Path Factorization Induced Network Flow

U S Rajput
Department of Mathematics and Astronomy,
Lucknow University, Lucknow (India)

Bal Govind Shukla
Department of Mathematics and Astronomy,
Lucknow University, Lucknow (India)

ABSTRACT
In path factorization Ushio [8] gave the necessary and sufficient conditions for P_k design. When k is an even number, the spectrum problem is completely solved [9, 1]. For odd value of k the problem was studied by several researchers [7, 10, 11, 5, 2, 6]. In all these papers [7, 10, 11, 5, 2, 6] Ushio Conjecture [8] played an important role. Here in this paper we obtain a feasible network flow consisting of path factors of a bipartite graph satisfying the conditions of path factorization.

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1. INTRODUCTION
A flow network is a directed graph with two distinguished vertices called source and sink in such a way that all edges connected to source are directed outward and all edges connected to sink are directed inward. Flow is called feasible flow if it satisfies the following two conditions as given by [3, 4].

(i) $0 \leq f(u,v) \leq c(u,v) \quad \forall (u,v) \in E$. These are the capacity constraints. (If a capacity is ∞, then there is no upper bound on the flow value on that edge.)

(ii) For all $v \in V - \{s,t\}$, the total flow into v is same as the total flow out of v:

$$\sum_{u:(u,v)\in E} f(u,v) = \sum_{w:(v,w)\in E} f(v,w).$$

Constraint (ii) is called flow conservation law for network flow graph. The conservation law holds at all vertices other than the source and the sink. The value of flow denoted by $val(f)$, is the net flow out of the source:

$$val(f) = \sum_{u:(s,u)\in E} f(s,u).$$

Let $K_{m,n}$ be the complete bipartite graph with two partite sets having m and n vertices. A spanning sub graph F of $K_{m,n}$ is called a path factor if each component of F is a path of order at least two. In particular a spanning sub graph F of $K_{m,n}$ is called a P_k −factor of $K_{m,n}$ if each component of F is isomorphic to P_k. If $K_{m,n}$ is expressed as an arc disjoint sum of P_k −factors, then this sum is called a P_k −factorization of $K_{m,n}$.

In path factorization induced network flow each path factor of complete bipartite graph $K_{m,n}$ will contribute to a flow path, and hence this network flow will be a collection of flow paths each of which is an undirected path factor. We assume capacity of each edge as one which is not shown in the figures drawn.

2. MATHEMATICAL ANALYSIS
In the study of P_k − path factorization of complete bipartite graph $K_{m,n}$, we find different mutually vertex/edge disjoint paths. Each path will be a flow path. For each value of k(even/odd) flow path can be developed. In this paper we consider particular cases of odd values of k for which spectrum problem is determined i.e. P_3, P_5, P_7 and P_9.

Network flow in P_3 −factorization: Let P_3 be a path on 3 vertices and $K_{m,n}$ be a complete bipartite graph with partite sets having m and n vertices. Ushio [7] gave the necessary and sufficient conditions for the existence of a P_3 −factorization of $K_{m,n}$, which are given in theorem 2.1 below.

Theorem 2.1: $K_{m,n}$has P_3 −factor if and only if

(i) $m + n \equiv 0(mod\ 3)$,
(ii) $m \leq 2n$,
(iii) $n \leq 2m$,
(iv) $\frac{3mn}{2(m+n)}$, is an integer.

To show the feasible network flow in P_3 −factorization of $K_{m,n}$, we consider $K_{1,2}$, i.e. $m = 1$ and $n = 2$(trivial case), Fig. 1.

Let t be the number of copies in P_3 −factorization, then

$$t = \frac{m+n}{3} = \frac{2+1}{3} = 1 \quad \text{(in this case)}$$

and r be the number of P_3 −factors in network flow factorization, i.e.

$$r = \frac{3mn}{2(m+n)} = \frac{3 \times 2 \times 1}{2(2+1)} = 1.$$

To make it network flow graph we add source and sink in P_3 −factorization of $K_{1,2}$(Fig − 2).

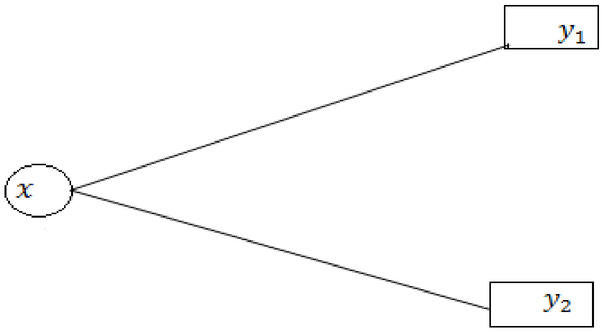

Fig. 1: Complete bipartite graph $K_{1,2}$.

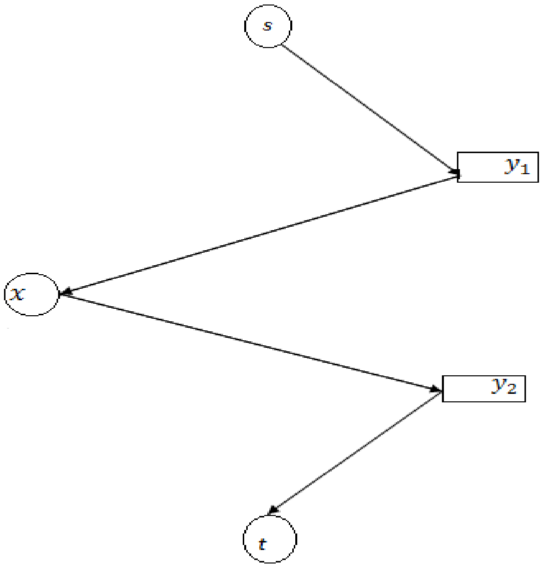

Fig. 2: Flow path sy_1xy_2t.

As shown above in Fig. 2, we find the disjoint feasible Network flow path between source s and sink t in P_3 −factorization of $K_{1,2}$. Similarly, for P_3 −factorization of $K_{2,4}$ (Fig. 3), we can find the disjoint network flow given in Fig. 4 and Fig. 5.

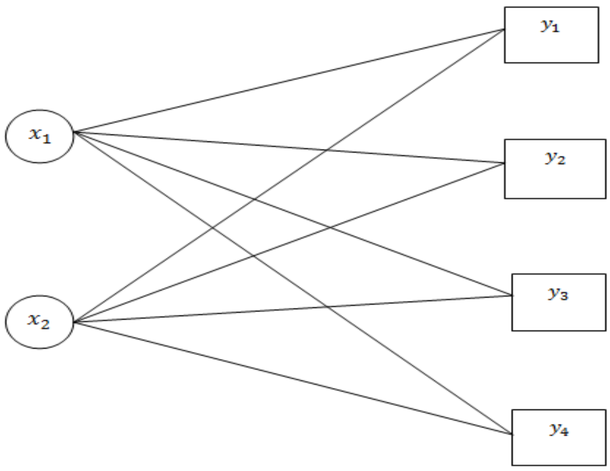

Fig. 3: Complete bipartite graph $K_{2,4}$.

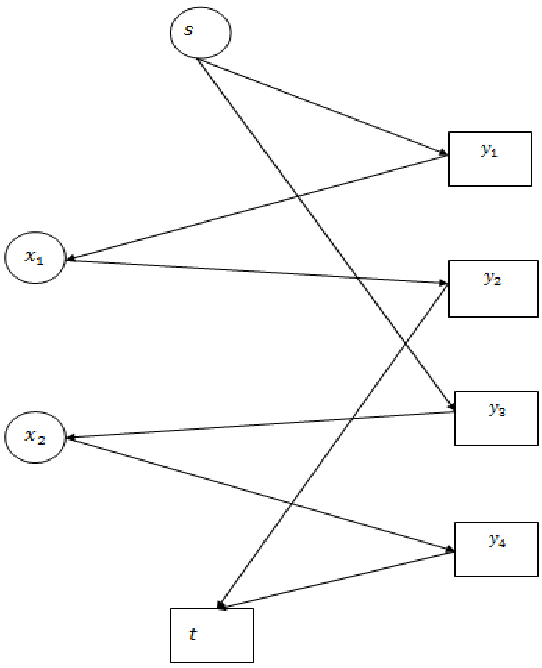

Fig. 4: Flow path $sy_1x_1y_2t, sy_3x_2y_4t.$

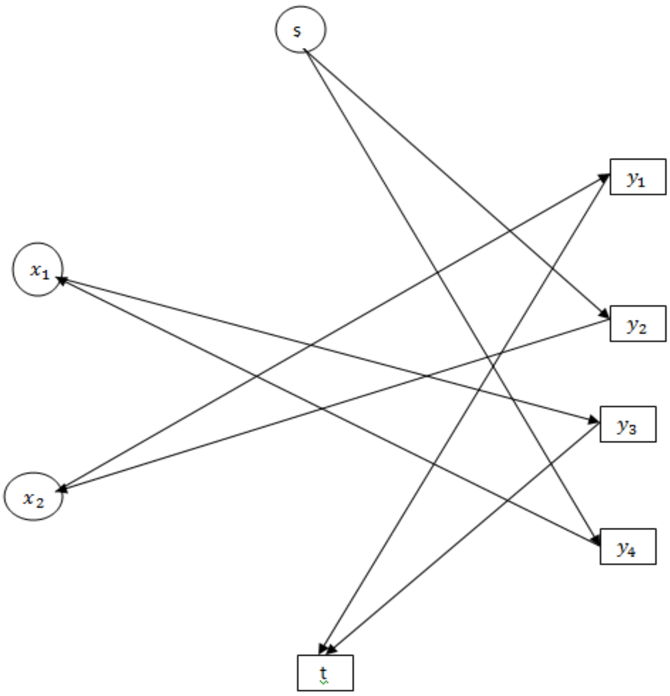

Fig. 5: Flow path $sy_2x_2y_1t, sy_4x_1y_3t.$

Network flows in P_5 −factorization: Wang J and Beliang Du [10] has shown that Ushio conjecture is true for P_5 −factorization of complete bipartite graph $K_{m,n}$. The necessary and sufficient conditions for the existence of a P_5 −factorization of $K_{m,n}$ are given in theorem 2.2 below,

Theorem 2.2: $K_{m,n}$ has P_5 −factorization if and only if,

(i) $3n \geq 2m,$

(ii) $3m \geq 2n,$

(iii) $m + n \equiv 0 (mod\ 5),$

(iv) $\frac{5mn}{[4(m+n)]}$, is an integer.

Here also we create the feasible network flow in P_5 −factorization of $K_{m,n}$. In particular if we take $m = 4$ and $n = 6$ (Fig-6), then $K_{4,6}$ has a P_5 −factorization network flow graph.

Here let t be the number of copies in a P_5 −factor graph, then in this case

$$t = \frac{m+n}{5}$$

$$= \frac{4+6}{5} = 2$$

and r be the number of disjoint P_5 −factors in graph, i.e.

$$r = \frac{5mn}{4(m+n)}$$

$$= \frac{5 \times 4 \times 6}{4(4+6)} = 3$$

To find network flow graph, we add source and sink in the graph of P_5 −factorization of $K_{4,6}$. We obtain flow graphs given in Fig. 7, Fig. 8 and Fig. 9. These represent P_5 disjoint flow paths.

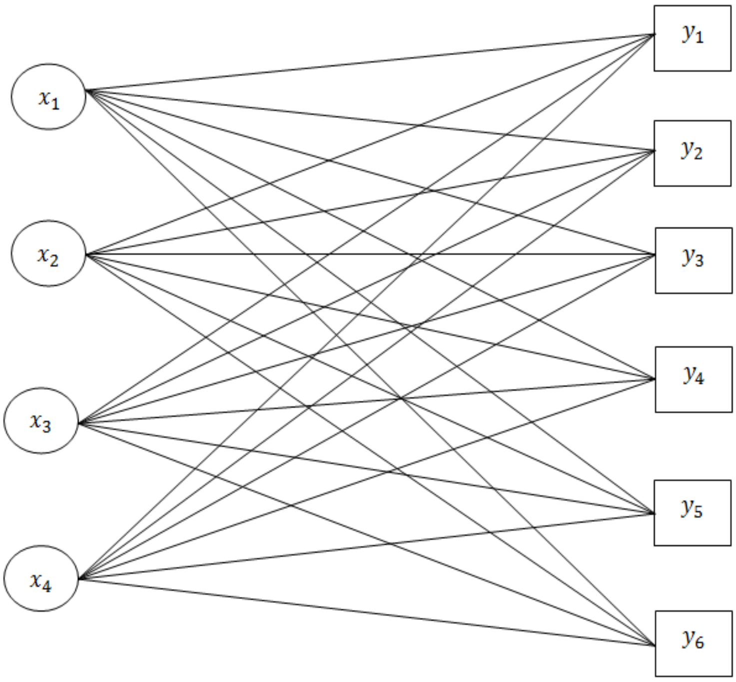

Fig. 6: Complete bipartite graph $K_{4,6}$.

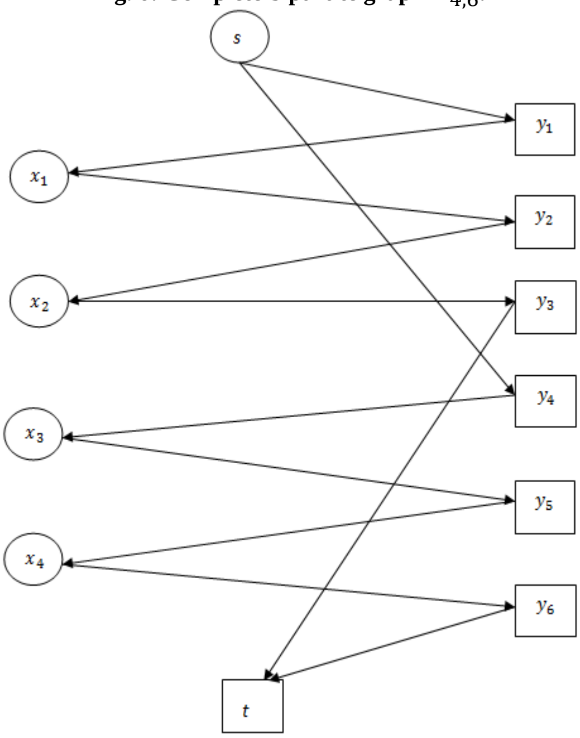

Fig. 7: Flow path $sy_1x_1y_2x_2y_3t, sy_4x_3y_5x_4y_6t$.

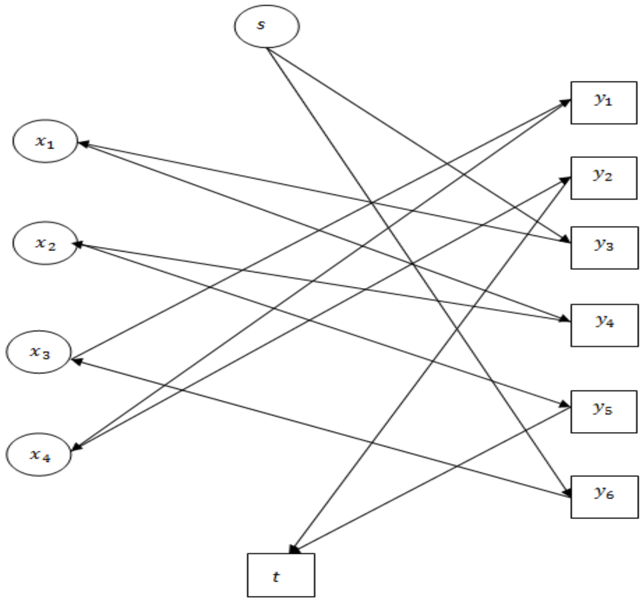

Fig. 8: Flow path $sy_3x_1y_4x_2y_5t, sy_6x_3y_1x_4y_2t.$

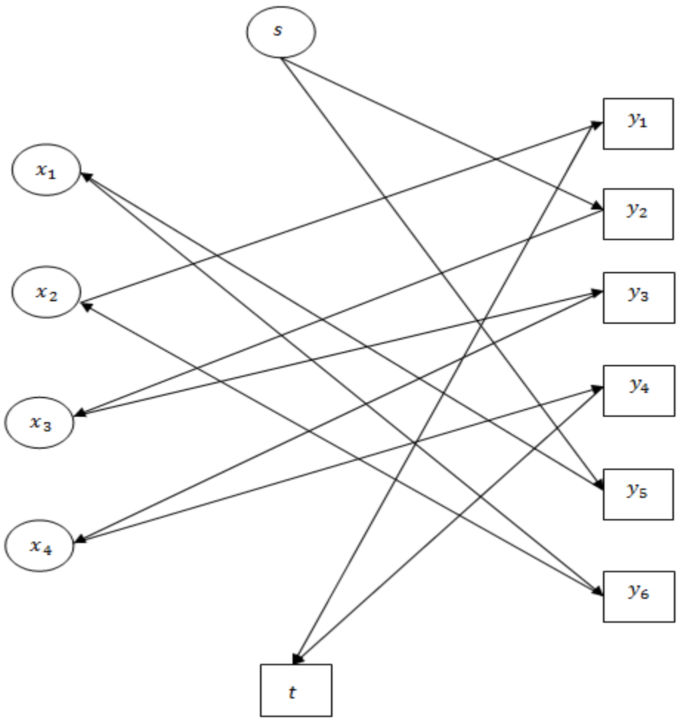

Fig. 9: Flow path $sy_5x_1y_6x_2y_1, y_2x_3y_3x_4y_4t.$

Disjoint Flow Paths in P_7 −factorization:

Wang [11], has shown that Ushio conjecture is true for P_7 −factorization of complete bipartite graph $K_{m,n}$, for which he gave necessary and sufficient conditions. These conditions are given in theorem 2.3, below.

Theorem 2.3: $K_{m,n}$ has P_7 −factorization if and only if,

(i) $4n \geq 3m,$

(ii) $4m \geq 3n,$

(iii) $m + n \equiv 0(mod\ 7),$

(iv) $\frac{7mn}{[6(m+n)]}$, an integer.

Here we show that theorem 2.3 is also helpful for finding the disjoint network flow paths in P_7 −factorization of $K_{m,n}$. Particularly consider, $K_{3,4}$ (Fig. 10).

Let t be the number of copies in P_7 −factor flow graph, i.e,

$$t = \frac{m+n}{7}$$

$$= \frac{4+3}{7} = 1$$

and r be the number of P_7 −factor in disjoint flow factorization, i.e.,

$$r = \frac{7mn}{6(m+n)}$$

$$= \frac{7 \times 4 \times 3}{6(4+3)} = 2.$$

To show flow paths we add source and sink in P_7 −factorization of $K_{3,4}$. We obtain two network flow graphs, which are shown in Fig. 11 and Fig. 12.

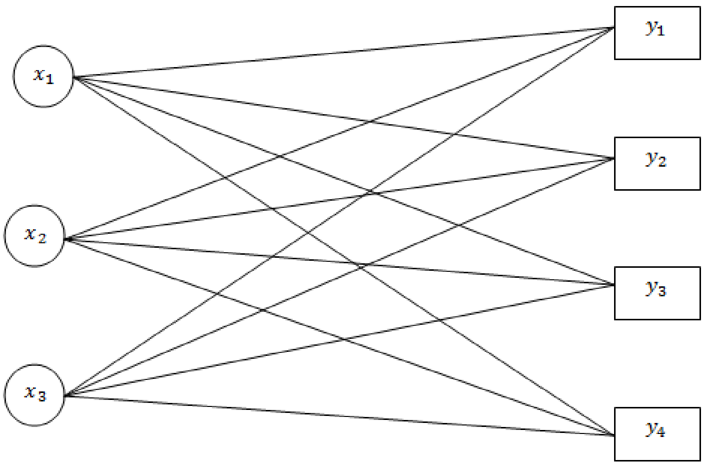

Fig. 10: Complete bipartite graph $K_{3,4}$.

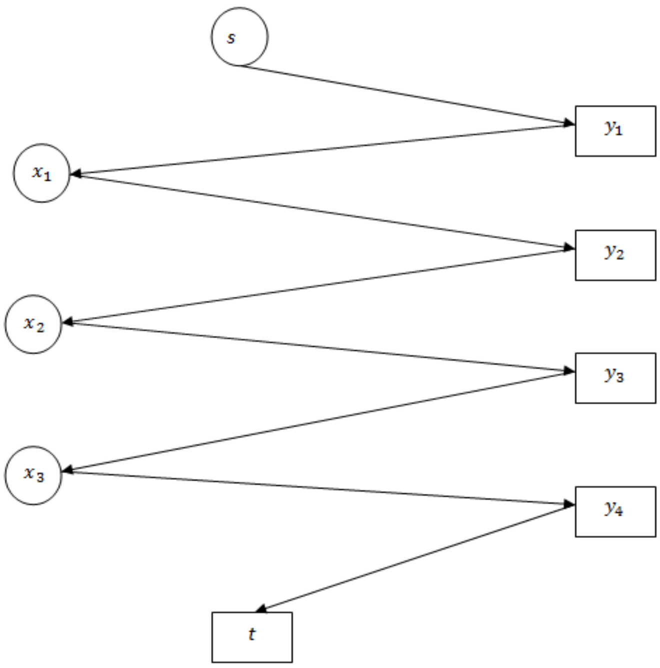

Fig. 11: Flow path $sy_1x_1y_2x_2y_3x_3y_4t$.

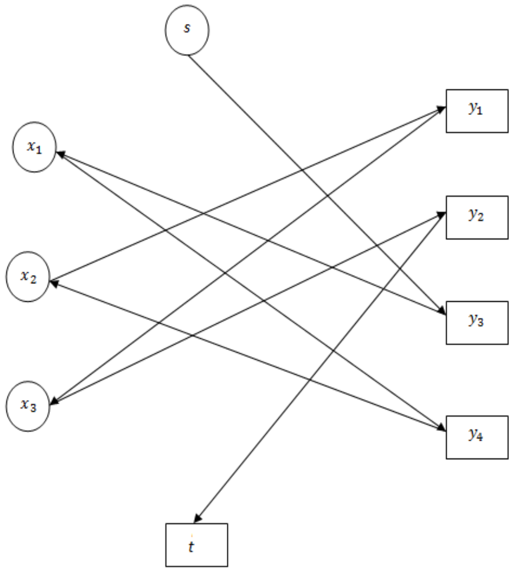

Fig. 12: Flow path $sy_3x_1y_4x_2y_1x_3y_2t$.

Disjoint Flow Paths in P_9 −factorization:

We [5] have studied P_9 −factorization of complete bipartite graph $K_{m,n}$ in which, we established the necessary and sufficient conditions given in theorem 2.4.

Theorem 2.4: If $K_{m,n}$has P_9 −factorization if and only if,

(i) $5n \geq 4m,$
(ii) $5m \geq 4n,$
(iii) $m + n \equiv 0(mod\ 9),$
(iv) $\frac{9mn}{[8(m+n)]}$, is an integer.

In $K_{m,n}$ take $m = 8$ and $n = 10$ (fig. 13). Since $K_{8,10}$ satisfy above theorem, therefore we can find P_9 −factorization flow graph.

Let t be the number of copies in P_9 −factor graph, i.e,

$$t = \frac{m+n}{9}$$

$$= \frac{8+10}{9} = 2$$

and r be the number of P_9 −factor in disjoint flow path in factorization, i.e.,

$$r = \frac{9mn}{8(m+n)}$$

$$= \frac{9 \times 10 \times 8}{8(10+8)}$$
$$= 5.$$

To show disjoint flow graph we add source and sink in P_9 −factorization graph of $K_{8,10}$, in Fig. 14 to Fig. 18. Here each factor of P_9 −factorization has 2 disjoint copies of flow paths.

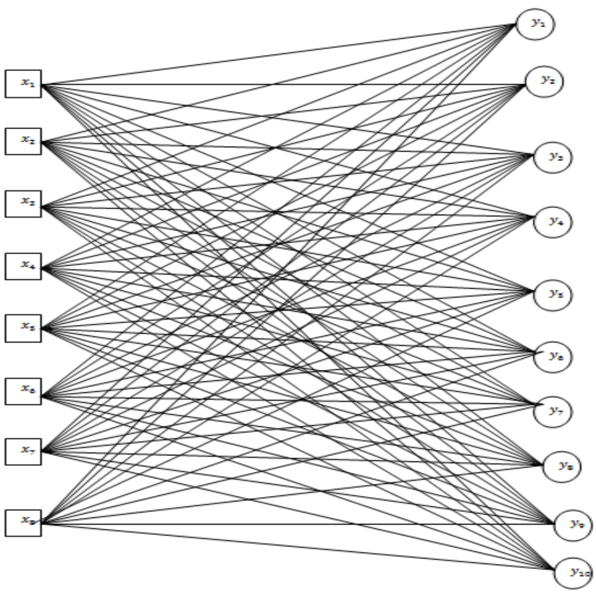

Fig. 13: Complete bipartite graph $K_{8,10}$

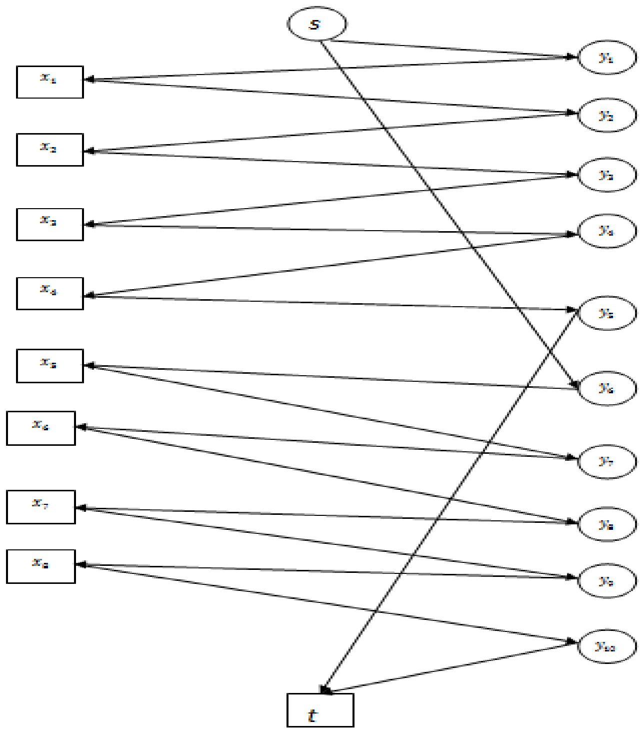

Fig. 14: Flow path $sy_1x_1y_2x_2y_3x_3y_4x_4y_5t, sy_6x_5y_7x_6y_8x_7y_9x_8y_{10}t$.

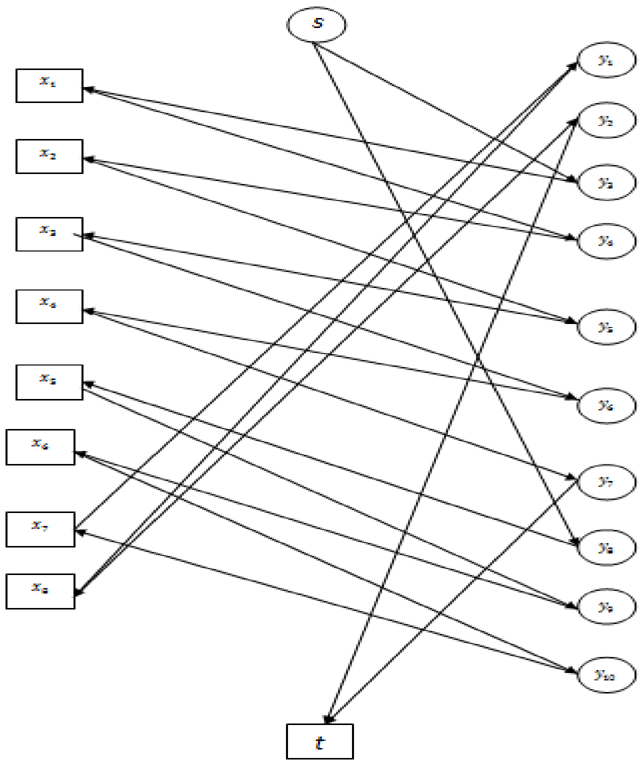

Fig. 15: Flow path $sy_3x_1y_4x_2y_5x_3y_6x_4y_7t, sy_8x_5y_9x_6y_{10}x_7y_1x_8y_2t$.

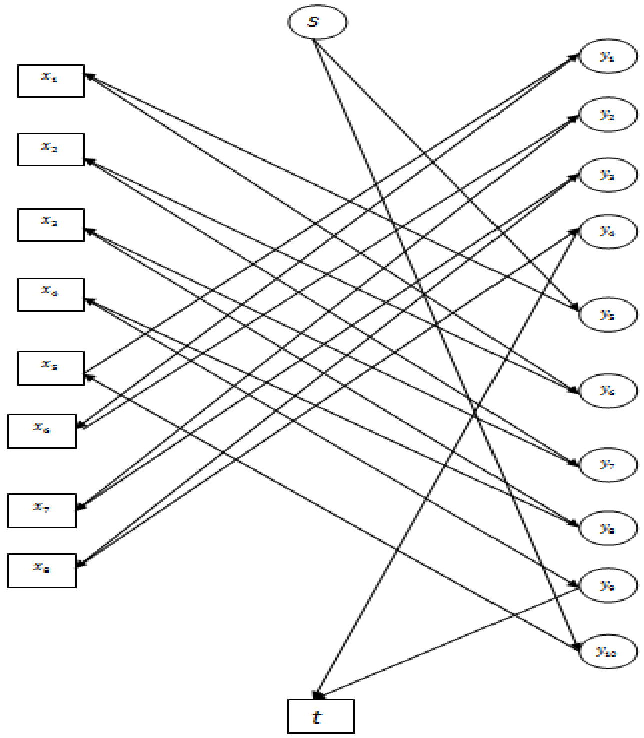

Fig. 16: Flow path $sy_5x_1y_6x_2y_7x_3y_8x_4y_9t,\ sy_{10}x_5y_1x_6y_2x_7y_3x_8y_4t,$

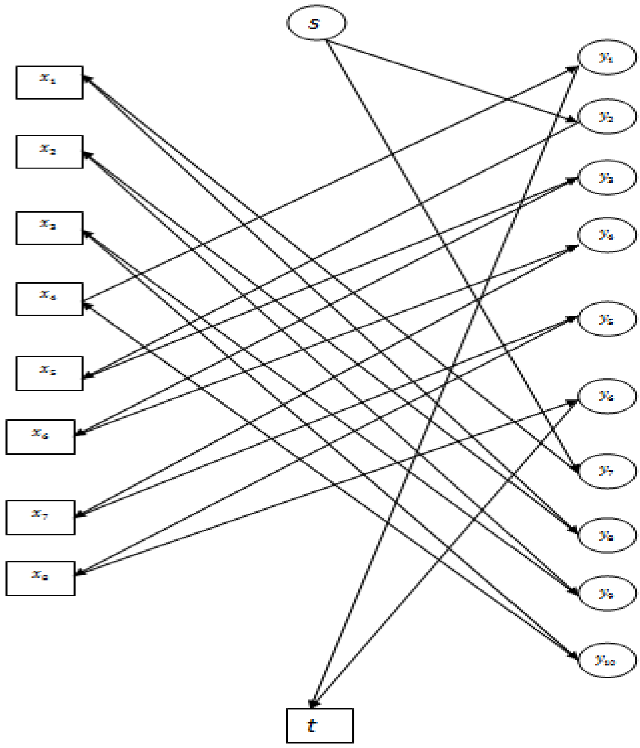

Fig. 17: Flow path $sy_7x_1y_8x_2y_9x_3y_{10}x_4y_1t, sy_2x_5y_3x_6y_4x_7y_5x_8y_6t.$

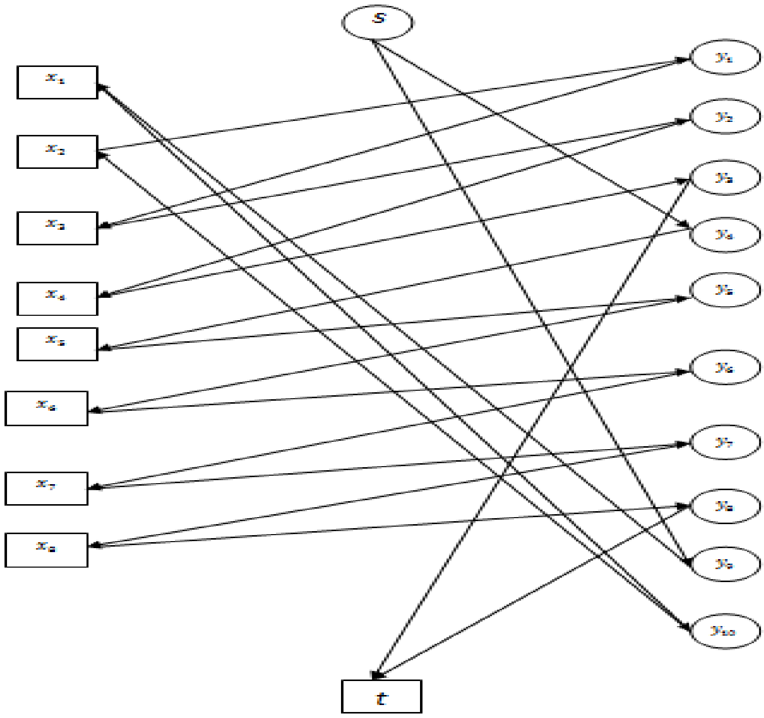

Fig. 18: Flow path $sy_9x_1y_{10}x_2y_1x_3y_2x_4y_3t, sy_4x_5y_5x_6y_6x_7y_7x_8y_8t.$

3. DISCUSSION

In this paper it is analyzed that P_k −factorization of complete bipartite grraph $K_{m,n}$ (for $k = 3, 5, 7$ and 9), will give the disjoint flow paths. Hence we can say that Ushio conjecture is helpful in finding the disjoint flow paths in a complete bipartite graph $K_{m,n}$.

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