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On γ-Connected Sets

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ABSTRACT

The definition of γ-connectedness due to Császár [3] is modified and its basic properties are studied. The images and preimages of γ- connected sets under $(\gamma - \gamma')$-continuous mapping are investigated. Sufficient conditions for a mapping to be $(\gamma - \gamma')$- continuous are given.

General Terms:

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γ- connectedness, $(\gamma - \gamma')$- continuous mapping, $(\gamma - \gamma')$- semi-connected, $(\gamma - \gamma')$- weakly semiconnected, $(\gamma - \gamma')$- connected, γ- strongly semi-locally connected.

1. INTRODUCTION

Let X be a set and $\Gamma(X)$ be the collection of all monotone mappings: $A \subseteq B \subseteq X$ implies $\gamma A \subseteq \gamma B$ for $\gamma \in \Gamma(X)$. According to Császár $A \subseteq X$ is γ-open [1] if $A \subseteq \gamma A$ and the collection μ_γ of all γ-open sets forms a generalized topology : μ_γ is closed under arbitrary unions, and (X, μ_γ) is called a generalized topological space (GT-space). A set is γ-closed if its complement in X is γ-open. For a set $A \subseteq X$, $c_\gamma A$ is the intersection of all γ-closed sets containing A. The sets $U, V \subseteq X$ are said to be γ-separated if $c_\gamma U \cap V = \emptyset$ and $U \cap c_\gamma V = \emptyset$. Note that γ-separated sets U and V are contained in $X' = \cup\{A : A \in \mu_\gamma\}$.
The set $X' = \cup\{A : A \in \mu_\gamma\}$ may not be equal to X. When $X' = X$, X is called a strong generalized topological space.

A set $S \subseteq X$ is γ-connected [3] if $S = U \cup V$, U and V are γ-separated, implies that $U = \emptyset$ or $V = \emptyset$. The space X is said to be γ-connected if it is γ-connected subset of itself. It follows that if a space X is not strong then it is trivially γ-connected.

This situation is at variance with the classical topology where the property of connectedness acts as a classification device. Császár definition acts as classification device only in the class of strong generalized topological spaces. In order to overcome this problem the definition of γ-connectedness is modified and the analogues of all the results in Sections 1 and 2 [3] are obtained. In Section 5 sufficient conditions are given so that a mapping becomes $(\gamma - \gamma')$-continuous.

2. PRILIMINARIES

LEMMA 2.1. *[3] For $U, V \subseteq X$, the following statements are equivalent:*
(a). U and V are $\gamma-$separated.
(b). There are $\gamma-$closed sets F_U and F_V such that $U \subseteq F_U \subseteq X - V$ and $V \subseteq F_V \subseteq X - U$.
(c). There are $\gamma-$open sets G_U and G_V such that $U \subseteq G_U \subseteq X - V$ and $V \subseteq G_V \subseteq X - U$.

LEMMA 2.2. *Let $S \subseteq X$, then $c_\gamma(S \cap X') = c_\gamma(S)$.*

PROOF. *To show $c_\gamma(S) \subseteq c_\gamma(S \cap X')$, it is sufficient to consider the points in X'. Let $x \in X'$ be such that $x \in c_\gamma(S)$. Let G be any $\gamma-$open set such that $x \in G$. Then $G \cap S \neq \phi$. Therefore, $G \cap (S \cap X') \neq \phi$, that is, $x \in c_\gamma(S \cap X')$.* □

3. γ- CONNECTED SETS

DEFINITION 3.1. *A set $S \subseteq X$ is said to be $\gamma-$connected if $S \cap X' = U \cup V$, U and V are $\gamma-$separated, implies $U = \phi$ or $V = \phi$. The space X is said to be $\gamma-$connected if it is $\gamma-$connected subset of itself.*

THEOREM 3.2. *For $\gamma \in \Gamma(X)$, the following statements are equivalent:*
(a). The space X is $\gamma-$connected.
(b). If $X' = G \cup G'$, $G \cap G' = \phi$, G and G' are $\gamma-$open, then either $G = \phi$ or $G' = \phi$.
(c). If $X = F \cup F'$, $F \cap F' \cap X' = \phi$, F and F' are $\gamma-$closed, then $F \cap X' = \phi$ or $F' \cap X' = \phi$.
(d). If H is $\gamma-$closed set such that $H \cap X'$ is $\gamma-$open, then $H \cap X' = \phi$ or $H \cap X' = X'$.

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PROOF. $(a) \Rightarrow (b)$. *By* 2.1 *G and* G' *are* $\gamma-$*separated.*
$(b) \Rightarrow (c)$. *Let* $G = X - F$ *and* $G' = X - F'$. *Then* $G = X' \cap F'$ *and* $G' = X' \cap F$. *Therefore, G and* G' *are* $\gamma-$*open,* $G \cup G' = X'$ *and* $G \cap G' = \phi$. *So* $G = \phi$ *or* $G' = \phi$. *Therefore,* $F \cap X' = \phi$ *or* $F' \cap X' = \phi$.
$(c) \Rightarrow (d)$. *Let* $F = H$ *and* $F' = X - (H \cap X')$. *Then* $F \cap F' \cap X' = \phi$, *F and* F' *are* $\gamma-$*closed. Therefore,* $F \cap X' = \phi$ *or* $F' \cap X' = \phi$. *Thus,* $H \cap X' = \phi$ *or* $H \cap X' = X'$.
$(d) \Rightarrow (b)$. *Let* $X' = G \cup G'$, $G \cap G' = \phi$, *G and* G' *are* $\gamma-$*open. Let* $H = X - G$, *then* $H \cap X' = G'$. *So* $H \cap X'$ *is* $\gamma-$*open. Then* $H \cap X' = \phi$ *or* $H \cap X' = X'$. *Therefore,* $G = X'$ *or* $G = \phi$, *that is,* $G = \phi$ *or* $G' = \phi$.
$(b) \Rightarrow (a)$. *Let* $X' = U \cup V$, *where U and V are* $\gamma-$*separated. Let* $G = X - c_\gamma U$ *and* $G' = X - c_\gamma V$. *Then* $G = V$ *and* $G' = U$ *are* $\gamma-$*open sets,* $G \cap G' = \phi$. *Thus,* $G = V = \phi$ *or* $G' = U = \phi$. □

EXAMPLE 1. *Let* $\mathbb{R}$ *be set of real numbers,* $A = [0, 1]$, $B = (1, 2]$. *Then the collection* $\mu = \{\phi, A, B, [0, 2]\}$ *is a generalized topology on* $\mathbb{R}$. *The corresponding* γ *giving* μ *by Lemma* 1.1 *[2] is given by* $\gamma A = \cup\{G \in \mu : G \subseteq A\}$. *It is easily seen that* $\mathbb{R}$ *is not* $\gamma-$*connected.*

LEMMA 3.3. *If S is* γ*-connected and* $S \cap X' \subseteq U \cup V$, *where U and V are* γ*- separated, then* $S \cap X' \subseteq U$ *or* $S \cap X' \subseteq V$.

PROOF. Clearly, $S \cap X' = (U \cap S) \cup (V \cap S)$, where $U \cap S$ and $V \cap S$ are γ-seperated. So $U \cap S = \phi$ or $V \cap S = \phi$. Therefore, $S \cap X' \subseteq U$ or $S \cap X' \subseteq V$. □

THEOREM 3.4. *If S is* γ*- connected,* $S \subseteq T \subseteq c_\gamma S$, *then T is* γ*- connected.*

PROOF. Let $T \cap X' = U \cup V$, where U and V are γ- separated. Then $S \cap X' \subseteq U \cup V$. Then by lemma 3.3, $S \cap X' \subseteq U$ or $S \cap X' \subseteq V$. Therefore,

$$c_\gamma(S \cap X') \subseteq c_\gamma U \subseteq X - V$$

or

$$c_\gamma(S \cap X') \subseteq c_\gamma V \subseteq X - U.$$

By Lemma 2.2, $c_\gamma S \subseteq X - V$ or $c_\gamma S \subseteq X - U$. Therefore, $T \cap X' \subseteq X - V$ or $T \cap X' \subseteq X - U$. This gives $V = \phi$ or $U = \phi$. □

COROLLARY 3.5. *If S is* γ*- connected then* $c_\gamma S$ *is* γ*- connected.*

THEOREM 3.6. *If* $S = \cup_{\lambda \in \wedge} S_\lambda$, S_λ *is* γ*- connected for* $\lambda \in \wedge$, *and* $\lambda, \lambda' \in \wedge, \lambda \neq \lambda', S_\lambda \cap X'$ *and* $S_{\lambda'} \cap X'$ *are not* γ*- separated, then S is* γ *- connected.*

PROOF. Let $S \cap X' = U \cup V$, where U and V are γ- seperated. Then $S_\lambda \cap X' \subseteq U \cup V$ then by lemma 3.3, $S_\lambda \cap X' \subseteq U$ or $S_\lambda \cap X' \subseteq V$. By hypothesis there does not exist $\lambda, \lambda' \in \wedge$ such that $\lambda \neq \lambda'$ and $S_\lambda \cap X' \subseteq U$ and $S_{\lambda'} \cap X' \subseteq V$. Therefore, $S_\lambda \cap X' \subseteq U$ for each $\lambda \in \wedge$ or $S_\lambda \cap X' \subseteq V$ for each $\lambda \in \wedge$. Thus, $S \cap X' \subseteq U$ or $S \cap X' \subseteq V$. That is, $U = \phi$ or $V = \phi$.

□

COROLLARY 3.7. *If* $S = \cup_{\lambda \in \wedge} S_\lambda$, S_λ *is* γ*- connected for* $\lambda \in \wedge$, *and* $S_\lambda \cap S_{\lambda'} \cap X' \neq \phi$ *for* $\lambda, \lambda' \in \wedge$, *then S is* γ*- connected.*

COROLLARY 3.8. *If* $S = \cup_{\lambda \in \wedge} S_\lambda$, S_λ *is* γ*- connected for* $\lambda \in \wedge$ *and* $\cap_{\lambda \in \wedge} S_\lambda \cap X' \neq \phi$, *then S is* γ*- connected.*

DEFINITION 3.9. *If* $A \subseteq X$ *and* $x \in A$, *the set* $A_x = \cup\{S \subseteq A : x \in S$, *and S is* $\gamma -$ *connected*$\}$ *is called the* $\gamma-$ *component of A containing x.*
Evidently, $A - X'$ *is contained in* A_x *for each* $x \in A$ *since the singleton* $\{x\}$ *is* γ*- connected and* $(A - X') \cup \{x\}$ *is* γ*- connected.*

THEOREM 3.10. *Every subset* $A \subseteq X$ *is the union of all its* $\gamma-$*components. Each* $\gamma-$*component is a maximal* $\gamma-$*connected subset of A and two distinct* $\gamma-$*components are disjoint on* $A \cap X'$. *The* $\gamma-$*components of a* $\gamma-$*closed set are* $\gamma-$*closed.*

PROOF. By definition every singleton is $\gamma-$connected, so $x \in A_x$ for $x \in A$, and $A = \cup_{x \in A \cap X'} A_x$. By corollary 3.8, A_x is $\gamma-$connected for $x \in A \cap X'$. If $x \in S \subseteq A$ is $\gamma-$connected, then $S \subseteq A_x$. Therefore, A_x is maximal $\gamma-$connected subset of A containing x. If $x, y \in A \cap X'$ and $A_x \cap A_y \cap X' \neq \phi$, then $A_x \cup A_y$ is $\gamma-$connected by corollary 3.8 and hence $A_x \cup A_y \subseteq A_x \cap A_y$ so that $A_x = A_y$. If A is $\gamma-$closed and $x \in A$, then $c_\gamma A_x \subseteq A$ is $\gamma-$connected by corollary 3.5 containing x. So $c_\gamma A_x \subseteq A_x$ and A_x is $\gamma-$ closed. □

4. IMAGES AND PREIMAGES OF γ- CONNRCTED SETS UNDER $(\gamma - \gamma')$- CONTINUOUS MAPPINGS

DEFINITION 4.1. *Let* $f : X \to Y$, *be a mapping and* $\gamma \in \Gamma(X)$ *and* $\gamma' \in \Gamma(Y)$.*Then f is said to be* $(\gamma - \gamma')$*- continuous [2] if* $f^{-1}(A)$ *is* γ*- open whenever A is* γ'*- open.*
f is $(\gamma - \gamma')$*- homeomorphism if f is bijective, f is* $(\gamma - \gamma')$*- continuous and* f^{-1} *is* $(\gamma' - \gamma)$*- continuous. f is* $(\gamma - \gamma')$*- open if* $f(A)$ *is* γ'*-open for every* γ*- open set A.*

REMARK 4.2. *Let* $f : X \to Y$ *be* $(\gamma - \gamma')$*- continuous and x be a point in* $X - X'$. *If* $f(x) \in Y'$, *then* $f^{-1}(Y')$ *contains x and so* $f^{-1}(Y')$ *would not be* γ*-open. Thus,* $f(x) \notin Y'$ *for each* $x \in X - X'$, *that is,* $f(X - X') \subseteq Y - Y'$.

THEOREM 4.3. *If* $S \subseteq X$ *is* γ*- connected and* $f : X \to Y$ *is a* $(\gamma - \gamma')$*- continuous mapping such that* $f(X') \subseteq Y'$, *then* $f(S)$ *is* γ'*- connected.*

PROOF. Let $f(S) \cap Y' = U' \cup V'$ with γ'- separated sets U' and V'. By 2.1 [3], $f^{-1}(U')$ and $f^{-1}(V')$ are γ- separated, and $S \cap f^{-1}(Y') \subseteq f^{-1}(U') \cup f^{-1}(V')$. Since $f(X') \subseteq Y'$, $S \cap X' \subseteq f^{-1}(U') \cup f^{-1}(V')$. Since S is γ- connected, by lemma 3.3 $S \cap X' \subseteq f^{-1}(U')$ or $S \cap X' \subseteq f^{-1}(V')$. Therefore, $f(S \cap X') \subseteq U'$ or $f(S \cap X') \subseteq V'$. By Remark 4.2 $f(S - X') \cap Y' = \emptyset$. Hence,$f(S) \cap Y' \subseteq U'$ or $f(S) \cap Y' \subseteq V'$. Thus, either $U' = \emptyset$ or $V' = \emptyset$. □

The condition $f(X') \subseteq Y'$ in the classical topology case holds trivially since $X' = X$ and $Y' = Y$.

THEOREM 4.4. *If f is* $(\gamma - \gamma')$*- open and injective,* $Y' \subseteq f(X')$, $S \subseteq X$ *and* $f(S)$ *is* γ'*- connected, then S is* γ*- connected.*

PROOF. Let $S \cap X' = U \cup V$ with γ- separated sets U and V. Then by 2.3 [3] $f(S \cap X') = f(U) \cup f(V)$ with γ'- separated sets $f(U)$ and $f(V)$. Since f is injective, $f(S) \cap f(X') = f(U) \cup f(V)$. Now $Y' \subseteq f(X')$ implies $f(S) \cap Y' \subseteq f(U) \cup f(V)$. Therefore, $f(U) = \emptyset$ or $f(V) = \emptyset$, that is, $U = \emptyset$ or $V = \emptyset$. □

THEOREM 4.5. *If f is* $(\gamma - \gamma')$*-open injective and* $Y' \subseteq f(X')$, *then for each* γ' *connected set* $C \subseteq Y$, $f^{-1}(C)$ *is* γ*- connected.*

PROOF. Let $f^{-1}(C) \cap X' = U \cup V$ with U and V γ- separated sets. Suppose $U \neq \emptyset$ and $V \neq \emptyset$. Then $f(U)$ and $f(V)$ are non-empty and γ'- separated sets by 2.3[3]. Since U and V are γ- separated, $U \subseteq X - c_\gamma V$. Then $f(U) \subseteq f(X - c_\gamma V) \subseteq Y'$ since f is $(\gamma - \gamma')$- open. Similarly, $f(V) \subseteq Y'$.Since $f(U) \subseteq C$ and $f(V) \subseteq C$, $f(U) \subseteq Y' \cap C$ and $f(V) \subseteq Y' \cap C$. Since $Y' = f(X')$, for each $y \in Y' \cap C$, there exists an $x \in X'$ such that $f(x) = y$. Since $x \in f^{-1}(C)$, $x \in f^{-1}(C) \cap X' = U \cup V$. Consequently, $f(x) = y$ belongs to either $f(U)$ or $f(V)$, that is, $C \cap Y' = f(U) \cup f(V)$. This means that C is not γ'- connected, a contradiction. □

5. SUFFICIENT CONDITIONS FOR $(\gamma - \gamma')$-CONTINUITY

DEFINITION 5.1. *A GT - space (X, μ_γ) is said to be T_1- space if for any two distinct points in X' there exists a γ- open set containing one of the points but not the other.*
Note that the above definition of T_1- space is different from the one in [4, 10].
The following definitions are generalizations of the corresponding notions in [5, 6, 7, 8, 9].

DEFINITION 5.2. *Let $f : X \to Y$ be a mapping from a GT-space X into a GT- space Y. Then f is said to be*
1. $(\gamma - \gamma')$- semiconnected if for each γ'- closed and γ'- connected set $K \subseteq Y$, $f^{-1}(K)$ is γ- closed and γ- connected.
2. $(\gamma - \gamma')$- weakly semiconnected if for each γ'- closed and γ'-connected set $K \subseteq Y$, $f^{-1}(K)$ is γ- closed.
3. $(\gamma - \gamma')$- connected if for each γ- connected set $A \subseteq X$, $f(A)$ is γ'- connected.

DEFINITION 5.3. *A GT- space (X, μ_γ) is said to be γ- strongly semi-locally connected if for each $x \in X'$ and γ- open set W containing x there exists a γ- open set $V \subseteq W$ containing x such that $X - V$ is γ- connected.*

THEOREM 5.4. *If $f : X \to Y$ is (γ, γ')- weakly semiconnected and onto γ'- strongly semi- locally connected space Y, then f is $(\gamma - \gamma')$- continuous.*

PROOF. Let $B \subseteq Y$ be γ'- open. It will be shown that $f^{-1}(B)$ is γ- open in X. For each point $b \in B$ there exists a γ'- open set $V_b \subseteq B$ containing b such that $Y - V_b$ is γ'- connected. Also $Y - V_B$ is γ'- closed. So $f^{-1}(Y - V_b)$ is γ- closed since f is $(\gamma - \gamma')$ weakly semiconnected. Also $f^{-1}(Y - V_b)$ contains no point of $f^{-1}(V_b)$. Thus, $X - f^{-1}(Y - V_b) = R_b$ is γ- open set containing $f^{-1}(V_b)$ and has the property that $f(R_b) = V_b$. Consequently, $\cup_{b\in B} R_b$ is γ- open and furthermore $f^{-1}(B) = \cup_{b\in B} R_b$. □

THEOREM 5.5. *If f is $(\gamma - \gamma')$- open injective, $(\gamma - \gamma')$-connected mapping, $Y' \subseteq f(X')$ and Y is a T_1- space, then f is $(\gamma - \gamma')$ semiconnected. Further, if f is onto and Y is γ'-strongly semi-locally connected, then f is $(\gamma - \gamma')$- continuous.*

PROOF. Let $B \subseteq Y$ be any γ'- closed and γ'- connected set. By 4.5 $f^{-1}(B)$ is a γ- connected set. Suppose $f^{-1}(B)$ is not γ- closed. Then there exists a point x in $c_\gamma(f^{-1}(B)) - f^{-1}(B)$. Since $f^{-1}(B)$ is γ- connected, $\{x\} \cup f^{-1}(B)$ is γ- connected by theorem 3.4 Since f is $(\gamma - \gamma')$- open and injective, $f(X - X') \subseteq Y - Y' \subseteq B$ since B is γ'- closed. Therefore, $X - X' \subseteq f^{-1}(B)$. So $x \in X'$ and $f(x) \in Y'$ and $f(x) \notin B$.
Now $f(\{x\} \cup f^{-1}(B)) \cap Y' = \{\{f(x)\} \cup f(f^{-1}(B))\} \cap Y' = \{f(x)\} \cup (B \cap Y')$. Since Y is a T_1- space. $\{f(x)\}$ and $B \cap Y'$ are γ'- separated sets. So $f(\{x\} \cup f^{-1}(B))$ is not γ'- connected. a contradiction. Therefore, f is $(\gamma - \gamma')$- semiconnected. The second part follows from 5.4 □

COROLLARY 5.6. *If $f : X \to Y$ is a bijection, X is γ-strongly semi-locally connected, Y is γ'-strongly semi-locally connected, f is $(\gamma - \gamma')$- weakly semiconnected and f^{-1} is $(\gamma' - \gamma)$-weakly semiconnected, then f is a $(\gamma' - \gamma)$- homeomorphism.*

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