A Result on Line Graphs and Hamiltonian Graphs

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ABSTRACT
In 1856, Hamiltonian introduced the Hamiltonian Graph where a Graph which is covered all the vertices without repetition and end with starting vertex. In this Paper I would like to prove that

If 'G' is a Complete and locally Complete graph, on n ≥ 3 vertices, which does not contain an induced $K_{1,3}$, then G is Hamiltonian.

Keywords: Graph, Hamiltonian Graph, Complete Graph, Neighborhood, Locally Complete Graph.

1. INTRODUCTION
Graphs, considered here, are finite, undirected and simple and complete Graphs being followed for terminology and notation. let $G = (V, E)$ be a graph, with V the set of vertices and E the set of edges. Suppose that W is a nonempty subset of V . The sub graph of G, whose vertex set is W and whose edge set is the set of those edges of G that have both ends in W , is called the sub graph of G *induced* by W and is denoted by $G[W]$. For any vertex ν in V , the *neighbour set* of ν is the set of all vertices adjacent to ν. This set is denoted by $N(ν)$. For a graph $G = (V, E)$, we shall denote

$$\delta(G) = \min_{v \in V} |N(v)| \qquad \Delta(G) = \max_{v \in V} |N(v)|$$

a graph G = (V , E) is locally complete , if for each vertex ν the graph G[N(ν)] is complete . With every graph G, having at least one edge, there exists associated a graph L(G), called the line graph of G, whose vertices, can be put in a one-to-one correspondence with the edges of G, in such a way that two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G are adjacent.

The neighborhood is often denoted $N_G(v)$ or (when the graph is unambiguous) $N(v)$. The same neighborhood notation may also be used to refer to sets of adjacent vertices rather than the corresponding induced sub graphs. The neighborhood described above does not include v itself, and is more specifically the **open neighborhood** of v; it is also possible to define a neighborhood in which v itself is included, called the **closed neighborhood** and denoted by $N_G[v]$. When stated without any qualification, a neighborhood is assumed to be open.

1.1 Definition: A graph – usually denoted G(V,E) or G = (V,E) – consists of set of vertices V together with a set of edges E. The number of vertices in a graph is usually denoted *n* while the number of edges is usually denoted *m*.

1.2 Definition: Vertices are also known as nodes, points and (in social networks) as actors, agents or players.

1.3 Definition: Edges are also known as lines and (in social networks) as ties or links. An edge

e = (u,v) is defined by the unordered pair of vertices that serve as its end points.

1.4 Example: The graph depicted in Figure 1 has vertex set V={a,b,c,d,e.f} and edge set

E = {(a,b),(b,c),(c,d),(c,e),(d,e),(e,f)}.

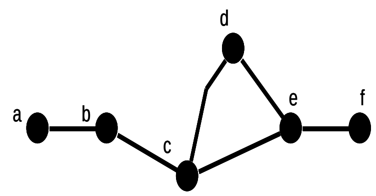

Figure 1.

1. 5 Definition: Two vertices *u* and *v* are *adjacent* if there exists an edge (*u*,*v*) that connects them.

1.6 Definition: An edge (u,v) is said to be *incident* upon nodes u and v.

1.7 Definition: An edge e = (u,u) that links a vertex to itself is known as a *self-loop* or *reflexive* tie.

1.8 Definition: Every graph has associated with it an *adjacency matrix*, which is a binary *n*×*n* matrix A in which $a_{ij} = 1$ and $a_{ji} = 1$ if vertex vi is adjacent to vertex vj, and aij = 0 and aji = 0 otherwise. The natural graphical representation of an adjacency matrix is a table, such as shown below.

	a	b	c	d	e	f
a	0	1	0	0	0	0
b	1	0	1	0	0	0
c	0	1	0	1	1	0
d	0	0	1	0	1	0
e	0	0	1	1	0	1
f	0	0	0	0	1	0

Adjacency matrix for graph in Figure 1.

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1.9 Definition: Examining either Figure 1 or given adjacency Matrix, we can see that not every vertex is adjacent to every other. A graph in which all vertices are adjacent to all others is said to be *complete*.

1.10 Definition: While not every vertex in the graph in Figure 1 is adjacent, one can construct a sequence of adjacent vertices from any vertex to any other. Graphs with this property are called *connected*.

1.11 Note: Reachability. Similarly, any pair of vertices in which one vertex can reach the other via a sequence of adjacent vertices is called *reachable*. If we determine reachability for every pair of vertices, we can construct a reachability matrix R such as depicted in Figure 2. The matrix R can be thought of as the result of applying transitive closure to the adjacency matrix A.

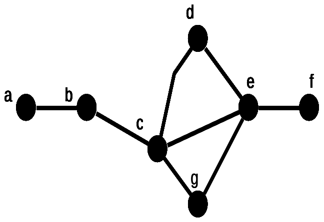

Figure: 2

1.12 Definition : A walk is closed if $v_o = v_n$.*degree* of the vertex and is denoted d(v).

1.13 Definition : A *tree* is a connected graph that contains no cycles. In a tree, every pair of points is connected by a unique path. That is, there is only one way to get from A to B.

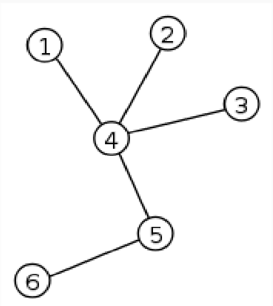

Figure 3: A labeled tree with 6 vertices and 5 edges

1.14 Definition: A *spanning tree* for a graph *G* is a sub-graph of *G* which is a tree that includes every vertex of *G*.

1.15 Definition: The length of a walk (and therefore a path or trail) is defined as the number of edges it contains. For example, in Figure 3, the path *a,b,c,d,e* has length 4.

1.16 Definition: The number of vertices adjacent to a given vertex is called the *degree* of the vertex and is denoted d(v).

1.17 Definition : In the mathematical field of graph theory, a bipartite graph (or bigraph) is a graph whose vertices can be divided into two disjoint sets *U* and *V* such that every edge connects a vertex in *U* to one in *V*; that is, *U* and *V* are independent sets. Equivalently, a bipartite graph is a graph that does not contain any odd-length cycles.

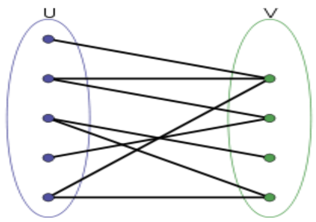

Figure 4: Example of a bipartite graph.

1.18 Definition : An Eulerian circuit in a graph G is circuit which includes every vertex and every edge of G. It may pass through a vertex more than once, but because it is a circuit it traverse each edge exactly once. A graph which has an Eulerian circuit is called an Eulerian graph. An Eulerian path in a graph G is a walk which passes through every vertex of G and which traverses each edge of G exactly once

1.19 Example : Königsberg bridge problem: The city of Königsberg (now Kaliningrad) had seven bridges on the Pregel River. People were wondering whether it would be possible to take a walk through the city passing exactly once on each bridge. Euler built the representative graph, observed that it had vertices of odd degree, and proved that this made such a walk impossible. Does there exist a walk crossing each of the seven bridges of Königsberg exactly once

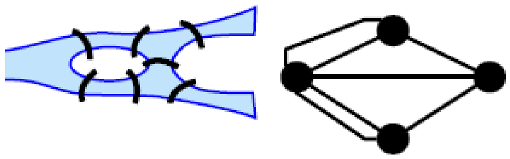

Figure 5: Konigsberg problem

2. COMPLETE GRAPHS, LOCALLY COMPLETE GRAPHS, HAMILTONIAN GRAPHS, LINE GRAPHS

In this section we have to prove that main theorem using definitions.

2.1 Definition: A Hamilton circuit is a path that visits every vertex in the graph exactly once and return to the starting vertex. Determining whether such paths or circuits exist is an NP-complete problem. In the diagram below, an example Hamilton Circuit would be

2.2 Example :

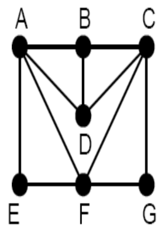

Figure 6: Hamilton Circuit would be AEFGCDBA.

2.3 Definition : Compete Graph: A simple graph in which there exists an edge between every pair of vertices is called a complete graph.

2.4 Definition : Let { v1, v2……vn}be the vertex set of a graph G, and for each 'α'.

let Ni^*denote the closed neighborhood of v_a. Let N_a be any subset of N_α^* containing v_a which generates a complete subgraph C_a of G. Then C_a is called a complete sub neighborhood of v_a, and the indexed family $C^* = \{C_1, C_2, \ldots, C_n\}$ is called a complete family for G if $G = \bigcup C^*$. A graph G is called locally complete iff G has at least one complete family.

2.5 Examples : It is easily seen that complete graphs, trees, and unicyclic graphs are also locally complete.

The complete bigraph $K_{3,2}$ is the smallest (nontrivial, connected) graph which fails to be locally complete.

2.6 Theorem : If 'G' is a Complete and locally Complete graph, on $n \geq 3$ vertices, which does not contain an induced $K_{1,3}$, then G is Hamiltonian.

Proof : Suppose that the Theorem is not true

let 'G' be a complete and locally Complete graph on at least three vertices,

which does not contain an induced $K_{1,3}$, but which is not Hamiltonian.

Clearly, 'G' contains a cycle.

Let 'C' be a largest cycle in 'G'.

Then, 'C' does not span 'G' and, since 'G' is complete, there exists a vertex ν, not on 'C', which is adjacent to a vertex u, lying on 'C'.

Let u_1 and u_2 be the vertices neighbouring 'u', on the cycle 'C'.

Since 'G' is locally complete, there exists a path 'P', in G[N(u)],

From 'ν' to the one of u_1 or u_2,

which does not include the other.

Without loss of generality,

we shall suppose that 'P' is a path from 'ν' to u_1 and that $u_2 \notin P$.

Now,

if $P \cap C = \{u_1\}$, then, by attaching 'P' to 'C' at u_1 and ν,

we could obtain a cycle larger than C.

Hence, we may assume that $P \cap C$ contains vertices other than u_1.

Also, we cannot have 'ν' adjacent to either u_1 or u_2, without producing

a cycle larger than 'C'.

Thus, since $\{u, u_1, u_2, \nu\}$ cannot induce a $K_{1,3}$ in 'G', then it must be that u_1u_2 is an edge of 'G'.

For the purpose of this proof,

we shall define a singular vertex to be a vertex

$w \in P \cap C - \{u1\}$, such that neither of the vertices, neighboring 'w' in C, belongsto N(u).

We shall consider two cases:

Case 1. Every vertex in $P \cap C - \{u_1\}$ is singular.

Then, for any vertex

$w \in P \cap C - \{u_1\}$, w is adjacent to u, but neither of the vertices w_1 and w_2,

neighboring 'w' on 'C', belongs to N(u).

Thus, since $\{w, w_1, w_2, u\}$ cannot induce a $K_{1,3}$ in 'G', then it must be that w_1w_2 is an edge of 'G'.

Now, traverse C, starting at u_2 and moving away from u and for each vertex

$w \in P \cap C - \{u_1\}$, by-pass w, by taking the edge w_1w_2.

Continue, until the vertex u_1 is reached. Then, follow P from u_1 to ν then to u and finish at u_2.

Then, we have passed through each vertex of $C \cup P$, exactly once, and have

thus constructed a cycle larger than C.

Case 2: $P \cap C - \{u1\}$ contains non singular vertices.

Then, follow P from ν toward u_1, until the first nonsingular vertex w is reached. Let w_1 and w_2 be the vertices neighbouring w along C.

Then, at least one of w_1 and w_2 is adjacent to u.

Without loss of generality, suppose that w_1 is adjacent to u.

Now,

form a new cycle C^1 , containing exactly the same vertices as C, as follows.

Delete the edges ww1, uu1 and uu2 and add the edges wu,w_1u and u_1u_2.

Note that if w is a neighbour of u_1 or u_2, then not all of these edges may be distinct

(e.g., if w1 = u1, then uu1 = uw1).

But now, the vertices neighbouring u in $C^1$are w and w1, and the subpath P^1 of P, from w to ν, does not include w_1

(as else w1, being a nonsingular vertex, would have been chosen earlier, instead of w).

Moreover, from the choice of w, it follows that P^1 cannot contain any

Non singular vertex with respect to C^1 and w (in the place of u1). Thus, relative

to P^1 and C^1 , we are back to the Case 1. Hence, in any case, C cannot have

been a largest cycle and, with this contradiction, Hence the Theorem is proved.

Remark. The above Theorem does not provide a necessary condition. For example,

let us consider the graph G = (V , E), where

V = $\{v_1, v_2, \ldots, v_6\}$

and E = $\{v_1v_2, v_1v_6, v_2v_3, v_2v_4, v_2v_5, v_2v_6, v_3v_4, v_3v_6, v_4v_5, v_4v_6, v_5v_6\}$.

Obviously, this graph is Complete, locally Complete, Hamiltonian, but

$G[\{v_1, v_2, v_3, v_5\}]$ is isomorphic to $K_{1,3}$.

If L(G) is the line graph of a graph 'G', then it is well known that L(G)

cannot contain K1,3 as an induced sub graph. Thus, we have the following

Corollary 1. Every complete and locally complete line graph, on $n \geq 3$

vertices, is Hamiltonian.

Corollary 2. If every edge of a complete graph 'G' lies in a triangle, then L(G)

is Hamiltonian.

Proof. If every edge of G lies in a triangle, then L(G) is locally complete and,

by Corollary 1, L(G) is Hamiltonian.

Corollary 3. If G is a complete and locally complete graph, on $n \geq 3$ vertices, then L(G) is Hamiltonian.

Proof. If G is complete and locally complete , on at least three vertices,

then every edge of G must lie in a triangle and, hence, L(G) is Hamiltonian

Corollary 4. If G is a complete graph with $\delta(G) \geq 3$, then L(L(G)) is Hamiltonian.

Corollary 5. If G is Hamiltonian, then L(G) is Hamiltonian.

Proof : This is a nice, basic result to see if a line graph is Hamiltonian. A graph is Hamiltonian if there exists a Hamiltonian cycle in the graph. It may be easier to find a Hamiltonian cycle in G than L(G), but from this proposition, we would get that L(G) is Hamiltonian.

3. REFERENCES

[1] Bondy J.A. and V. Chvátal, A Method in Graph Theory, Discr. Math. 15 1976), pp 111-136.

[2] Dirac G.A., Some Theorems on Abstract Graphs, Proc. Lond. Math. Soc. 2 (1952), pp 69-81.

[3] Garey M.R and D.S. Jhonson, Computers and Intractability: A Guide to the Theory of NPCompleteness, W.H. Freeman and Company, New York.

[4] O. Ore, Note on Hamiltonian Circuits, Am. Mat. Monthly 67 (1960), pp 55.

[5] West D.B, Introduction to Graph Theory, Prentice-Hall, Inc., New Jersey.

[6] Williams – Nash, C.St.J.A. "Hamiltonian Arcs and Circuits" in recent trends in graph theory ed. By M. Capobianco et al., Springer – verlag, Berlin, 1971.

[7] V.Chvatal "New directions in Hamiltonian Graph theory" In New directions in graph theory ed. By F.Harary, Academic press, N.Y.London, 1973.

[8] Venu Madhava Sarma.S and T.V. Pradeep Kumar International Journal of Mathematical Archive-2(12), 2011,Page 2538-2542.

[9] Venu Madhava Sarma.S International Journal of Computer Application, Issue 2, Volume 1(February 2012) , Page 21-31.

[10] Venu Madhava Sarma.S and T.V.Pradeep Kumar International J. of Math. Sci. & Engg. Appls. (IJMSEA) ISSN 0973-9424, Vol. 6 No. III (May, 2012), pp. 47-54.

[11] Venu Madhava Sarma.S and T.V.Pradeep Kumar proceedings of Two Day UGC National seminar on "Modern Trends in Mathematics and Physical Sciences (NSMTMPS – 2012) dated 20th , 21st Jan, 2012.

[12] D. P. Geller, *The square root of a digraph,* J. Combinatorial Theory, 5 (1968), 320-321.

[13] F. Harary, *Graph Theory* , Addison-Wesley, Reading, Mass., 1969.

[14] A. Mukhopadhyay, *The square root of a graph,* J. Combinatorial Theory, 2 (1967), 290-295.