Solving Polluted Lakes System by using Perturbation-Iteration Method

M. Khalid
Department of Mathematical Sciences
Federal Urdu University of Arts, Sciences & Techonology
University Road, Karachi-75300, Pakistan

Mariam Sultana
Department of Mathematical Sciences
Federal Urdu University of Arts, Sciences & Technology
University Road, Karachi-75300, Pakistan

Faheem Zaidi
Department of Mathematical Sciences
Federal Urdu University of Arts, Sciences & Technology
University Road, Karachi-75300, Pakistan

Fareeha Sami Khan
Department of Mathematical Sciences
Federal Urdu University of Arts, Sciences & Technology
University Road, Karachi-75300, Pakistan

ABSTRACT

Water pollution is a major global problem that requires ongoing evaluation and revision of water resource policies at all levels, in order to create a healthy living environment. Differential equations are an effective way to analyze such situations. In this paper a system of linear equations with interconnecting pipes is considered for analyzing the pollution of system of lakes through differential equations. Perturbation-iteration method is used to compute an approximate solution of three input models i.e. periodic, linear step model and exponentially decaying model. The fourth order Runge-Kutta method ($RK4$) numerical solution of the lakes system problem is used as a reference to compare with the analytical approximations showing the high accuracy of the results.

Keywords:

Water Pollution, Pollution of System of Lakes, Perturbation Iteration Method, Analytical Approximations

1. INTRODUCTION

Perturbation methods are being used effectively for over many decades in solving many algebraic equations, differential equations and integro-differential equations. They have also opened new doors to understand the structure of many applied problems. But still a very major problem with this method is the selection of parameter. This small parameter selection whether introduced artificially or it is the part of physical problem, always restricts the range of validation of the results.

To overcome this parameter selection problem, Boyaci & Pakdemirli [1–3] developed a new perturbation-iteration method for algebraic equations. Then it was applied to differential equations and system of non linear differential equations [4] and obtained accurate convergence compared to exact solutions if available. Recently this method has been applied on many different problems and differential equations such as Bratu type equations, linear and nonlinear differential equations and systems, heat equations, Fredholm and Volterra integral equations and obtained satisfactory results [4–8] Basic idea of perturbation method is to combine Taylor series expansion and perturbation expansion and generate a new iteration series. [1, 2] introduced expansion and correction terms of only first derivatives in the Taylor Series expansion, i.e. $n = 1$, $m = 1$ and one correction term in the perturbation.

The algorithm is named $PIA(1, 1)$. Consider the following system of first-order differential equations.

$$H_k(\dot{x}_k, x_j, \epsilon, t) = 0; \ k = 1, 2, .., K; \ j = 1, 2, .., K \quad (1)$$

where k is a representative of the number of differential equations in the system. For example, $k = 1$ for a single equation. In the open form, the system of equations is

$$
\begin{aligned}
H_1 &= H_1(\dot{x}_1, x_1, x_2, x_3, ..., x_K, \epsilon, t) = 0 \\
H_2 &= H_2(\dot{x}_2, x_1, x_2, x_3, ..., x_K, \epsilon, t) = 0 \\
H_3 &= H_3(\dot{x}_3, x_1, x_2, x_3, ..., x_K, \epsilon, t) = 0 \\
&\quad . \\
&\quad . \\
&\quad . \\
H_k &= H_k(\dot{x}_k, x_1, x_2, x_3, ..., x_K, \epsilon, t) = 0
\end{aligned}
\quad (2)
$$

Assume an approximate solution of the system

$$x_{k,n+1} = x_{k,n} + \epsilon x_{k,n}^c \quad (3)$$

with one correction term in the perturbation expansion. The subscript n represents the nth iteration over this approximate solution. The system can be approximated with a Taylor series expansion in the neighborhood of $\epsilon = 0$ as

$$H_k = \sum_{m=0}^{M} \frac{1}{m!} \left[\left(\frac{d}{d\epsilon} \right)^m H_k \right]_{\epsilon=0} \epsilon^m; \ k = 1, 2, .., K \quad (4)$$

where

$$\frac{d}{d\epsilon} = \frac{\partial \dot{x}_{k,n+1}}{\partial \epsilon}\frac{\partial}{\partial \dot{x}_{k,n+1}} + \sum_{j=1}^{K}\left(\frac{\partial x_{j,n+1}}{\partial \epsilon}\frac{\partial}{\partial x_{j,n+1}}\right) + \frac{\partial}{\partial \epsilon} \quad (5)$$

is defined for the $(n+1)^{th}$ iterative equations

$$F_k(\dot{x}_{k,n+1}, x_{j,n+1}, \epsilon, t) = 0 \quad (6)$$

substituting Eq.(5) into Eq. (4), an iterative equation is obtained;

$$H_k = \sum_{m=0}^{M}\frac{1}{m!}\left[\left(\dot{x}_{k,n}^c\frac{\partial}{\partial \dot{x}_{k,n+1}} + \sum_{j=1}^{K}\left(x_{j,n}^c\frac{\partial}{\partial x_{j,n+1}}\right)+\right.\right.$$
$$\left.\left.\frac{\partial}{\partial \epsilon}\right)^m H_k\right]_{\epsilon=0} \times \epsilon^m = 0 \ where \ k = 1,2,..,K \quad (7)$$

Which is a first-order differential equation and can be solved for the correction terms $x_{k,n}^c$. Then, using Eq.(3), the $(n+1)^{th}$ iteration solution can be found. Iterations are terminated after a satisfactory approximation is obtained.

Note that for a more general algorithm, n correction terms instead of one can be taken in expansion (3) which would then be a $PIA(n,m)$ algorithm. The algorithm can also be used generally and be applied to a differential equation system having arbitrary order of derivatives. Within the scope of this paper, only a case $n = m = 1$ is considered for the sake of simplicity, because more Algebra is involved in constructing iteration solutions for $PIA(1,2)$ and $PIA(1,3)$ as compared to $PIA(1,1)$. We have computed the numerical results by the well-known symbolic software "Mathematica 9.0"

2. MODEL OF SYSTEM OF LAKES

Model of pollution of system of lakes is discussed in detail in [9]. Each lake is treated as a large chamber connected to channels or pipes between the compartments as shown in figure 1. Arrows represent the direction of flow through each channel or pipeline. When a pollutant enters the first lake, to know the rate of pollutant at which it enters the lake per unit time, $p(t)$ is introduced in the system of equations. The function $p(t)$ maybe a constant or it may vary with time.

Let the amount of pollution in each lake i be denoted by $f_i(t)$ at any time $t \geq 0$, where $i = 1,2,3$. It is being assumed that the amount of pollutant is equally distributed in each lake and the volume V_i of each lake i remain constant. Also the type of pollutant remains constant and not transforming into other kinds of pollutant. Then the concentration of the lake is given by

$$C_i(t) = \frac{f_i(t)}{V_i} \quad (8)$$

Initially each lake $f_i(t)$ is considered to be free of pollution i.e. $f_i(0) = 0$ for $i = 1,2,3$. So the following conditions are obtained:

$$\begin{aligned} Lake \quad &1: F_{13} = F_{21} + F_{31}\\ Lake \quad &2: F_{21} = F_{32}\\ Lake \quad &3: F_{31} + F_{32} = F_{13} \end{aligned} \quad (9)$$

Flux of pollutant flowing from lake i to lake j at any time t measures the rate of flow of the concentration of pollutant and is given
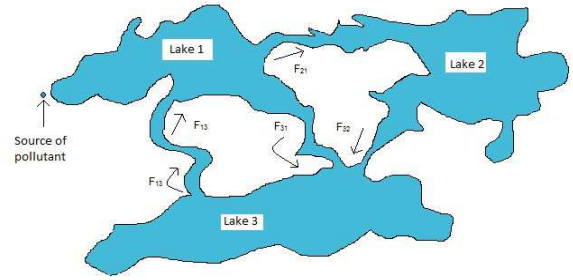

Fig. 1. System of three lakes connected with channels

by

$$Flux_{ji} = f_iC_i(t) = \frac{(flow \ rate)_{ji}f_i(t)}{V_i} \quad (10)$$

Where $(flow \ rate)_{ji}$ is a constant from lake i to lake j. It can easily be seen that, Rate of change of pollutant= Input rate $-$ output rate. From the above discussion, system of first-order differential equations is obtained as

$$\frac{dx}{dt} = \frac{F_{13}}{V_3}z(t) + p(t) - \frac{F_{31}}{V_1}x(t) - \frac{F_{21}}{V_1}x(t) \quad (11a)$$

$$\frac{dy}{dt} = \frac{F_{21}}{V_1}x(t) - \frac{F_{32}}{V_2}y(t) \quad (11b)$$

$$\frac{dz}{dt} = \frac{F_{31}}{V_1}x(t) + \frac{F_{32}}{V_2}y(t) - \frac{F_{13}}{V_3}z(t) \quad (11c)$$

Now consider the models under discussion in specific cases.

3. CASE 1: PERIODIC MODEL

The Periodic model proves that the pollutant introduced in the lake is periodical. Example of this first case can be a factory producing waste and dumping it in the lakes. As working hours of factory is in the day time so mixing of pollutant stops during night. Let $p(t) = 1 + \sin(t)$ in this case. Parameters are taken from Biazar [9]. $V_1 = 2900 \, mi^3$, $V_2 = 850 \, mi^3$, $V_3 = 1180 \, mi^3$, $F_{21} = 18 \, mi^3/year$, $F_{32} = 18 \, mi^3/year$, $F_{31} = 20 \, mi^3/year$, $F_{13} = 38 \, mi^3/year$. So system Eq.(11a to 11c) becomes

$$\frac{dx}{dt} = \frac{38}{1180}z(t) + (1 + \sin t) - \frac{38}{2900}x(t) \quad (12a)$$

$$\frac{dy}{dt} = \frac{18}{2900}x(t) - \frac{18}{850}y(t) \quad (12b)$$

$$\frac{dz}{dt} = \frac{20}{2900}x(t) + \frac{18}{850}y(t) - \frac{38}{1180}z(t) \quad (12c)$$

With the initial condition $x(0) = 0$, $y(0) = 0$, $z(0) = 0$. As the system is solved using $PIA(1,1)$ therefore introducing perturbation parameter, Eq. (12a to 12c) becomes

$$F_1 = \dot{x}_{n+1} - \frac{38}{1180}z_{n+1}\epsilon - (1+\sin t)\epsilon + \frac{38}{2900}x_{n+1} = 0$$
(13a)

$$F_2 = \dot{y}_{n+1} - \frac{18}{2900}x_{n+1}\epsilon + \frac{18}{850}y_{n+1} = 0 \qquad (13b)$$

$$F_3 = \dot{z}_{n+1} - \frac{20}{2900}x_{n+1}\epsilon - \frac{18}{850}y_{n+1}\epsilon + \frac{38}{1180}z_{n+1} = 0$$
(13c)

So for Eq.(7) this system becomes

$$\dot{x}_n + \frac{20}{2900}x_n + \frac{18}{2900}x_n + \dot{x}_n^c + \frac{38}{2900}x_n^c\epsilon -$$
$$\frac{38}{1180}z_n\epsilon - \epsilon\sin t - \epsilon = 0 \qquad (14a)$$

$$\dot{y}_n + \frac{18}{850}y_n + \dot{y}_n^c\epsilon + \frac{18}{850}y_n^c\epsilon - \frac{18}{2900}x_n\epsilon = 0 \qquad (14b)$$

$$\dot{z}_n + \frac{38}{1180}z_n + \dot{z}_n^c\epsilon + \frac{38}{1180}z_n^c\epsilon - \frac{20}{2900}x_n\epsilon -$$
$$\frac{18}{850}y_n\epsilon = 0 \qquad (14c)$$

For the convenience of calculations ϵ will be equal to one, but it's not necessary to take it equal to unity always. On taking $\epsilon = 1$, then Eq.(14a) to (14c) becomes

$$\dot{x}_n^c + \frac{38}{2900}x_n^c = -\dot{x}_n - \frac{20}{2900}x_n + \frac{18}{2900}x_n +$$
$$\frac{38}{1180}z_n + \sin t + 1 \qquad (15a)$$

$$\dot{y}_n^c + \frac{18}{850}y_n^c = -\dot{y}_n - \frac{18}{850}y_n + \frac{18}{2900}x_n \qquad (15b)$$

$$\dot{z}_n^c + \frac{38}{1180}z_n^c = -\dot{z}_n - \frac{38}{1180}z_n + \frac{20}{2900}x_n + \frac{18}{850}y_n \qquad (15c)$$

Initial guess is same as the initial condition i.e. $x_{10} = 0$; $y_{10} = 0$; $x_{10} = 0$. On solving with perturbation iteration method following iterations have been obtained

$$x_{11} = \frac{-1450e^{\frac{-19t}{1450}}}{3995459}\Big[2075311 - 2102861e^{\frac{19t}{1450}} +$$
$$27550e^{\frac{19t}{1450}}\cos t - 361e^{\frac{19t}{1450}}\sin t\Big]$$

$$y_{11} = 0$$

$$z_{11} = 0$$

$$x_{12} = \frac{-1450e^{\frac{-19t}{1450}}}{3995459}\Big[2075311 - 2102861e^{\frac{19t}{1450}} +$$
$$27550e^{\frac{19t}{1450}}\cos t - 361e^{\frac{19t}{1450}}\sin t\Big]$$

$$y_{12} = -\frac{425e^{-\frac{9t}{425}}}{1436778487093346}\Big[-120141838562743 +$$
$$195761040073452e^{\frac{199t}{24650}} - 75619920373334e^{\frac{9t}{425}} +$$
$$718862625e^{\frac{9t}{425}}\cos t + 20964552291e^{\frac{9t}{425}}\sin t\Big]$$

$$z_{12} = -\frac{2950e^{-\frac{19t}{590}}}{11374711823354683}\Big[-41842326531549 +$$
$$104858917223795e^{\frac{817t}{42775}} - 63017794035206e^{\frac{19t}{590}} +$$
$$1203342960e^{\frac{19t}{590}}\cos t + 26548645394e^{\frac{19t}{590}}\sin t\Big]$$

$$x_{13} = \frac{-1450e^{\frac{-19t}{1450}}}{3995459}\Big[2075311 - 2102861e^{\frac{19t}{1450}} +$$
$$27550e^{\frac{19t}{1450}}\cos t - 361e^{\frac{19t}{1450}}\sin t\Big] +$$
$$\frac{725e^{\frac{-19t}{590}}}{102853582882157243806670 9}\Big[$$
$$-2595663600067560019825 5 -$$
$$3102627271560408161932 5e^{\frac{817t}{42775}} +$$
$$56982594394546947738 0e^{\frac{19t}{590}} -$$
$$124242445099147529112 7te^{\frac{817t}{42775}} +$$
$$31432173273234020 0e^{\frac{19t}{590}}\cos t -$$
$$1837654860015062 0e^{\frac{19t}{590}}\sin t\Big]$$

$$y_{13} = -\frac{425e^{-\frac{9t}{425}}}{1436778487093346}\Big[-120141838562743 +$$
$$195761040073452e^{\frac{199t}{24650}} - 75619920373334e^{\frac{9t}{425}} +$$
$$718862625e^{\frac{9t}{425}}\cos t + 20964552291e^{\frac{9t}{425}}\sin t\Big]$$

$$z_{13} = -\frac{2950e^{-\frac{19t}{590}}}{11374711823354683}\Big[-41842326531549 +$$
$$104858917223795e^{\frac{817t}{42775}} - 63017794035206e^{\frac{19t}{590}} +$$
$$1203342960e^{\frac{19t}{590}}\cos t + 26548645394e^{\frac{19t}{590}}\sin t\Big] +$$
$$\frac{2655e^{-\frac{19t}{590}}}{14316408969641629826269 07}\Big[$$
$$-101136728960709162023 34 +$$
$$368012307712496904174 65e^{\frac{553t}{50150}} -$$
$$346191570509053423662 90e^{\frac{817t}{42775}} +$$
$$793152851503691403117 4e^{\frac{19t}{590}} +$$
$$70660689654119985e^{\frac{19t}{590}}\cos t -$$
$$4703622674393601e^{\frac{19t}{590}}\sin t\Big]$$

x_{14}, y_{14}, z_{14} are graphically shown in figures.

4. CASE 2: LINEAR INPUT MODEL:

Linear input describes the steady behavior of the pollutant addition into the lake. At time zero the pollutant concentration is also zero but as the time increases the addition of pollutant starts and remains steady afterwards. It can also be understood better by an example that if a manufacturing plant starts production and dump its raw wastage on a constant rate then $p(t) = 100t$. Eq. (11a) to Eq. (11c)
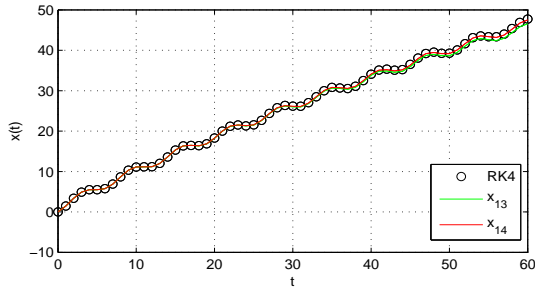
Fig. 2.   Graphical representation of pollutant dumping periodically in Lake 1 with Perturbation iteration and $RK4$ solutions
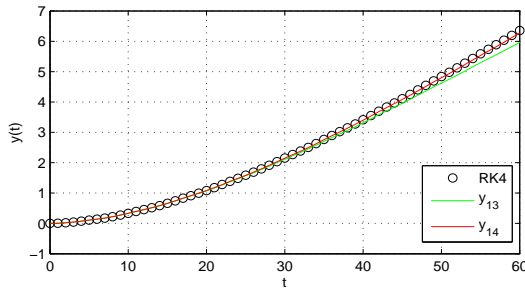

Fig. 3.   Graph of mixing pollutant in Lake 2. Comparison of $RK4$ and $PIA(1,1)$ approximation is shown.
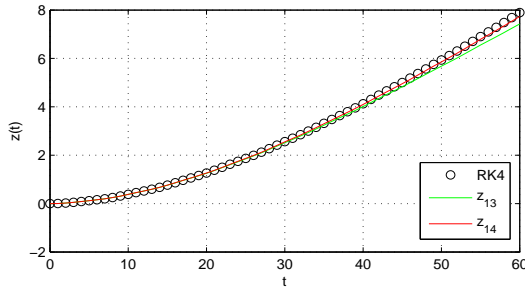

Fig. 4.   Graphically pollutant mixing periodically in Lake 3 is presented by $RK4$ and $PIA(1,1)$ approximations.

becomes

$$\frac{dx}{dt} = \frac{38}{1180}z(t) + 100t - \frac{38}{2900}x(t) \tag{16a}$$

$$\frac{dy}{dt} = \frac{18}{2900}x(t) - \frac{18}{850}y(t) \tag{16b}$$

$$\frac{dz}{dt} = \frac{20}{2900}x(t) + \frac{18}{850}y(t) - \frac{38}{1180}z(t) \tag{16c}$$

With the initial condition $x(0) = 0$, $y(0) = 0$, $z(0) = 0$. As the system is solved using $PIA(1,1)$ therefore introducing perturbation parameter, Eq. (16a) to (16c) becomes

$$F_1 = \dot{x}_{n+1} - \frac{38}{1180}z_{n+1}\epsilon - 100t\epsilon + \frac{38}{2900}x_{n+1} = 0 \tag{17a}$$

$$F_2 = \dot{y}_{n+1} - \frac{18}{2900}x_{n+1}\epsilon + \frac{18}{850}y_{n+1} = 0 \tag{17b}$$

$$F_3 = \dot{z}_{n+1} - \frac{20}{2900}x_{n+1}\epsilon - \frac{18}{850}y_{n+1}\epsilon + \frac{38}{1180}z_{n+1} = 0 \tag{17c}$$

So for Eq. (7) this system becomes

$$\dot{x}_n + \frac{20}{2900}x_n + \frac{18}{2900}x_n + \dot{x}_n^c\epsilon +$$
$$\frac{38}{2900}x_n^c\epsilon - \frac{38}{1180}z_n\epsilon - 100t\epsilon = 0 \tag{18a}$$

$$\dot{y}_n + \frac{18}{850}y_n + \dot{y}_n^c\epsilon + \frac{18}{850}y_n^c\epsilon - \frac{18}{2900}x_n\epsilon = 0 \tag{18b}$$

$$\dot{z}_n + \frac{38}{1180}z_n + \dot{z}_n^c\epsilon + \frac{38}{1180}z_n^c\epsilon - \frac{20}{2900}x_n\epsilon -$$
$$\frac{18}{850}y_n\epsilon = 0 \tag{18c}$$

On taking $\epsilon = 1$, then Eq. (18a) to Eq. (18c) becomes

$$\dot{x}_n^c + \frac{38}{2900}x_n^c = -\dot{x}_n - \frac{20}{2900}x_n - \frac{18}{2900}x_n +$$
$$\frac{38}{1180}z_n + 100t \tag{19a}$$

$$\dot{y}_n^c + \frac{18}{850}y_n^c = -\dot{y}_n - \frac{18}{850}y_n + \frac{18}{2900}x_n \tag{19b}$$

$$\dot{z}_n^c + \frac{38}{1180}z_n^c = -\dot{z}_n - \frac{38}{1180}z_n + \frac{20}{2900}x_n + \frac{18}{850}y_n \tag{19c}$$

Initial guess is same as the initial condition i.e. $x_{10} = 0$; $y_{10} = 0$; $x_{10} = 0$. On solving with perturbation iteration method following iterations have been obtained

$$x_{11} = \frac{145000e^{-\frac{19t}{1450}}}{361}\left[1450 - 1450e^{\frac{19t}{1450}} + 19e^{\frac{19t}{1450}}t\right]$$

$$y_{11} = 0$$

$$z_{11} = 0$$

$$x_{12} = \frac{145000e^{-\frac{19t}{1450}}}{361}\left[1450 - 1450e^{\frac{19t}{1450}} + 19e^{\frac{19t}{1450}}t\right]$$

$$y_{12} = \frac{42500e^{-\frac{9t}{425}}}{646551}\left[-2608225 + 6812100e^{\frac{199t}{24650}} - 4203875e^{\frac{9t}{425}} + 34029e^{\frac{9t}{425}}t\right]$$

$$z_{12} = \frac{590000e^{-\frac{19t}{590}}}{294937}\left[-17405 + 105125e^{\frac{817t}{42775}} - 87720e^{\frac{19t}{590}} + 817te^{\frac{19t}{590}}\right]$$

$$x_{13} = \frac{145000e^{\frac{-19t}{1450}}}{361}\left[1450 - 1450e^{\frac{19t}{1450}} + 19e^{\frac{19t}{1450}}t\right] +$$
$$\frac{725000e^{\frac{-19t}{590}}}{12682291}\left[1026895 + 11879125e^{\frac{817t}{42775}} - 12906020e^{\frac{19t}{590}} + 118465te^{\frac{817t}{42775}} + 70262te^{\frac{19t}{590}}\right]$$
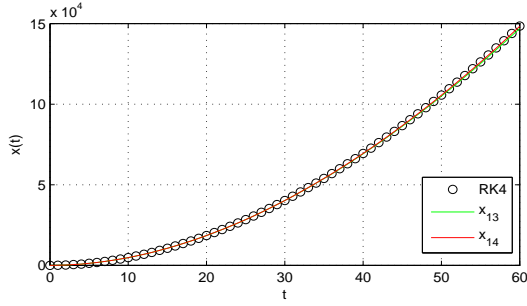
Fig. 5.   At different time intervals the addition of linear input pollution with Perturbation iteration method and $RK4$ method in Lake 1 is shown
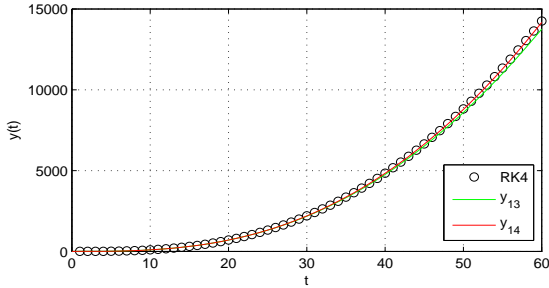

Fig. 6.   Linear input of pollutant in Lake 2 is shown at different time intervals and compared by $RK4$ and $PIA(1,1)$ approximations.
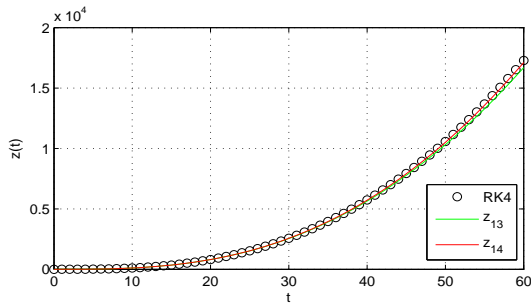

Fig. 7.   Graphical representation of linear input pollutant in Lake 3 with Perturbation iteration method and $RK4$ solutions

$$y_{13} = \frac{42500 e^{-\frac{9t}{425}}}{646551}\left[-2608225 + 6812100 e^{\frac{199t}{24650}} - 4203875 e^{\frac{9t}{425}} + 34029t e^{\frac{9t}{425}} \right]$$

$$z_{13} = \frac{590000 e^{-\frac{19t}{590}}}{294937}\left[-17405 + 105125 e^{\frac{817t}{42775}} - 87720 e^{\frac{19t}{590}} + 817t e^{\frac{19t}{590}} \right]$$

x_{14}, y_{14}, z_{14} are graphically shown in figures.

5.   CASE 3: EXPONENTIALLY DECAYING MODEL

This model is assumed when heavy dumping of pollutant is under consideration i.e. $p(t) = 200e^{-10t}$. Example of such a case is if

any industry situated in a city collects and stores its wastage and dumps it after a few days' period. So Eq.(11a) to Eq.(11c) takes the form

$$\frac{dx}{dt} = \frac{38}{1180}z(t) + 200 e^{-10t} - \frac{38}{2900}x(t) \qquad (20a)$$

$$\frac{dy}{dt} = \frac{18}{2900}x(t) - \frac{18}{850}y(t) \qquad (20b)$$

$$\frac{dz}{dt} = \frac{20}{2900}x(t) + \frac{18}{850}y(t) - \frac{38}{1180}z(t) \qquad (20c)$$

With the initial condition $x(0) = 0$, $y(0) = 0$, $z(0) = 0$. As the system is solved using $PIA(1,1)$ therefore introducing perturbation parameter, Eq. (20a) to (20c) becomes

$$F_1 = \dot{x}_{n+1} - \frac{38}{1180}z_{n+1}\epsilon + 200 e^{-10t}\epsilon + \frac{38}{2900}x_{n+1} = 0 \qquad (21a)$$

$$F_2 = \dot{y}_{n+1} - \frac{18}{2900}x_{n+1}\epsilon + \frac{18}{850}y_{n+1} = 0 \qquad (21b)$$

$$F_3 = \dot{z}_{n+1} - \frac{20}{2900}x_{n+1}\epsilon - \frac{18}{850}y_{n+1}\epsilon + \frac{38}{1180}z_{n+1} = 0 \qquad (21c)$$

So for Eq. (7) this system becomes

$$\dot{x}_n + \frac{20}{2900}x_n + \frac{18}{2900}x_n + \dot{x}_n^c + \frac{38}{2900}x_n^c\epsilon - \frac{38}{1180}z_n\epsilon + 200\epsilon e^{-10t} = 0 \qquad (22a)$$

$$\dot{y}_n + \frac{18}{850}y_n + \dot{y}_n^c\epsilon + \frac{18}{850}y_n^c\epsilon - \frac{18}{2900}x_n\epsilon = 0 \qquad (22b)$$

$$\dot{z}_n + \frac{38}{1180}z_n + \dot{z}_n^c\epsilon + \frac{38}{1180}z_n^c\epsilon - \frac{20}{2900}x_n\epsilon - \frac{18}{850}y_n\epsilon = 0 \qquad (22c)$$

On taking $\epsilon = 1$, then Eq. (22a) to Eq. (22c) becomes

$$\dot{x}_n^c + \frac{38}{2900}x_n^c = -\dot{x}_n - \frac{20}{2900}x_n - \frac{18}{2900}x_n + \frac{38}{1180}z_n - 200 e^{-10t} \qquad (23a)$$

$$\dot{y}_n^c + \frac{18}{850}y_n^c = -\dot{y}_n - \frac{18}{850}y_n + \frac{18}{2900}x_n \qquad (23b)$$

$$\dot{z}_n^c + \frac{38}{1180}z_n^c = -\dot{z}_n - \frac{38}{1180}z_n + \frac{20}{2900}x_n + \frac{18}{850}y_n \qquad (23c)$$

Initial guess is same as the initial condition i.e. $x_{10} = 0$; $y_{10} = 0$; $x_{10} = 0$. On solving with perturbation iteration method following iterations have been obtained

$$x_{11} = \frac{290000 e^{-10t}}{14481}\left[-1 + e^{\frac{14481t}{1450}} \right]$$

$$y_{11} = 0$$

$$z_{11} = 0$$

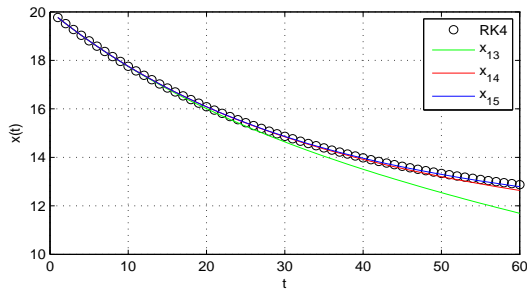$$x_{12} = \frac{290000 e^{-10t}}{14481}\left[-1 + e^{\frac{14481t}{1450}} \right]$$

Fig. 8. Heavy dumping of pollutant in Lake 1.Solutions obtained by Perturbation iteration method and $RK4$ method
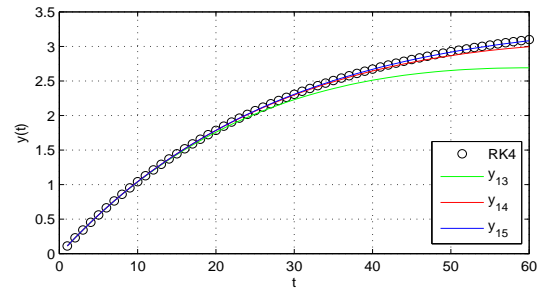
$$y_{12} = \frac{85000e^{-10t}}{1357930031}\Big[199 - 246177e^{\frac{4241t}{425}} + 245978e^{\frac{14481t}{1450}}\Big]$$

$$z_{12} = \frac{590000e^{-10t}}{69577975737}\Big[1634 - 854379e^{\frac{5881t}{590}} + 852745e^{\frac{14481t}{1450}}\Big]$$

$$x_{13} = \frac{290000e^{-10t}}{14481}\Big[-1 + e^{\frac{14481t}{1450}}\Big] + \frac{145000e^{-10t}}{43325022665842371}\Big[-13349780 + 3649817378205e^{\frac{5881t}{590}} - 3649804028425e^{\frac{14481t}{1450}} + 69577975737te^{\frac{14481t}{1450}}\Big]$$

$$y_{13} = \frac{85000e^{-10t}}{1357930031}\Big[199 - 246177e^{\frac{4241t}{425}} + 245978e^{\frac{14481t}{1450}}\Big]$$

$$z_{13} == \frac{590000e^{-10t}}{69577975737}\Big[1634 - 854379e^{\frac{5881t}{590}} + 852745e^{\frac{14481t}{1450}}\Big] + \frac{33271398000e^{-\frac{19t}{590}}}{2657041681} + \frac{1800e^{-\frac{19t}{590}}}{1357930031}\Big[-\frac{117410e^{-\frac{5881t}{590}}}{5881} - \frac{12345776550e^{\frac{553t}{50150}}}{553} + \frac{10521708950e^{\frac{817t}{42775}}}{817}\Big]$$

x_{14}, y_{14}, z_{14} and x_{15}, y_{15}, z_{15} are graphically shown in figures.

6. CONCLUSION

In this paper, a perturbation iteration method has been used to analyze the pollution rate in system of lakes. Results are compared by $RK4$ method as shown in figures. Three different types of pollutant have been assumed and the convergence of this system is checked that yielded satisfying results. This system of equations has already been analyzed by different methods such as HPM and VIM but this method is more accurate and easy to solve such type of systems

Fig. 9. Graphically shown heavy dumping of pollutant in Lake 2 with Perturbation iteration method and $RK4$ solutions
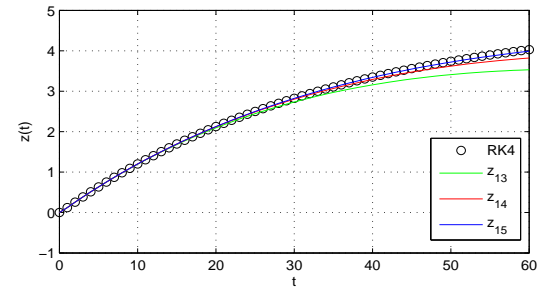

Fig. 10. Graphical comparison of heavy dumping of pollutant in Lake 3 by $RK4$ and $PIA(1,1)$.

because more than one corrected terms can accelerate the process of convergence. However we have only considered $PIA(1,1)$ and get the required convergence on second iteration in some models. PIA method obtains its coefficients using an easily computable straightforward procedure that can be implemented into programs like Matlab or Mathematica.

7. CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests regarding the publication of this paper.

8. ACKNOWLEDGMENT

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