Exact Traveling Wave Solutions for Fitzhugh-Nagumo (FN) Equation and Modified Liouville Equation

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ABSTRACT

In this paper, we employ the $\exp(-\varphi(\xi))$-expansion method to find the exact traveling wave solutions involving parameters of nonlinear evolution equations Fitzhugh-Nagumo (FN) equation and Modified Liouville equation. When these parameters are taken to be special values, the solitary wave solutions are derived from the exact traveling wave solutions. It is shown that the proposed method provides a more powerful mathematical tool for constructing exact traveling wave solutions for many other nonlinear evolution equations.

Keywords:

The $\exp(-\varphi(\xi))$-expansion method; Fitzhugh-Nagumo (FN) equation; Modified Liouville equation; Traveling wave solutions; Solitary wave solutions; Kink-antikink shaped.

AMS subject classifications: 35A05, 35A20, 65K99, 65Z05, 76R50, 70K70

1. INTRODUCTION

Many models in mathematics and physics are described by nonlinear differential equations. Nowadays, research in physics devotes much attention to nonlinear partial differential evolution model equations, appearing in various fields of science, especially fluid mechanics, solid-state physics, plasma physics, and nonlinear optics. Large varieties of physical, chemical, and biological phenomena are governed by nonlinear partial differential equations. One of the most exciting advances of nonlinear science and theoretical physics has been the development of methods to look for exact solutions of nonlinear partial differential equations. Exact solutions to nonlinear partial differential equations play an important role in nonlinear science, especially in nonlinear physical science since they can provide much physical information and more insight into the physical aspects of the problem and thus lead to further applications. Nonlinear wave phenomena of dispersion, dissipation, diffusion, reaction and convection are very important in nonlinear wave equations. In recent years, quite a few methods for obtaining explicit traveling and solitary wave solutions of nonlinear evolution equations have been proposed. A variety of powerful methods, tanh - sech method [1]-[3], extended tanh - method [4]-[6], sine - cosine method [7]-[9], homogeneous balance method [10, 11],F-expansion method [12]-[14], exp-function method [15, 16], trigonometric function series method [17], $(\frac{G'}{G})-$ expansion method [18]-[21], Jacobi elliptic function method [22]-[25], The $\exp(-\varphi(\xi))$-expansion method[26]-[28] and so on.
The objective of this article is to apply The $\exp(-\varphi(\xi))$-expansion method for finding the exact traveling wave solution of Fitzhugh-Nagumo (FN) equation and Modified Liouville equation which play an important role in biology and mathematical physics.
The rest of this paper is organized as follows: In Section 2, we give the description of The $\exp(-\varphi(\xi))$-expansion method In Section 3, we use this method to find the exact solutions of the nonlinear evolution equations pointed out above. In Section 4, conclusions are given.

2. DESCRIPTION OF METHOD

Consider the following nonlinear evolution equation

$$F(u, u_t, u_x, u_{tt}, u_{xx}, ...) = 0, \tag{1}$$

where F is a polynomial in $u(x,t)$ and its partial derivatives in which the highest order derivatives and nonlinear terms are involved. In the following,we give the main steps of this method
Step 1. We use the wave transformation

$$u(x,t) = u(\xi), \qquad \xi = x - ct, \tag{2}$$

where c is a positive constant, to reduce Eq.1to the following ODE:

$$P(u, u', u'', u''', ...) = 0, \tag{3}$$

where P is a polynomial in $u(\xi)$ and its total derivatives, while $' = \frac{d}{d\xi}'$.
Step 2. Suppose that the solution of ODE3 can be expressed by a polynomial in $exp(-\varphi(\xi))$ as follows

$$u(\xi) = a_m \left(exp(-\varphi(\xi))\right)^m + ..., \;\; a_m \neq 0, \tag{4}$$

where $\varphi(\xi)$ satisfies the ODE in the form

$$\varphi'(\xi) = exp(-\varphi(\xi)) + \mu exp(\varphi(\xi)) + \lambda, \tag{5}$$

the solutions of ODE 5 are
when $\lambda^2 - 4\mu > 0, \mu \neq 0,$

$$\varphi(\xi) = ln\left(\frac{-\sqrt{\lambda^2-4\mu}\; tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(\xi + C_1)\right) - \lambda}{2\mu}\right), \tag{6}$$

when $\lambda^2-4\mu>0, \mu=0$,

$$\varphi(\xi)=-ln\left(\frac{\lambda}{exp\left(\lambda\left(\xi+C_1\right)\right)-1}\right), \tag{7}$$

when $\lambda^2-4\mu=0, \mu\neq 0, \lambda\neq 0$,

$$\varphi(\xi)=ln\left(-\frac{2\left(\lambda\left(\xi+C_1\right)+2\right)}{\lambda^2\left(\xi+C_1\right)}\right), \tag{8}$$

when $\lambda^2-4\mu=0, \mu=0, \lambda=0$,

$$\varphi(\xi)=ln\left(\xi+C_1\right), \tag{9}$$

when $\lambda^2-4\mu<0$,

$$\varphi(\xi)=ln\left(\frac{\sqrt{4\mu-\lambda^2}\,tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\ \left(\xi+C_1\right)\right)-\lambda}{2\mu}\right), \tag{10}$$

where $a_m, ..., \lambda, \mu$ are constants to be determined later,

Step 3. Substitute Eq.4 along Eq.5 into Eq.3 and collecting all the terms of the same power $exp\left(-m\varphi(\xi)\right)$, $m=0,1,2,3,....$ and equating them to zero, we obtain a system of algebraic equations, which can be solved by Maple or Mathematica to get the values of a_i.

Step 4. substituting these values and the solutions of Eq.5 into Eq.3 we obtain the exact solutions of Eq.3.

3. APPLICATION

Here, we will apply the $\exp(-\varphi(\xi))$-expansion method described in sec.2 to find the exact traveling wave solutions and then the solitary wave solutions for the following nonlinear systems of evolution equations.

3.1 Example 1: Fitzhugh-Nagumo (FN) equation

The nonlinear well-known Fitzhugh-Nagumo (FN) equation reads [29], [30] and [34]

$$u_{xx}-u(1-u)(\alpha-u)-u_t=0, \tag{11}$$

where α is an arbitrary constant. When $\alpha=-1$, the FN equation reduces to the Newell-Whitehead (NW) equation. The FN equation 11 is an important nonlinear reaction-diffusion equation and usually is used to model the transmission of nerve impulses [31], [32], also is used in circuit theory, biology and the area of population genetics [33] as mathematical models. In addition, this equation arises in heat and mass transfer. Sayed and Gharib have found traveling wave solution class for FN and NW equations by using an improved sine-cosine method and Wu's elimination method [30]. Also Abbasbandy has employed the Homotopy Analysis Method (HAM) to obtain the solitary solutions of FN equation [34].

Using the wave transformation $u(x,t)=u(\xi)$, $\xi=kx+\omega t$, to reduce Eq.f1 to be the following ODE:

$$k^2u''-\omega u'+u(u-1)(\alpha-u)=0, \tag{12}$$

Balancing u'' and u^3 in Eq.12 yields, $N+2=3N \Longrightarrow N=1$. Consequently, we have the formal solution:

$$u(\xi)=a_0+a_1exp(-\varphi(\xi)), \tag{13}$$

where a_0 and a_1 are constants to be determined, such that $a_1\neq 0$. It is easy to see that

$$u'=-\frac{a_1}{\left(\mathrm{e}^{\phi(\xi)}\right)^2}-a_1\mu-\frac{a_1\lambda}{\mathrm{e}^{\phi(\xi)}}, \tag{14}$$

$$u''=2\,\frac{a_1}{\left(\mathrm{e}^{\phi(\xi)}\right)^3}+2\,\frac{a_1\mu}{\mathrm{e}^{\phi(\xi)}}+3\,\frac{a_1\lambda}{\left(\mathrm{e}^{\phi(\xi)}\right)^2}+a_1\lambda\,\mu+\frac{a_1\lambda^2}{\mathrm{e}^{\phi(\xi)}}. \tag{15}$$

Substituting Eq.13 and its derivatives in Eq.12 and equating the coefficient of different power's of $exp(\varphi(\xi))$ to zero, we get

$$exp(-3\varphi(\xi)):\ 2\,k^2a_1-{a_1}^3=0, \tag{16}$$

$$exp(-2\varphi(\xi)):\ 3\,k^2a_1\lambda+{a_1}^2+\alpha\,{a_1}^2-3\,a_0{a_1}^2+\omega\,a_1=0, \tag{17}$$

$$\begin{aligned}exp(-\varphi(\xi)):\ &2\,k^2a_1\mu+k^2a_1\lambda^2-\alpha\,a_1+2\,a_0a_1+2\,\alpha\,a_0a_1\\ &-3\,{a_0}^2a_1+\omega\,a_1\lambda=0,\end{aligned} \tag{18}$$

$$\begin{aligned}exp(0\varphi(\xi)):\ &k^2a_1\lambda\,\mu-\alpha\,a_0+{a_0}^2+\alpha\,{a_0}^2\\ &-{a_0}^3+\omega\,a_1\mu=0.\end{aligned} \tag{19}$$

Eqs.16-19 yields

Case 1.

$$k=\pm\sqrt{\frac{1}{2}}a_1,\ \mu=\frac{1}{4}\,\frac{-1+{a_1}^2\lambda^2}{{a_1}^2},\ \omega=\frac{1}{2}\,a_1-\alpha\,a_1,$$

$$a_0=\frac{1}{2}\,a_1\lambda+\frac{1}{2},\ a_1=a_1.$$

Case 2.

$$k=\pm\sqrt{\frac{1}{2}}a_1,\ \mu=-\frac{1}{4}\,\frac{\alpha^2-{a_1}^2\lambda^2}{{a_1}^2},\ \omega=-a_1+\frac{1}{2}\,\alpha\,a_1,$$

$$a_0=\frac{1}{2}\,\alpha+\frac{1}{2}\,a_1\lambda,\ a_1=a_1.$$

Case 3.

$$k=\pm\sqrt{\frac{1}{2}}a_1,\ \mu=-\frac{1}{4}\,\frac{1+\alpha^2-2\,\alpha-{a_1}^2\lambda^2}{{a_1}^2},$$

$$\omega=\frac{1}{2}\,a_1+\frac{1}{2}\,\alpha\,a_1,\ a_0=\frac{1}{2}\,\alpha+\frac{1}{2}+\frac{1}{2}\,a_1\lambda, a_1=a_1.$$

Let us now discuss the following case:

Case 1.

When $\lambda^2-4\mu>0, \mu\neq 0$,

$$u=\frac{1}{2}\,a_1\lambda+\frac{1}{2}+\frac{2\mu a_1}{-\sqrt{\lambda^2-4\mu}\,tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\left(\xi+C_1\right)\right)-\lambda}. \tag{20}$$

When $\lambda^2-4\mu>0, \mu=0$,

$$u=\frac{1}{2}\,a_1\lambda+\frac{1}{2}+\frac{\lambda a_1}{exp\left(\lambda\left(\xi+C_1\right)\right)-1}. \tag{21}$$

When $\lambda^2-4\mu=0, \mu\neq 0, \lambda\neq 0$,

$$u=\frac{1}{2}\,a_1\lambda+\frac{1}{2}-\frac{2a_1\left(\lambda\left(\xi+C_1\right)+2\right)}{\lambda^2\left(\xi+C_1\right)}. \tag{22}$$

When $\lambda^2 - 4\mu = 0, \mu = 0, \lambda = 0,$

$$u = \frac{1}{2} + \frac{a_1}{\xi + C_1}. \quad (23)$$

When $\lambda^2 - 4\mu < 0,$

$$u = \frac{1}{2}\,a_1\lambda + \frac{1}{2} + \frac{2\mu a_1}{\sqrt{4\mu - \lambda^2}\,tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\,(\xi + C_1)\right) - \lambda}. \quad (24)$$

Case 2.
When $\lambda^2 - 4\mu > 0, \mu \neq 0,$

$$u = \frac{1}{2}\,\alpha + \frac{1}{2}\,a_1\lambda + \frac{2\mu a_1}{-\sqrt{\lambda^2 - 4\mu}\,tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\,(\xi + C_1)\right) - \lambda}. \quad (25)$$

When $\lambda^2 - 4\mu > 0, \mu = 0,$

$$u = \frac{1}{2}\,\alpha + \frac{1}{2}\,a_1\lambda + \frac{\lambda a_1}{exp\,(\lambda\,(\xi + C_1)) - 1}. \quad (26)$$

When $\lambda^2 - 4\mu = 0, \mu \neq 0, \lambda \neq 0,$

$$u = \frac{1}{2}\,\alpha + \frac{1}{2}\,a_1\lambda - \frac{2a_1\,(\lambda\,(\xi + C_1) + 2)}{\lambda^2\,(\xi + C_1)}. \quad (27)$$

When $\lambda^2 - 4\mu = 0, \mu = 0, \lambda = 0,$

$$u = \frac{1}{2}\,\alpha + \frac{a_1}{\xi + C_1}. \quad (28)$$

When $\lambda^2 - 4\mu < 0,$

$$u = \frac{1}{2}\,\alpha + \frac{1}{2}\,a_1\lambda + \frac{2\mu a_1}{\sqrt{4\mu - \lambda^2}\,tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\,(\xi + C_1)\right) - \lambda}. \quad (29)$$

Case 3.
When $\lambda^2 - 4\mu > 0, \mu \neq 0,$

$$u = \frac{1}{2}\,\alpha + \frac{1}{2} + \frac{1}{2}\,a_1\lambda + \frac{2\mu a_1}{-\sqrt{\lambda^2 - 4\mu}\,tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\,(\xi + C_1)\right) - \lambda}. \quad (30)$$

When $\lambda^2 - 4\mu > 0, \mu = 0,$

$$u = \frac{1}{2}\,\alpha + \frac{1}{2} + \frac{1}{2}\,a_1\lambda + \frac{\lambda a_1}{exp\,(\lambda\,(\xi + C_1)) - 1}. \quad (31)$$

When $\lambda^2 - 4\mu = 0, \mu \neq 0, \lambda \neq 0,$

$$u = \frac{1}{2}\,\alpha + \frac{1}{2} + \frac{1}{2}\,a_1\lambda - \frac{2a_1\,(\lambda\,(\xi + C_1) + 2)}{\lambda^2\,(\xi + C_1)}. \quad (32)$$

When $\lambda^2 - 4\mu = 0, \mu = 0, \lambda = 0,$

$$u = \frac{1}{2}\,\alpha + \frac{1}{2} + \frac{a_1}{\xi + C_1}. \quad (33)$$

When $\lambda^2 - 4\mu < 0,$

$$u = \frac{1}{2}\,\alpha + \frac{1}{2} + \frac{1}{2}\,a_1\lambda + \frac{2\mu a_1}{\sqrt{4\mu - \lambda^2}\,tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\,(\xi + C_1)\right) - \lambda}. \quad (34)$$

3.2 Example 2. Modified Liouville equation

Now, let us consider the Modified Liouville equation.

$$a^2 u_{xx} - u_{tt} + be^{\beta\,u} = 0, \quad (35)$$

respectively, where a, β and b are non zero and arbitrary coefficients. Using the wave transformation $u(x,t) = u(\xi), \xi = kx + \omega t$, $v = e^{\beta\,u}$, to reduce Eq.35 to be in the form:

$$(\frac{k^2a^2}{\beta} - \frac{\omega^2}{\beta})v''v - (\frac{k^2a^2}{\beta} - \frac{\omega^2}{\beta})v'^2 + bv^3 = 0. \quad (36)$$

Balancing $v''v$ and v^3 in Eq.36 yields, $N + 2 + N = 3N \Longrightarrow N = 2$. Consequently, we have the formal solution:

$$v = a_0 + a_1 e^{-\phi(\xi)} + a_2 e^{-2\,\phi(\xi)}, \quad (37)$$

$$v' = -\frac{a_1}{(e^{\phi(\xi)})^2} - a_1\mu - \frac{a_1\lambda}{e^{\phi(\xi)}} - 2\,\frac{a_2}{(e^{\phi(\xi)})^3} - 2\,\frac{a_2\mu}{e^{\phi(\xi)}} - 2\,\frac{a_2\lambda}{(e^{\phi(\xi)})^2}, \quad (38)$$

$$v'' = 2\,\frac{a_1}{(e^{\phi(\xi)})^3} + 2\,\frac{a_1\mu}{e^{\phi(\xi)}} + 3\,\frac{a_1\lambda}{(e^{\phi(\xi)})^2} + a_1\lambda\,\mu + \frac{a_1\lambda^2}{e^{\phi(\xi)}}$$
$$+6\,\frac{a_2}{(e^{\phi(\xi)})^4} + 8\,\frac{a_2\mu}{(e^{\phi(\xi)})^2},$$
$$+10\,\frac{a_2\lambda}{(e^{\phi(\xi)})^3} + 2\,a_2\mu^2 + 6\,\frac{a_2\mu\,\lambda}{e^{\phi(\xi)}} + 4\,\frac{a_2\lambda^2}{(e^{\phi(\xi)})^2}. \quad (39)$$

Substituting Eq.37 and its derivatives in Eq.36 and equating the coefficient of different power's of $exp(\varphi(\xi))$ to zero, we get

$$exp(-6\varphi(\xi)) : \; 2\left(\frac{k^2a^2}{\beta} - \frac{\omega^2}{\beta}\right)a_2{}^2 + ba_2{}^3 = 0, \quad (40)$$

$$exp(-5\varphi(\xi)) : \; \left(\frac{k^2a^2}{\beta} - \frac{\omega^2}{\beta}\right)(4\,a_1a_2 + 2\,a_2{}^2\lambda)$$
$$+3\,ba_1a_2{}^2 = 0, \quad (41)$$

$$exp(-4\varphi(\xi)) : \; \left(\frac{k^2a^2}{\beta} - \frac{\omega^2}{\beta}\right)(5\,a_1\lambda\,a_2 + 6\,a_2a_0 + a_1{}^2)$$
$$+b\left(3\,a_0a_2{}^2 + 3\,a_1{}^2a_2\right) = 0, \quad (42)$$

$$exp(-3\varphi(\xi)) : \; \left(\frac{k^2a^2}{\beta} - \frac{\omega^2}{\beta}\right)(2\,a_1\mu\,a_2 + a_1\lambda^2a_2 + 10\,a_2\lambda\,a_0$$
$$-2\,a_2{}^2\mu\,\lambda + a_1{}^2\lambda + 2\,a_0a_1 + b\left(6\,a_0a_1a_2 + a_1{}^3\right) = 0, \quad (43)$$

$$exp(-2\varphi(\xi)) : \; \left(\frac{k^2a^2}{\beta} - \frac{\omega^2}{\beta}\right)(3\,a_1\lambda\,a_0 + 8\,a_2\mu\,a_0$$
$$+4\,a_2\lambda^2a_0 - a_1\lambda\,\mu\,a_2$$
$$-2\,a_2{}^2\mu^2) + b\,(3\,a_0{}^2a_2 + 3\,a_0a_1{}^2) = 0, \quad (44)$$

$$exp(-\varphi(\xi)) : \left(\frac{k^2a^2}{\beta} - \frac{\omega^2}{\beta}\right)(2\,a_1\mu\,a_0 - a_1{}^2\lambda\,\mu + a_1\lambda^2a_0$$
$$-2\,a_2\mu^2a_1 + 6\,a_2\mu\,\lambda\,a_0 + 3\,ba_0{}^2a_1 = 0, \quad (45)$$

$$exp(0\varphi(\xi)) : \; \left(\frac{k^2a^2}{\beta} - \frac{\omega^2}{\beta}\right)(a_1\lambda\,\mu\,a_0 + 2\,a_2\mu^2a_0 - a_1{}^2\mu^2)$$
$$+ba_0{}^3 = 0. \quad (46)$$

Eqs.40-46 yields

$$a_0 = -2\,\frac{\mu\,(k^2a^2 - \omega^2)}{b\beta},\; a_1 = \left(-2\,\frac{\lambda\,(k^2a^2 - \omega^2)}{b\beta}\right),$$

$$a_2 = -2\,\frac{k^2a^2-\omega^2}{b\beta}.$$

Let us now discuss the following case:
When $\lambda^2-4\mu>0, \mu\neq 0$,

$$v_1 = -2\,\frac{\mu\,(k^2a^2-\omega^2)}{b\beta} - 2A\,\frac{\lambda\,(k^2a^2-\omega^2)}{b\beta} - 2\,\frac{k^2a^2-\omega^2}{b\beta}A^2, \tag{47}$$

for this

$$u_1 = \tfrac{1}{\beta}\,ln(v_1). \tag{48}$$

since

$$A = \left[\frac{2\mu}{-\sqrt{\lambda^2-4\mu}\;tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\,(\xi+C_1)\right)-\lambda}\right] \tag{49}$$

When $\lambda^2-4\mu>0, \mu=0$,

$$v_2 = -2\,\frac{\mu\left(k^2a^2-\omega^2\right)}{b\beta} - \left(2\,\frac{\lambda\left(k^2a^2-\omega^2\right)}{b\beta}\right)\frac{\lambda}{exp(\lambda(\xi+C_1))-1} -2\,\frac{k^2a^2-\omega^2}{b\beta}\left[\frac{\lambda}{exp\left(\lambda\left(xi+C_1\right)\right)-1}\right]^2, \tag{50}$$

for this

$$u_2 = \tfrac{1}{\beta}\,ln(v_2). \tag{51}$$

When $\lambda^2-4\mu=0, \mu\neq 0, \lambda\neq 0$,

$$v_3 = -2\,\frac{\mu\left(k^2a^2-\omega^2\right)}{b\beta} + \left(2\,\frac{\lambda\left(k^2a^2-\omega^2\right)}{b\beta}\right)\frac{2(\lambda(\xi+C_1)+2)}{\lambda^2(\xi+C_1)} -2\,\frac{k^2a^2-\omega^2}{b\beta}\left[\frac{2\left(\lambda\left(\xi+C_1\right)+2\right)}{\lambda^2\left(\xi+C_1\right)}\right]^2, \tag{52}$$

$$u = \tfrac{1}{\beta}\,ln(v_3). \tag{53}$$

When $\lambda^2-4\mu=0, \mu=0, \lambda=0$,

$$v_4 = -2\,\frac{\mu\left(k^2a^2-\omega^2\right)}{b\beta} - \left(2\,\frac{\lambda\left(k^2a^2-\omega^2\right)}{b\beta}\right)\frac{1}{\xi+C_1} -2\,\frac{k^2a^2-\omega^2}{b\beta}\left[\frac{1}{\xi+C_1}\right]^2, \tag{54}$$

for this

$$u_4 = \tfrac{1}{\beta}\,ln(v_4). \tag{55}$$

When $\lambda^2-4\mu<0$,

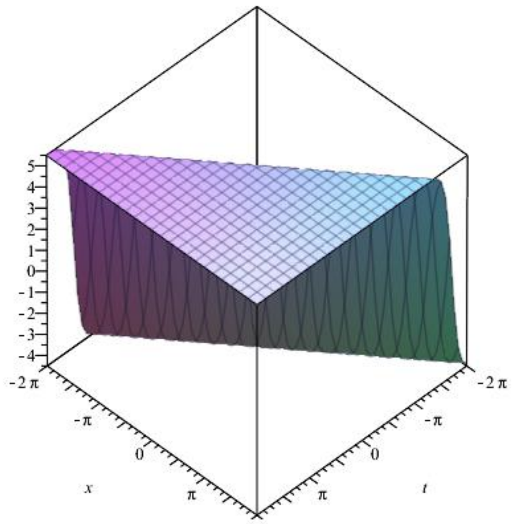
$$v_5 = -2\,\frac{\mu\left(k^2a^2-\omega^2\right)}{b\beta} - \left(2B\,\frac{\lambda\left(k^2a^2-\omega^2\right)}{b\beta}\right) -2B^2\,\frac{k^2a^2-\omega^2}{b\beta}, \tag{56}$$

for this

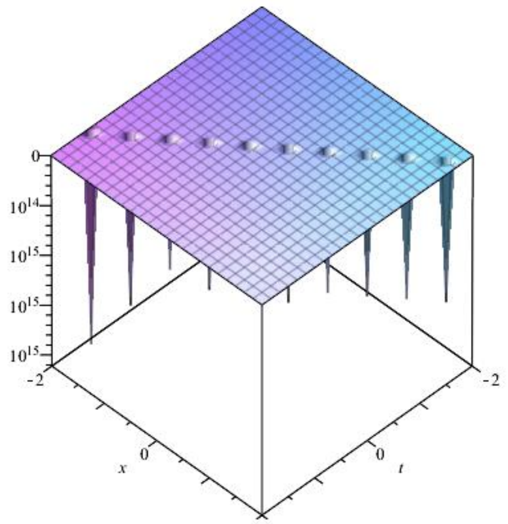
$$u_5 = \tfrac{1}{\beta}ln(v_5). \tag{57}$$

Since

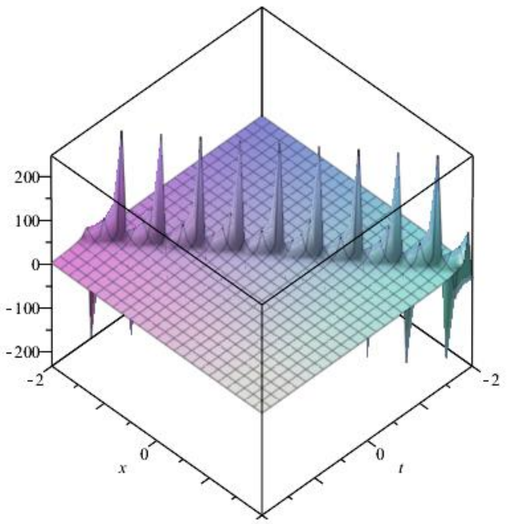
$$B = \left[\frac{2\mu}{\sqrt{4\mu-\lambda^2}\;tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\;\;(\xi+C_1)\right)-\lambda}\right]. \tag{58}$$



[Eq.30]

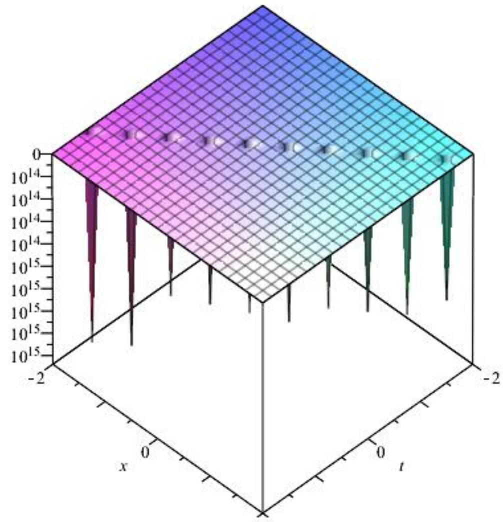


[Eq.31]

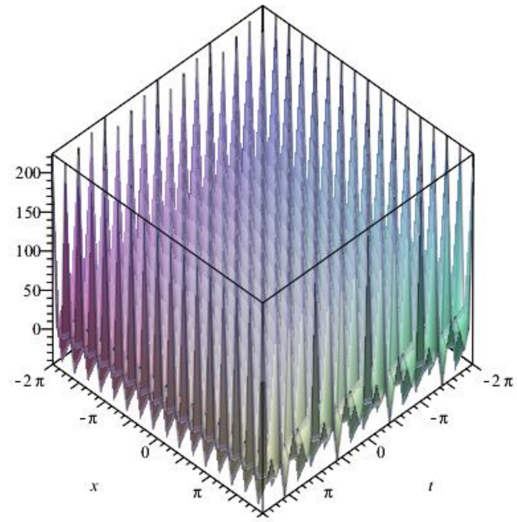


[Eq.32]

Fig. 1. The figures of solution of Eqs.30, 31 and 32

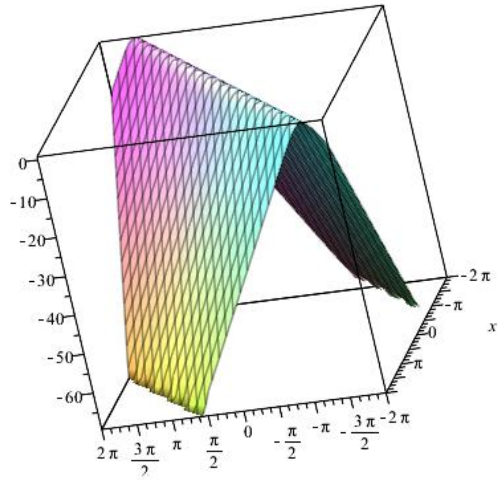


[Eq.33]

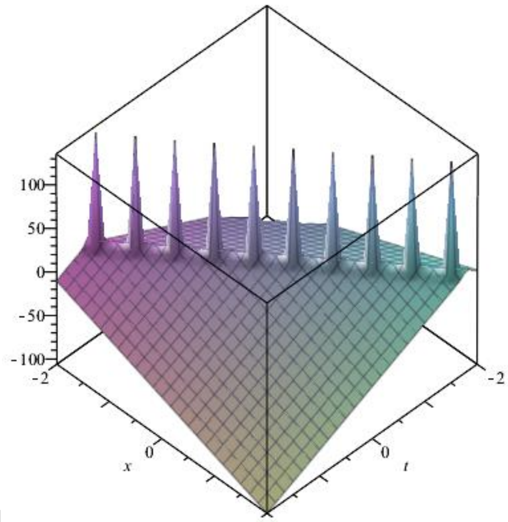


[Eq.34]

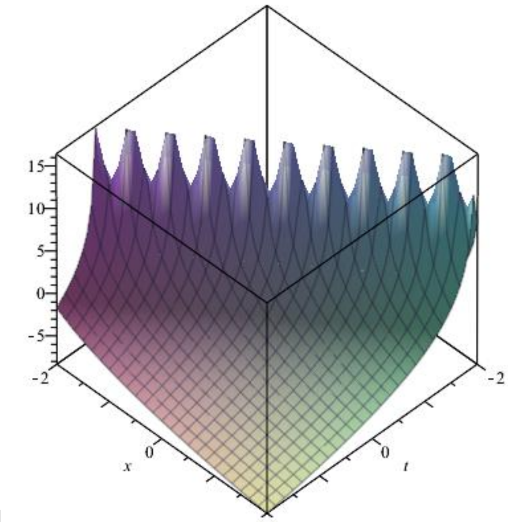
Fig. 2. The figures of solution of Eqs. 33 and 34



[Eq.48]



[Eq.51]

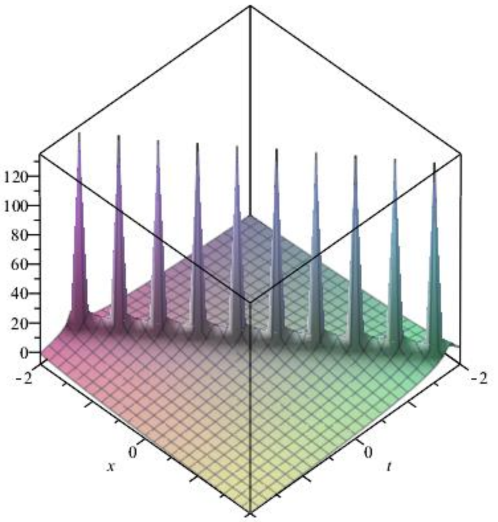


[Eq.53]

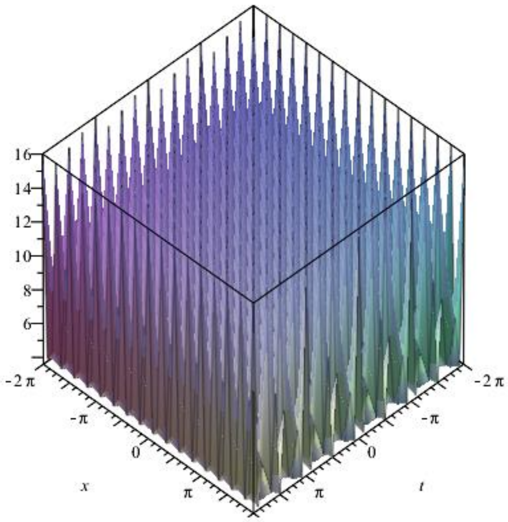
Fig. 3. The figures of solution of Eqs.48, 51 and 53

4. CONCLUSION

The $\exp(-\varphi(\xi))$-expansion method has been applied in this paper to find the exact traveling wave solutions and then the solitary wave solutions of two nonlinear evolution equations, namely, Fitzhugh-Nagumo (FN) equation and Modified Liouville equation . Let us compare between our results obtained in the present article with the well-known results obtained by other authors using different methods as follows: Our results of Fitzhugh-Nagumo (FN) equation and Modified Liouville equation are new and different from those obtained in [29]-[34] and fig. 1, 2, 3, 4 show the solitary traveling wave solution of Fitzhugh-Nagumo (FN) equation and Modified Liouville equation. We can conclude that the $\exp(-\varphi(\xi))$-expansion method is is a very powerful and efficient technique in finding exact solutions for wide classes of nonlinear problems and can be applied to many other nonlinear evolution equations in mathematical physics. Another possible merit is that the reliability of the method and the reduction in the size of computational domain give this method a wider applicability.



[Eq.55]



[Eq.57]

Fig. 4. The figures of solution of Eqs. 55 and 55

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