Inverse Flexible Weibull Extension Distribution

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ABSTRACT
In this paper, a new two parameters model is introduced. We called it the inverse flexible Weibull extension (IFW) distribution. Several properties of this distribution have been discussed. The maximum likelihood estimators of the parameters are derived. Two real data sets are analyzed using the new model, which show that the new model fits the data better than some other very well known models.

Keywords
Inverse Weibull distribution; Hazard function; Moments; Maximum likelihood estimators; Median and mode.

1. INTRODUCTION
The Weibull distribution [1] is one of the most important and famous distributions used in the modeling of lifetimes of components of engineering applications, physical systems and many different fields. In previous years, many authors provided many new extensions for the Weibull distribution. Mudholkar and et al. [2] proposed the exponentiated Weibull distribution. The inverse Weibull distribution is discussed by Drapella[3], Mudholkar and Kollia[4] and Jiang et al.[5], among others. The inverse Weibull is the distribution of the reciprocal of a random variable, which has a Weibull distribution. A three parameter modified Weibull extension is proposed by Xie et al. [6]. Sarhan et al.[7] has defined the exponentiated modified Weibull extension distribution. Bebbington et al.[8] introduced a new two parameter distribution referred to as a flexible Weibull extension, which has a hazard function that can be increasing, decreasing or bathtub shaped. A flexible Weibull extension distribution has cumulative distribution function (cdf) given by

$$G(x) = \left[1 - e^{-e^{\alpha x - \beta/x}}\right],\ x > 0, \tag{1}$$

and its probability density function (pdf) takes the following form

$$g(x) = \left(\alpha + \frac{\beta}{x^2}\right) e^{\alpha x - \beta/x} e^{-e^{\alpha x - \beta/x}},\ x > 0. \tag{2}$$

El-Gohary et al. [9] proposed the exponentiated flexible Weibull extension (EFW) distribution. In this paper we propose a new two parameter distribution which is the distribution of the reciprocal of a random variable has the flexible Weibull extension distribution as was done for the inverse weibull (IW) distribution. We referred to it by the inverse flexible Weibull extension (IFW) distribution.

The paper is organized as follows. In Section 2, we present the IFW distribution, and provide its cumulative distribution function, the probability density function , the survival function and the hazard function. Some statistical properties such as the quantile , the median, the mode and the moments are obtained in Section 3. Section 4 obtains the parameter estimation using MLE method. In Section 5, a numerical result are obtained by using two real data sets. Finally, a conclusion for the results is given in Section 6.

2. INVERSE FLEXIBLE WEIBULL EXTENSION DISTRIBUTION
In this section, we introduce the inverse flexible Weibull extension distribution.

2.1 IFW Specifications
In this subsection, we define the inverse flexible Weibull extension distribution according to the following theorem.

Theorem 1. *Let a non-negative random variable Y has the flexible Weibull extension distribution, symbolically we write $Y \sim FW(\alpha, \beta)$. Define a new random variable $X = 1/Y$, then the random variable X has the inverse flexible Weibull extension distribution, symbolically we write $X \sim IFW(\alpha, \beta)$. The cumulative distribution function and the probability density function of X are respectively given by*

$$F_X(x) = e^{-e^{\alpha/x - \beta x}},\ x > 0,\ \alpha,\ \beta > 0, \tag{3}$$

and

$$f_X(x) = \left(\beta + \frac{\alpha}{x^2}\right) e^{\alpha/x - \beta x} e^{-e^{\alpha/x - \beta x}},\ x > 0,\ \alpha,\ \beta > 0. \tag{4}$$

Proof: Since

$$\begin{aligned} \boldsymbol{F_X(x)} &= \mathbf{P}(\mathbf{X} \le \mathbf{x}) = \mathbf{P}\left(\mathbf{Y} \ge \frac{\mathbf{1}}{\mathbf{x}}\right) \\ &= \mathbf{1} - \mathbf{P}\left(\mathbf{Y} < \frac{\mathbf{1}}{\mathbf{x}}\right) = \mathbf{1} - \boldsymbol{G}\left(\frac{\mathbf{1}}{\boldsymbol{x}}\right). \end{aligned} \tag{5}$$

Substituting from (1) into (5), we find (3). By derivation the cdf of X given in (3) with respect to x, we find the pdf of X given in (4), which complete the proof.

Since the cdf of IFW is in closed form, we can use it to generate simulated data by using the following formula

$$\mathrm{x} = \frac{1}{2\beta}\left[-\ln\left(-\ln(\mathrm{U})\right) + \sqrt{\left[\ln\left(-\ln(\mathrm{U})\right)\right]^2 + 4\alpha\beta}\right],$$

where U is a random variable which follows a standard uniform distribution on (0,1) interval.

2.2 Survival and Hazard Rate Functions
If $\mathbf{X} \sim \mathbf{IFW}(\boldsymbol{\alpha}, \boldsymbol{\beta})$, then the survival function and the hazard rate function of $\mathbf{X}$ are given respectively by

$$S(x) = 1 - F(x) = 1 - e^{-e^{\alpha/x - \beta x}}, \tag{6}$$

and

$$h(x) = \frac{f(x)}{S(x)} = \frac{\left(\beta + \frac{\alpha}{x^2}\right) e^{\alpha/x - \beta x} e^{-e^{\alpha/x - \beta x}}}{1 - e^{-e^{\alpha/x - \beta x}}}. \tag{7}$$

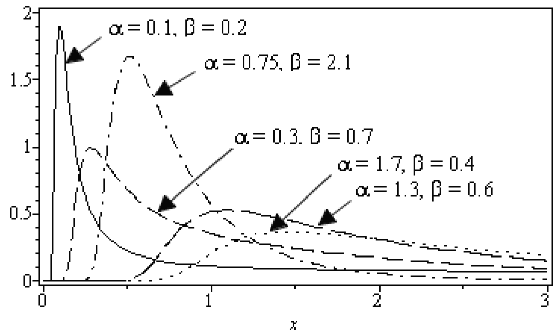

Fig 1: The pdf of the IFW distribution at different values of its parameters.

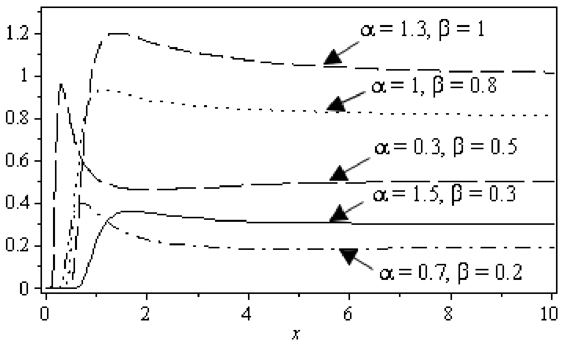

Fig 2: The hazard function of the IFW distribution at different values of its parameters.

3. STATISTICAL PROPERTIES

In this section, we will derive some of statistical properties for the IFW, specially moments, modes, quantiles and median.

3.1 Quantile and Median of IFW

In this subsection, we will present the forms of the quantile, the mode and the median of IFW as closed forms. The quantile x_q of the $X\sim \mathrm{IFW}(\alpha,\beta)$ is given by

$$x_q = \frac{1}{2\beta}\left[-ln\left(-\ln(q)\right)+\sqrt{\left[ln\left(-ln(q)\right)\right]^2+4\alpha\beta}\right], 0 < q < 1. \quad (8)$$

Sitting $q=\frac{1}{2}$ in (8), we get the median of $\mathrm{IFW}(\alpha,\beta)$ distribution as

$$Med(X) = \frac{1}{2\beta}\left[-ln\left(-\ln(\tfrac{1}{2})\right)+\sqrt{\left[ln\left(-ln(\tfrac{1}{2})\right)\right]^2+4\alpha\beta}\right]. \quad (9)$$

3.2 The Mode

In this subsection, we will derive the mode of the $\mathrm{IFW}(\alpha,\beta)$ distribution by derivation its pdf with respect to x and equate it to zero. The mode is the solution of the following equation with respect to x

$$\left(\frac{\alpha}{x^2}+\beta\right)\left[e^{\alpha/x-\beta x}-1\right]-\frac{2\alpha}{\alpha x+\beta x^2}=0\,. \quad (10)$$

It is not possible to get an analytic solution in x to (9) in the general case. It has to be obtained numerically by using methods such as fixed-point or bisection method.

3.3 The Moments

In this subsection, we will derive the r^{th} moments of the $\mathrm{IFW}(\alpha,\beta)$ distribution as infinite series expansion.

Theorem 2. *If $X\sim IFW(\alpha,\beta)$, then the r^{th} moment of X is given by*

$$\mu^{(r)} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^j\alpha^k\Gamma(r-k-1)}{k!\ j!\ \beta^{r-k-1}(j+1)^{r-2k+1}}\left[\frac{(r-k)(r-k-1)}{\beta}+\alpha(j+1)^2\right]. \quad (11)$$

Proof: The r^{th} moment of the positive random variable X with pdf $f(x;\alpha,\beta)$ is given by

$$\mu^{(r)} = \int_0^{\infty} x^r f(x;\alpha,\beta)dx. \quad (12)$$

Substituting from (4) into (12), we get

$$\mu^{(r)} = \int_0^{\infty} x^r\left(\beta+\frac{\alpha}{x^2}\right)e^{\alpha/x-\beta x}e^{-e^{\alpha/x-\beta x}}dx.$$

$$=\beta\int_0^{\infty} x^r e^{\alpha/x-\beta x}e^{-e^{\alpha/x-\beta x}}dx+\alpha\int_0^{\infty} x^{r-2}e^{\alpha/x-\beta x}e^{-e^{\alpha/x-\beta x}}dx.$$

Let

$$I_1=\int_0^{\infty}x^r e^{\alpha/x-\beta x}e^{-e^{\alpha/x-\beta x}}dx,\ \ I_2=\int_0^{\infty}x^{r-2}e^{\alpha/x-\beta x}e^{-e^{\alpha/x-\beta x}}dx.$$

Then

$$\mu^{(r)}=\beta I_1+\alpha I_2. \quad (13)$$

Using the series expansion of $e^{-e^{\alpha/x-\beta x}}$, one gets

$$I_1=\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\int_0^{\infty}x^r e^{(j+1)\alpha/x}e^{-(j+1)\beta x}dx.$$

Using the series expansion of $e^{(j+1)\alpha/x}$, we have

$$I_1=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^j\alpha^k(j+1)^k}{j!\ \ k!}\int_0^{\infty}x^{r-k}e^{-(j+1)\beta x}dx.$$

Using the substitution $y=(j+1)\beta x$ in the above integral, then we can get

$$I_1=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^j\alpha^k}{j!\ \ k!\,\beta^{r-k+1}(j+1)^{r-2k+1}}\int_0^{\infty}y^{r-k}e^{-y}dy.$$

From the definition of complete Gamma function, one gets

$$I_1=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^j\alpha^k\Gamma(r-k+1)}{j!\ \ k!\,\beta^{r-k+1}(j+1)^{r-2k+1}}. \quad (14)$$

Similarly, we can obtain I_2 as follows

$$I_2=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^j\alpha^k\Gamma(r-k-1)}{j!\ \ k!\,\beta^{r-k-1}(j+1)^{r-2k-1}}. \quad (15)$$

Substituting from (14) and (15) into (13), we find (11), which completes the proof.

3.4 Moment Generating Function

In this subsection, we derived the moment generating function of $\mathrm{IFW}(\alpha,\beta)$ distribution as infinite series expansion according to the following theorem.

Theorem 3. *If $\boldsymbol{X\sim IFW(\alpha,\beta)}$, then the moment generating function $\boldsymbol{M_X(t)}$ is given by*

$$M_X(t) = \sum_{r=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\left\{\frac{(-1)^j\alpha^k\Gamma(r-k-1)t^r}{r!\ k!\ j!\ \beta^{r-k-1}(j+1)^{r-2k+1}}\left[\alpha(j+1)^2 + \frac{(r-k)(r-k-1)}{\beta}\right]\right\}. \quad (16)$$

Proof: We start with the well known definition of the moment generating function given by

$$M_X(t) = \int_0^{\infty} e^{xt} f(x;\alpha,\beta)dx.$$

Using the series expansion of e^{xt} , we have

$$M_X(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\int_0^{\infty} x^r f(x;\alpha,\beta)dx = \sum_{r=0}^{\infty}\frac{t^r}{r!}\mu^{(r)}. \quad (17)$$

Substituting from (11) into (17), we find (16), which completes the proof.

4. ESTIMATION AND INFERENCE

In this section, we discuss the estimation of the model parameters by using the method of maximum likelihood. Also the asymptotic confidence intervals of these parameters will be derived.

4.1 Maximum Likelihood Estimators

We will derive the maximum likelihood estimators (MLEs) of the unknown parameters α and β. Assume that $x_1, x_2, \ldots, x_n$ be a random sample of size n from IFW(α, β), then the likelihood function l of this sample is

$$l = \prod_{i=1}^{n} f(x_i;\alpha,\beta). \quad (18)$$

Substituting from (4) into (18), we get

$$l = \prod_{i=1}^{n}\left\{\left(\beta + \frac{\alpha}{x_i^2}\right)e^{\alpha/x_i - \beta x_i}e^{-e^{\alpha/x_i-\beta x_i}}\right\}.$$

The log-likelihood function $L = \ln(l)$ is given by

$$L = \alpha\sum_{i=0}^{n}\frac{1}{x_i} - \beta\sum_{i=0}^{n}x_i - \sum_{i=0}^{n}e^{\alpha/x_i-\beta x_i} + \sum_{i=0}^{n}ln\left(\beta + \frac{\alpha}{x_i^2}\right).$$

The first partial derivatives of log of the likelihood L with respect to α and β are obtained as follows

$$\frac{\partial L}{\partial\alpha} = \sum_{i=1}^{n}\frac{1}{x_i} - \sum_{i=1}^{n}\frac{e^{\alpha/x_i-\beta x_i}}{x_i} + \sum_{i=1}^{n}\frac{1}{\beta x_i^2+\alpha}$$

and

$$\frac{\partial L}{\partial\beta} = -\sum_{i=1}^{n}x_i + \sum_{i=1}^{n}x_i e^{\alpha/x_i-\beta x_i} + \sum_{i=1}^{n}\frac{x_i^2}{\beta x_i^2+\alpha}.$$

The normal equations can be obtained by setting the first partial derivatives of L to zero's. That is, the normal equations take the following form:

$$\sum_{i=1}^{n}\frac{1}{x_i} - \sum_{i=1}^{n}\frac{e^{\alpha/x_i-\beta x_i}}{x_i} + \sum_{i=1}^{n}\frac{1}{\beta x_i^2+\alpha} = 0$$

and

$$-\sum_{i=1}^{n}x_i + \sum_{i=1}^{n}x_i e^{\alpha/x_i-\beta x_i} + \sum_{i=1}^{n}\frac{x_i^2}{\beta x_i^2+\alpha} = 0.$$

The normal equations do not have explicit solutions and they have to be obtained numerically.

4.2 Asymptotic Confidence Bounds

In this subsection, we derive the asymptotic confidence intervals of the unknown parameters α and β when $\alpha, \beta > 0$ [10].

The simplest large sample approach is to assume that the MLEs $(\hat{\alpha}, \hat{\beta})$ are approximately multivariate normal with mean (α, β) and covariance matrix I_0^{-1} see [11], where I_0^{-1} is the inverse of the observed information matrix which defined by

$$I_0^{-1} = \begin{pmatrix}\frac{\partial^2 L}{\partial\alpha^2} & \frac{\partial^2 L}{\partial\alpha\partial\beta}\\ \frac{\partial^2 L}{\partial\beta\partial\alpha} & \frac{\partial^2 L}{\partial\beta^2}\end{pmatrix}^{-1} = \begin{pmatrix}Var(\hat{\alpha}) & Cov(\hat{\alpha},\hat{\beta})\\ Cov(\hat{\beta},\hat{\alpha}) & Var(\hat{\beta})\end{pmatrix}. \quad (19)$$

The second partial derivatives include in I_0^{-1} are given as follows

$$\frac{\partial^2 L}{\partial\alpha^2} = -\sum_{i=1}^{n}\frac{e^{\alpha/x_i-\beta x_i}}{x_i^2} - \sum_{i=1}^{n}\frac{1}{(\beta x_i^2+\alpha)^2},$$

$$\frac{\partial^2 L}{\partial\alpha\partial\beta} = \sum_{i=1}^{n}e^{\alpha/x_i-\beta x_i} - \sum_{i=1}^{n}\frac{x_i^2}{(\beta x_i^2+\alpha)^2}$$

and

$$\frac{\partial^2 L}{\partial\beta^2} = -\sum_{i=1}^{n}x_i^2 e^{\alpha/x_i-\beta x_i} - \sum_{i=1}^{n}\frac{x_i^4}{(\beta x_i^2+\alpha)^2}.$$

We can derive the $(1-\alpha)100\%$ confidence intervals of the parameters α and β by using variance covariance matrix as in the following forms

$$\hat{\alpha} \pm Z_{\frac{\delta}{2}}\sqrt{Var(\hat{\alpha})}\quad ,\hat{\beta} \pm Z_{\frac{\delta}{2}}\sqrt{Var(\hat{\beta})}$$

where $Z_{\frac{\delta}{2}}$ is the upper $(\delta/2)$ th percentile of the standard normal distribution.

5. DATA ANALYSIS

In this section we analyze two real data sets to illustrate that the IFW can be a good lifetime model, comparing with many known distributions such as flexible Weibull, inverse Weibull, generalized inverse Weibull and exponentiated generalized inverse Weibull distributions (FW,IW,GIW,EGIW). We have fitted all selected distributions in each example, we calculated the Kolmogorov Smirnov (K-S) distance test statistic and its corresponding p-value, the log-likeihood values (L), Akaike information criterion (AIC), correct Akaike information criterion (CAIC) and Bayesian information criterion (BIC) test statistic.

Example 5.1. The data set in Table 1, gives the lifetimes of 50 devices that were provided by (Aarset, 1987) [12]. The MLEs of the unknown parameters and the Kolmogorov-Smirnov (K-S) test statistic with its corresponding p-value for the five tested models are given in Table 2. The fitted survival and failure rate functions are shown in Fig 3 and Fig 4 respectively. The K-S test statistic value for IFW model is 0.276 and the corresponding p-value is $\mathbf{7.38 \times 10^{-4}}$. We observe that the IFW model has the lowest K-S value and the highest p-value for these data among all the models considered except EGIW model, which means that the new model fits the data better than the FW, IW and GIW models.

Table 1. Life time of 50 devices, see Aarset (1987).

0.1	0.2	1	1	1	1	1	2	3	6
7	11	12	18	18	18	18	18	21	32
36	40	45	46	47	50	55	60	63	63
67	67	67	67	72	75	79	82	82	83
84	84	84	85	85	85	85	85	86	86

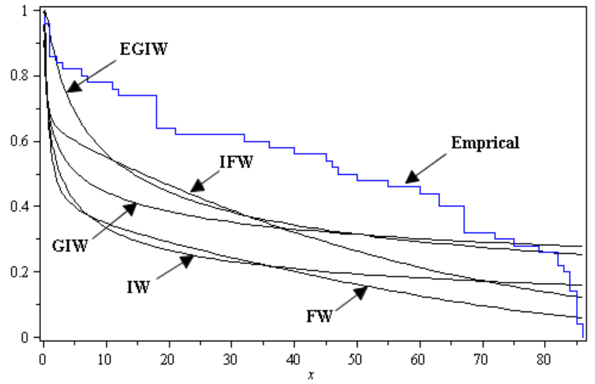

Fig 3: The empirical and fitted survival functions of selected models for Aarset data.

Substituting the MLEs of the unknown parameters into (19), we get estimation of the variance covariance matrix as the following:

$$I_0^{-1} = \begin{pmatrix} 1.2229 \times 10^{-3} & -4.4751 \times 10^{-6} \\ -4.4751 \times 10^{-6} & 1.0002 \times 10^{-5} \end{pmatrix}.$$

The approximate 95% two sided confidence intervals of the unknown parameters α and β are given respectively as [0.0964, 0.2335] and [0, 0.02998].

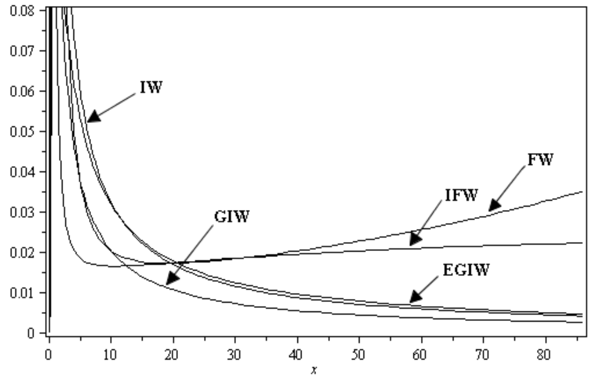

Fig 4: The fitted hazard functions of selected models for Aarset data.

In Fig 5 and Fig 6 we plot the profiles of the log-likelihood function of α and β for Aarset data. From Fig 5 and Fig 6 we show that the likelihood equations have a unique solution.

Table 2. The MLEs, K-S and p-values for Aarset data.

The model	MLE of the parameters $\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	K-S value	P-value
FW	0.012	0.70	- -	- -	0.469	4.29×10^{-9}
IW	1.043	0.397	- -	- -	0.435	5.95×10^{-9}
GIW	0.596	0.274	1.27	- -	0.324	3.72×10^{-5}
EGIW	1.008	0.61	2.14	0.75	0.254	2.47×10^{-3}
IFW	0.165	0.024	- -	- -	0.276	7.38×10^{-4}

Table 3. The log-likelihood, AIC, CAIC and BIC values for Aarset data.

The model	L	AIC	CAIC	BIC
FW	-281.07	566.14	566.396	569.964
IW	-287.48	580.951	581.473	586.687
GIW	-254.92	517.839	518.727	525.487
EGIW	-250.81	505.620	505.880	509.448
IFW	-242.57	488.914	489.169	492.738

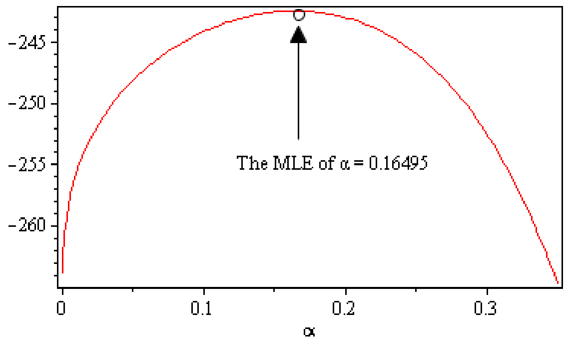

Fig 5: The profile of the log-likelihood function of α for Aarset data.

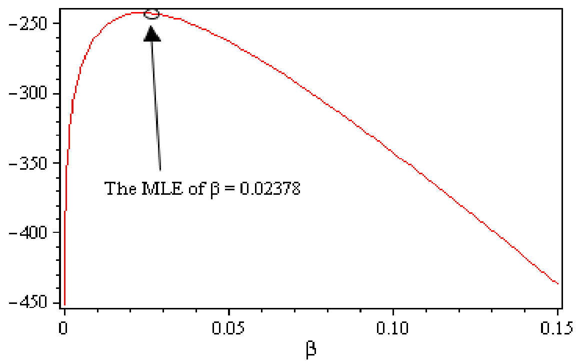

Fig 6: The profile of the log-likelihood function of β for Aarset data.

Example 5.2. Table 4, gives the data set corresponding to remission times (in months) of 128 bladder cancer patients reported in Lee and Wang (2003) [13]. The fitted survival and failure rate functions are shown in Fig 7 and Fig 8 respectively. From Fig. 7. we can observed that, the IFW distribution fits the data set better than all other distributions considered here, because its fitted curve is closer to the empirical curve. In fact, based on the values of the L, AIC, BIC, CAIC and K-S test statistic given in Table 5 and Table 6, we observe that the IFW distribution provides the best fit for these data among all the models used here. To show that the likelihood equations have a unique solution, we plot the profiles of the log-likelihood function of α and β for Lee and Wang data. in Fig 7 and Fig 8.

Table 4. Remission times of 128 bladder cancer patients, see Lee and Wang.

0.08	2.09	3.48	4.87	6.94	8.66	13.11	23.63
0.20	2.23	3.52	4.98	6.97	9.02	13.29	0.40
2.26	3.57	5.06	7.09	9.22	13.80	25.74	0.50
2.46	3.64	5.09	7.26	9.47	14.24	25.82	0.51
2.54	3.70	5.17	7.28	9.74	14.76	26.31	0.81
2.62	3.82	5.32	7.32	10.06	14.77	32.15	2.64
3.88	5.32	7.39	10.34	14.83	34.26	0.90	2.69
4.18	5.34	7.59	10.66	15.96	36.66	1.05	2.69
4.23	5.41	7.62	10.75	16.62	43.01	1.19	2.75
4.26	5.41	7.63	17.12	46.12	1.26	2.83	4.33
5.49	7.66	11.25	17.14	79.05	1.35	2.87	5.62
7.87	11.64	17.36	1.40	3.02	4.34	5.71	7.93
1.46	18.10	11.79	4.40	5.85	8.26	11.98	19.13
1.76	3.25	4.50	6.25	8.37	12.02	2.02	13.31
4.51	6.54	8.53	12.03	20.28	2.02	3.36	12.07
6.76	21.73	2.07	3.36	6.93	8.65	12.63	22.69

Table 5. The MLEs, K-S and p-values for Lee and Wang data.

The model	MLE of the parameters				K-S value	P-value
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$		
FW	0.054	0.915	- -	- -	0.390	1.1×10^{-17}
IW	16.14	0.464	- -	- -	0.503	4.0×10^{-29}
GIW	0.75	0.34	1.79	- -	0.369	7.2×10^{-16}
EGIW	1.006	0.5	1.05	2	0.608	2.4×10^{-42}
IFW	0.126	0.143	- -	- -	0.333	5.3×10^{-13}

Table 6. The log-likelihood, AIC, CAIC and BIC values for Lee and Wang data.

The model	L	AIC	CAIC	BIC
FW	-525.53	1055.07	1055.16	1060.77
IW	-500.12	1004.25	1004.33	1009.94
GIW	-495.18	966.362	996.56	1004.92
EGIW	-488.05	984.09	984.42	995.5
IFW	-453.61	911.22	911.31	916.92

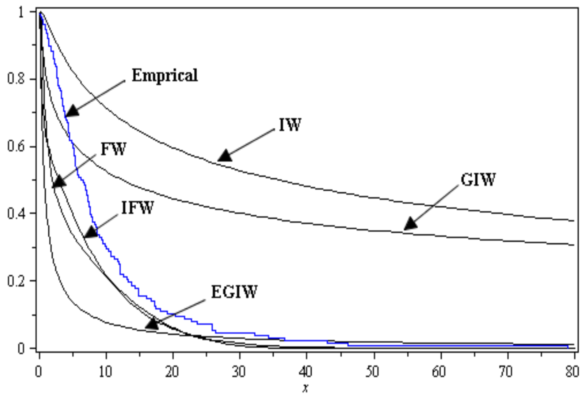

Fig 7: The empirical and fitted survival functions of selected models for Lee and Wang data.

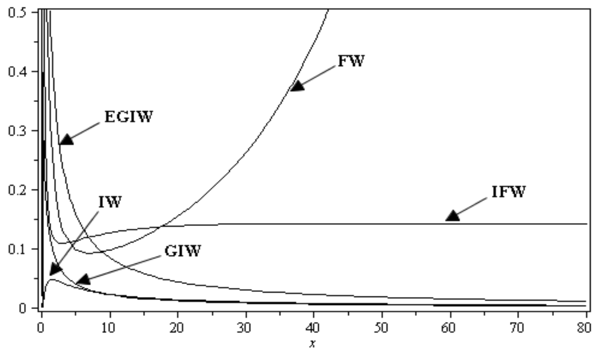

Fig 8: The fitted hazard functions of selected models for Lee and Wang data.

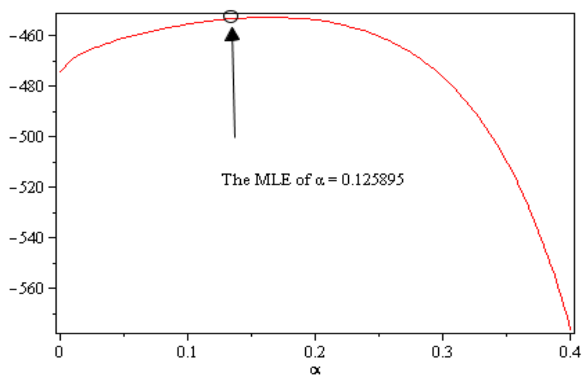

Fig 9: The profile of the log-likelihood function of α for Lee and Wang data.

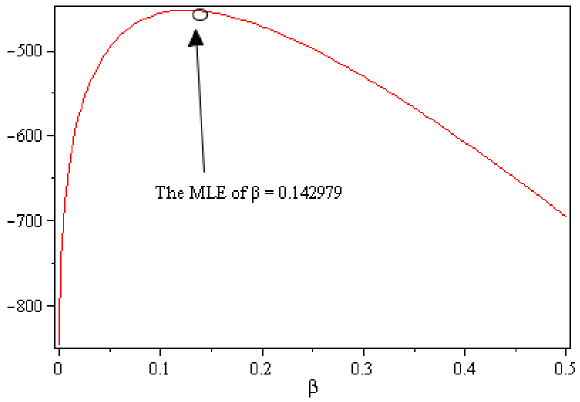

Fig 10: The profile of the log-likelihood function of β for Lee and Wang data.

Substituting the MLEs of the unknown parameters into (19), we get estimation of the variance covariance matrix as the following:

$$I_0^{-1} = \begin{pmatrix} 8.624 \times 10^{-4} & -3.461 \times 10^{-5} \\ -3.461 \times 10^{-5} & 1.407 \times 10^{-4} \end{pmatrix}.$$

The approximate 95% two sided confidence intervals of the unknown parameters α and β are given respectively as [0.0683, 0.1834] and [0.1197, 0.1662].

6. CONCLUSIONS

In this paper, we proposed a new two parameters model we called it the inverse flexible Weibull extension distribution. Some statistical properties of this distribution have been derived and discussed. The quantile, median, and mode of IFW are derived in closed forms. The maximum likelihood estimators of the parameters are derived and we obtained the observed Fisher information matrix. Two real data sets are analyzed using the new distribution and it is compared with the flexible Weibull**,** inverse Weibull, generalized inverse Weibull and exponentiated generalized inverse Weibull distributions. It is evident from the comparisons that the new distribution is the best distribution for fitting these data sets compared to other distributions considered here.

7. REFERENCES

[1] Weibull, W. A. 1951. Statistical distribution function of wide applicability. Journal of Applied Mechanics, 18, 293-6.

[2] Mudholkar, G. S., Srivastava, D. K. 1993. Exponentiated Weibull family for analyzing bathtub failure-rate data. IEEE Transactions on Reliability, 42, 299-302.

[3] Drapella, A. 1993. Complementary Weibull distribution unknown or just forgotten. Qual Reliab Eng Int, 9:383-385.

[4] Mudholkar, G. S., Kollia, G. D. 1994. Generalized Weibull family: a structural analysis. Commun Stat Ser A, 23:1149-1171.

[5] Jiang, R., Zuo, M. J. and Li, H. X. 1999. Weibull and Weibull inverse mixture models allowing negative weights". Reliab Eng Syst Saf, 66:227-234.

[6] Xie, M., Tang, Y. and Goh, T. N. 2002. A modified Weibull extension with bathtub-shaped failure rate function". Reliability Engineering and System Safety, 76, 279--285.

[7] Sarhan, A. M., Apaloo. J. 2013. Exponentiated modified Weibull extension distribution. Reliability Engineering and System Safety, 112, 137-144.

[8] Bebbington, M., Lai, C. D. and Zitikis, R. 2007. A flexible Weibull extension. Reliability Engineering and System Safety, 92, 719-726.

[9] El-Gohary, A., EL-Bassiouny. A. H. and El-Morshedy, M. Exponentiated Flexible Weibull Extension Distribution. Submitted. Jokull (to appear).

[10] Wongo, D. R. 1993. Maximum likelihood methods for fitting the Burr Type XII distribution to multiply (progressively) censored life test data", Metrika 40, 203-210.

[11] Garg, M. L., Rao, B. R. and Redmond, K. 1970. Maximum likelihood estimation of the parameters of the Gompertz survival function. J. Roy. Stat. Soc. Ser. Appl. Stat. 19, 152-159.

[12] Aarset, M. V. 1987. How to identify bathtub hazard rate. IEEE Transactions on Reliability, 36, 106-108.

[13] LEE, E. T., WANG, J. W. 2003 . Statistical Methods for Survival Data Analysis . Wiley, New York, 3rd edition.