i\,n^i\right)\\
&= n(n-1)^2\sum_{i=1}^{r-1} i\,n^i + (n-1)^3\sum_{j=1}^{r-2}\left(n^{2j}\sum_{i=1}^{r-(j+1)} i\,n^i\right)\\
&= n\left(n+(r-1)n^{r+1}-rn^r\right)+\frac{n\left(n^{2r}-n^2+(r-1)n^r-(r-1)n^{r+2}\right)}{n+1}\\
&= \frac{n^2(n^r-1)(n^{r-1}-1)}{n+1}.\end{aligned}$$

□

7. CONCLUSION

The results presented in this paper complement the results from [3]. Here we consider the total irregularity of simple undirected graphs under several graph operations. We presented the exact expression of the total irregularity of subdivision graph, triangle parallel graph, superposition graph, total graph, double graph, extended double graph, subdivision-vertex corona, subdivision-edge corona and subdivision-edge neighbourhood corona and sharp upper bounds of the total irregularity of thorn graph, subdivision-vertex neighbourhood corona and the hierarchical product of graphs. It is an open problem if the presented upper bounds on the total irregularity of thorn graph, subdivision-vertex neighbourhood corona and the hierarchical product of graphs are the best possible.

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