Loss of Uniqueness of the Boundary Value Problem Involving the Mini-Drucker-Prager CLoE Model

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ABSTRACT
The loss of positiveness of the second order work (SOW) induce the loss of uniqueness of the solution of the small strain boundary value problem as it is shown in the literature, and therefore, the onset of strain localization bands in the studied material. This paper is devoted to study the mini-CloE Drüker-Prager model. The results showed that non-associated model, although isotropic, can be the seat of strain localization in contrary to its counterpart associated isotropic model. In addition, the anisotropy is a factor encouraging the onset of strain localization. In fact, it makes the associated model subject of losing the positiveness of the SOW and accentuates the negativity of the SOW of the non-associated model. These results are similar with those established for the mini-CLoE von Mises and Mohr Coulomb models and those known for the elastic-plastic materials.

Key words
Bifurcation, second order work, mini-CLoE, Drücker-Prager limit surface.

1. INTRODUCTION
The localization of the strain in the narrow bands is a common phenomenon for various materials. It expresses the transition from the diffuse mode of the strain to a localized one when these materials are sufficiently deformed in the inelastic phase.

In theory the strain localization is often seen as a loss of uniqueness of the solution (bifurcation) of the boundary value problem. Theoretical developments showed that if the second order work is negative then the solution loses its uniqueness. This paper focuses on the loss of uniqueness in the case of the Drücker-Prager min-Cloe model in relation with the various features of the model.

2. CLOE MODEL
Cloe model (acronym of the concepts of consistency and Explicit localization) developed by Chambon [1,2] is one of the incremental nonlinear constitutive equations. It is characterized by the fact that the strain rate is not separated into an elastic part and inelastic one. The development of this model aims the following two purposes:

- The consistency to the boundary surface reflecting the absence of discontinuity of the response of the law, and lack of access to stress states outside the boundary.
- The definition of an explicit localization criterion without using a linearization.

Drücker-Prager mini-CLoE model
Theoretical developments of the Cloe model are difficult to implement. So, heuristic models so-called 'mini-Cloe' associating to the limit surface the classical boundary surfaces like: Von Mises [3], Mohr Coulomb[4] and Drücker-Prager [5], have been developed. These models with low number of parameters help to better understand how CLoE laws work. This paper is devoted to the study of the Drücker-Prager mini-CLoE model.

2.1. Constitutive equation
CLoE models are constructed on a unique and non-linear relation between the stress rate and the strain rate. A non-linear relation is considered:

$$\underset{\sim}{\tilde{\sigma}} = \underset{\approx}{A} : \left(\underset{\sim}{\dot{\varepsilon}} + \underset{\sim}{b}' \left\| \underset{\sim}{\dot{\varepsilon}} \right\| \right) \qquad (1)$$

$\underset{\sim}{\tilde{\sigma}}$ is the objective derivative of the Cauchy stress, $\underset{\sim}{\dot{\varepsilon}}$ is the strain rate with norm $\left\| \underset{\sim}{\dot{\varepsilon}} \right\| = \sqrt{\underset{\sim}{\dot{\varepsilon}} : \underset{\sim}{\dot{\varepsilon}}}$.

$\underset{\approx}{A}$ and $\underset{\sim}{b}'$ are constitutive tensors of fourth and second-order. These tensors depend on the assumed state variables of the material considered.

2.2. Limit surface
CLoE laws include explicitly a limit surface which splits the stress space into two areas: the area of permissible stress states and the area of prohibited stress states.
The studied mini-CLoE law in the paper integrates the Drücker-Prager yield surface as a limit surface. This surface is a cone centered on the trisector of stress space (Figure 1). The equation of this limit surface, in the case of non-cohesive materials, can be written in the form:

$$f = II_{\hat{\sigma}} - I_{\sigma} \tan \alpha_{\varphi} = 0 \qquad (2)$$

Where I_{σ} is the first invariant of the Cauchy stress and $II_{\hat{\sigma}}$ is the second invariant of the deviatoric part of the Cauchy stress:

$I_{\sigma} = \sigma_{11} + \sigma_{22} + \sigma_{33}$, $\underset{\sim}{\hat{\sigma}} = \underset{\sim}{\sigma} - \frac{I_{\sigma}}{3} \underset{\approx}{I}$. ($\underset{\approx}{I}$ is unit second-order tensor) and $II_{\hat{\sigma}} = \sqrt{(\hat{\sigma}_{11} - \hat{\sigma}_{22})^2 + (\hat{\sigma}_{11} - \hat{\sigma}_{33})^2 + (\hat{\sigma}_{22} - \hat{\sigma}_{33})^2}$.

α_{φ} : The half angle of the cone, connected to the internal friction angle in compression φ_c by: $\tan \alpha_{\varphi} = \sqrt{2} \frac{2 \sin \varphi_c}{3 - \sin \varphi_c}$.

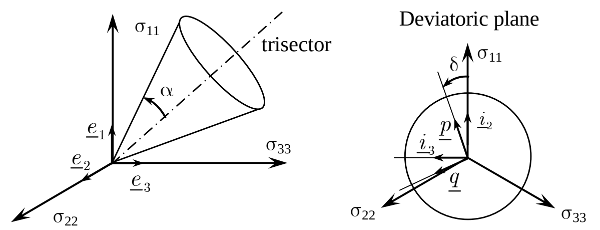

Figure 1. Drücker-Prager limit surface in stress space.

We first define the three basis which we will use:

The first one is the basis of the tensors associated with the components of the axis $\underline{i}$, $\underline{j}$ and $\underline{k}$ of the physical space which we choose coincide with the directions of the principal stresses.

The second order tensor $\underline{\sigma}$ is then expressed in the basis denoted $\mathcal{E}$, defined by the second order tensors $(\underline{e}_1, \underline{e}_2, \underline{e}_3, \underline{e}_4, \underline{e}_5, \underline{e}_6)$.

$$\underline{e}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \underline{e}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \underline{e}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\underline{e}_4 = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \underline{e}_5 = \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix} \text{ and } \underline{e}_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

The second basis is defined by involving stress invariants. This basis, noted $\mathcal{I}$, defines the plane orthogonal to the trisector containing the considered stress state. It is defined by the following second order tensors $(\underline{i}_1, \underline{i}_2, \underline{i}_3, \underline{e}_4, \underline{e}_5, \underline{e}_6)$:

$$\underline{i}_1 = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} \end{pmatrix},\ \underline{i}_2 = \begin{pmatrix} \frac{2}{\sqrt{6}} & 0 & 0 \\ 0 & \frac{-1}{\sqrt{6}} & 0 \\ 0 & 0 & \frac{-1}{\sqrt{6}} \end{pmatrix} \text{ and } \underline{i}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{-1}{\sqrt{2}} \end{pmatrix}$$

These tensors are the six elements of an orthogonal basis for the second order tensors.

The third basis, called deviatoric basis and denoted $\mathcal{D}$, is defined to reflect the evolution of the phase δ in the deviatoric plane. The basis $\mathcal{D}$ is defined by tensors $(\underline{i}_1, \underline{p}, \underline{q}, \underline{e}_4, \underline{e}_5, \underline{e}_6)$, where $\underline{p} = \frac{\hat{\underline{\sigma}}}{\|\hat{\underline{\sigma}}\|}$, $\underline{q}$ is such that $\|\underline{q}\| = 1$ and $\underline{p} \cdot \underline{q} = 0$.

Lode angle δ is defined such that $\underline{p} = \cos\delta\,\underline{i}_2 + \sin\delta\,\underline{i}_3$ and $\underline{q} = -\sin\delta\,\underline{i}_2 + \cos\delta\,\underline{i}_3$.

To simplify calculation, a change of the basis in the principal stresses space is performed, from the $\mathcal{E}$ stress basis to the $\mathcal{D}$ deviatoric stress basis.

This change of the basis is justified by the fact that the Drücker-Prager limit surface is independent of the third stress invariant, i.e. the Lode angle. This change of the basis is carried out only on the subspace of dimension 3 related to the principal stresses, and can then be reduced to the following formulation:

$$\underline{\sigma}_{|\mathcal{D}} = \underline{\underline{Q}}_{(\mathcal{E}\to\mathcal{D})} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \end{pmatrix} = \begin{pmatrix} \frac{I_\sigma}{\sqrt{3}} \\ \frac{II_{\hat{\sigma}}}{\sqrt{3}} \\ 0 \end{pmatrix} = \begin{pmatrix} s_{11} \\ s_{22} \\ 0 \end{pmatrix} \quad (3)$$

With $\underline{\underline{Q}}_{(\mathcal{E}\to\mathcal{D})}$ a fourth-order transition tensor from the $\mathcal{E}$ basis to the $\mathcal{D}$ basis defined as following:

$$\underline{\underline{Q}}_{(\mathcal{E}\to\mathcal{D})} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{2\cos\delta}{\sqrt{6}} & -\frac{2\sin\delta}{\sqrt{6}} & 0 & 0 & 0 \\ \frac{1}{\sqrt{3}} & -\frac{\cos\delta}{\sqrt{6}} + \frac{\sin\delta}{\sqrt{2}} & \frac{\cos\delta}{\sqrt{2}} + \frac{\sin\delta}{6} & 0 & 0 & 0 \\ \frac{1}{\sqrt{3}} & -\frac{\cos\delta}{\sqrt{6}} - \frac{\sin\delta}{\sqrt{2}} & -\frac{\cos\delta}{\sqrt{2}} + \frac{\sin\delta}{6} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (4)$$

2.3. Flow rule

The flow rule involves the dilatancy angle. It is defined by:

$$g = II_{\hat{\sigma}} - I_\sigma \tan\alpha_\psi = 0 \quad (5)$$

With α_ψ being the dilatancy angle.

If $f = g$ (i.e. $\varphi=\psi$), the concerned material is called associated material by analogy to classical plasticity.

It is possible to write a simple formulation of the flow rule based on strain rates:

$$\dot{\underline{\varepsilon}} = \kappa.\frac{\partial g}{\partial \underline{\sigma}} \quad (6)$$

2.4. Invertibility

It reflects the fact that for a given stress rate there exists one and only one corresponding strain rate. Chambon et al. [6] showed that invertibility is ensured if $\|\underline{b}'\| < 1$. This is sketched by the Gudehus[7] graphical representation in figure 2.

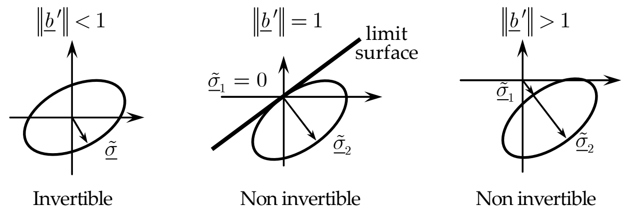

Figure 2. Invertibility of the CLoE model.

2.5. Consistency at the limit surface

For the stress state lying on the limit surface, consistency means that no outer stress rate response can be generated by the constitutive equation. This condition is written as:

$$\frac{\partial f}{\partial \underline{\sigma}} \cdot \tilde{\underline{\sigma}} < 0 \quad (7)$$

Another form of this condition involving the constitutive tensors $\underline{\underline{A}}$ and $\underline{b}'$ is established by Chambon et al. [8]:

$$\underline{\underline{A}} : \frac{\partial f}{\partial \underline{\sigma}} = -\alpha \underline{b}' \quad (8)$$

The constitutive tensors $\underline{\underline{A}}$ and $\underline{b}'$

• The tensor $\underline{b}'$

For an isotropic stress state, the tensor is assumed to be zero (i.e. the behavior is reversible). For a stress state lying on the limit surface, the tensor $\underline{b}'$ is determined from the flow rule $\|\underline{b}'_{\lim}\| = 1$. We deduce that:

$$^{T}\underline{b}'_{\lim} = \begin{pmatrix} \frac{-\tan\alpha_\psi}{\sqrt{1+\tan^2\alpha_\psi}} & \frac{-1}{\sqrt{1+\tan^2\alpha_\psi}} & 0 & 0 & 0 & 0 \end{pmatrix} \quad (9)$$

The evolution of the tensor $\underline{b}'$ is chosen as linear function of the invariants ratio ($II_{\hat{\sigma}} / I_\sigma$) between the isotropic state on the trisector of stress space ($\underline{b}' = \underline{0}$) and the final state of the limit surface ($\underline{b}' = \underline{b}'_{\lim}$).

$$\underline{b}' = \underline{b}'_{\lim} - \frac{1}{\tan\alpha_\varphi} \cdot \frac{II_{\hat{\sigma}}}{I_\sigma} \quad (10)$$

• The tensor $\underline{\underline{A}}$

For an isotropic stress state, $\underline{\underline{A}}$ is chosen similar to the isotropic elastic tensor:

$$\underline{\underline{A}}_{iso} = \begin{pmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 2g_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2g_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2g_3 \end{pmatrix} \quad (11)$$

Where $\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$ and $\mu = \frac{E}{2(1+\nu)}$ are the Lamé coefficients expressed in terms of Young's modulus E and Poisson's ratio ν. The shear moduli g_i are functions of the elastic shear modulus $G = \frac{E}{2(1+\nu)}$ and of intrinsic parameters ω_i used to calibrate the onset of the strain localization and the orientation of the shear band [9]:

$$g_i = G\left(1 - \omega_i \frac{-1}{\tan\alpha_\varphi} \cdot \frac{II_{\hat{\sigma}}}{I_\sigma}\right) \quad (12)$$

For a stress state on the limit surface, a rotation θ of the isotropic tensor $\underline{\underline{A}}$ is necessary to comply with the consistency condition at the limit surface. Therefore, we perform a change of basis of the isotropic tensor, from of the principal stresses basis $\mathcal{E}$ to the deviatoric basis $\mathcal{D}$, followed by a rotation θ therein.

$$\underline{\underline{A}}_{|D} = \underline{\underline{Q}}_{(\mathcal{E}\to\mathcal{D})} \underline{\underline{A}}_{iso} \underline{\underline{Q}}_{(\mathcal{E}\to\mathcal{D})}^{T} = \begin{pmatrix} 3\lambda+2\mu & 0 & 0 & 0 & 0 & 0 \\ 0 & 2\mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 2g_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2g_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2g_3 \end{pmatrix} \quad (13)$$

$$\underline{\underline{A}}_{\theta|D} = \underline{\underline{R}}_\theta \, \underline{\underline{A}}_{|D} = \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \underline{\underline{A}}_{|D} \quad (14)$$

Hence the expression of $\underline{\underline{A}}_{\theta|D}$:

$$\underline{\underline{A}}_{\theta|D} = \begin{pmatrix} (3\lambda+2\mu)\cos\theta & 2\mu\sin\theta & 0 & 0 & 0 & 0 \\ -(3\lambda+2\mu)\sin\theta & 2\mu\cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 2g_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2g_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2g_3 \end{pmatrix} \quad (15)$$

The evolution of the rotation of angle θ is chosen as that of $\underline{b}'$ between θ = 0 and θ = $\theta_{\lim}$:

$$\theta = \theta_{\lim} \frac{-1}{\tan\alpha_\varphi} \frac{II_{\hat{\sigma}}}{I_\sigma} \quad (16)$$

The $\theta_{\lim}$ angle is obtained from the consistency equation (8) to the limit surface:

$$\tan\theta_{\lim} = \frac{\tan\alpha_\varphi \left(R + \frac{\tan\alpha_\varphi}{\tan\alpha_\psi}\right)}{R + \tan\alpha_\varphi . \tan\alpha_\psi}, \; R = \frac{3\lambda+2\mu}{2\mu} = \frac{1+\nu}{1-2\nu} \quad (17)$$

3. EXISTENCE AND UNIQUENESS OF THE SOLUTION

Within the framework of elastic-plastic laws, under the assumption of small transformations, Hill [10] for associated materials and Bigoni and Hueckel [11] for non-associated materials showed that the uniqueness of the solution of the boundary value problem speed is ensured if the SOW is positive. Chambon and Caillerie [6] did the same for CLoE laws in small deformations. In the light of those results, the study of the existence and uniqueness of the solution of the rate boundary value problem in this paper is done through the study of the sign of the SOW.

3.1. Second order work (SOW)

The second order work is defined by:

$$W = \underline{\dot{\varepsilon}} : \underline{\dot{\sigma}} = \underline{\dot{\varepsilon}} : \underline{\underline{A}} : \underline{\dot{\varepsilon}} + \underline{\dot{\varepsilon}} : \underline{\underline{A}} \cdot \underline{b}' \|\underline{\dot{\varepsilon}}\| \quad (18)$$

We consider a strain rate $\underline{\dot{\varepsilon}}$ of norm equal to 1 ($\|\underline{\dot{\varepsilon}}\| = 1$) :

$$\underline{\dot{\varepsilon}} = a_1 \underline{i_1} + a_2 \underline{p} + a_3 \underline{q} + a_4 \underline{i_4} + a_5 \underline{i_5} + a_6 \underline{i_6} \quad (19)$$

$$a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 = 1 \quad (20)$$

This leads to the general formulation of the second order work:

$$\begin{aligned} W = {} & a_1^2 A_{11} + a_2^2 A_{22} + a_1 a_2 (A_{12} + A_{21}) + a_1 (b'_1 A_{11} + b'_2 A_{12}) \\ & + a_2 (b'_1 A_{21} + b'_2 A_{22}) + 2a_3^2 \mu + 2a_4^2 g_1 + 2a_5^2 g_2 + 2a_6^2 g_3 \end{aligned} \quad (21)$$

Where:

$$\underline{b}' = b'_1 \underline{i_1} + b'_2 \underline{p}\,, \quad A_{11} = 2\mu R\cos\theta,\; A_{21} = -2\mu R\sin\theta, \quad A_{12} = 2\mu\sin\theta \text{ and } A_{22} = 2\mu\cos\theta \quad (22)$$

3.2. Study of second order work sign

For the isotropic stress state, we have: $\underline{b}' = \underline{0}$, θ = 0 and $g_i = \mu$, therefore:

$$W = a_1^2 A_{11} + 2\mu(a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2) \quad (23)$$

The SOW is then strictly positive, regardless of the load.

For other stress states, minimizing the SOW under the condition (20) is performed by introducing a Lagrange multiplier λ. The expression *L* to be minimized is:

$$L = W - \lambda(\|\dot{\underline{\varepsilon}}\| - 1) = 0 \quad (24)$$

Hence the system of equations:

$$\begin{cases} \partial L / \partial a_1 = 2a_1(A_{11} + \lambda) + a_2(A_{12} + A_{21}) + b_1' A_{11} + b_2' A_{12} = 0 \\ \partial L / \partial a_2 = 2a_2(A_{22} + \lambda) + a_1(A_{12} + A_{21}) + b_1' A_{21} + b_2' A_{22} = 0 \\ \partial L / \partial a_3 = 2a_3(2\mu + \lambda) = 0 \\ \partial L / \partial a_4 = 2a_4(2g_1 + \lambda) = 0 \\ \partial L / \partial a_5 = 2a_5(2g_2 + \lambda) = 0 \\ \partial L / \partial a_6 = 2a_6(2g_3 + \lambda) = 0 \\ \partial L / \partial \lambda = a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 - 1 = 0 \end{cases}$$

Therefore the extremals are obtained for:

– **Situation 1 :** λ ≠ -2μ and λ ≠ -2g (ω = 0 or ω ≠ 0) and $a_3^2 = a_4^2 = a_5^2 = a_6^2 = 0$ and $a_1^2 + a_2^2 = 1$

$$\begin{aligned} W &= a_1^2 A_{11} + a_2^2 A_{22} + a_1 a_2 (A_{12} + A_{21}) \\ &\quad + a_1(b_1' A_{11} + b_2' A_{12}) + a_2(b_1' A_{21} + b_2' A_{22}) \end{aligned} \quad (25)$$

In this case, isotropic or anisotropic nature of the model does not affect the SOW.

To investigate the influence of non-associated nature of the model, we write:

$$b_1' = -r\cos(\gamma),\ b_2' = -r\sin(\gamma) \ \text{ with } \gamma = \pi/2 - \alpha_\psi \quad (26)$$

And: $a_1 = \cos(\gamma + \varpi)$, $a_2 = \sin(\gamma + \varpi)$ (27)

With ϖ measuring the "offset" from the flow rule.

And the SOW is:

$$\begin{aligned} W/2\mu &= R.\cos(\theta + \gamma + \varpi)\left[\cos(\gamma + \varpi) - r\cos\gamma\right] \\ &\quad + \sin(\theta + \gamma + \varpi)\left[\sin(\gamma + \varpi) - r\sin\gamma\right] \end{aligned} \quad (28)$$

Figures 3 and 4 represent the evolution of the SOW with respect to the strain rate (i.e. according to ϖ) of the associated model on the limit surface (r = 1) for various values of Poisson's ratio ν and different values of φ the friction function (within the allowable margin [12]). The simulations show that the minimum value of the SOW is null and it is for the direction of the strain rate corresponding to the flow rule (ϖ = 0). The boundary value problem keeps its uniqueness and the strain localization is therefore excluded.

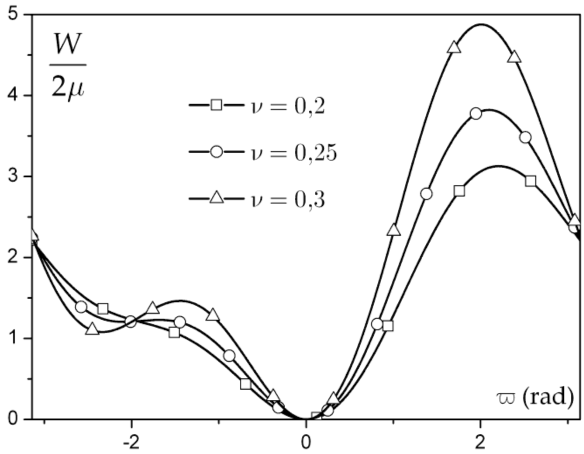

Figure 3. SOW on limit surface versus strain rate for associated model for different values of Poisson ratio *ν*.

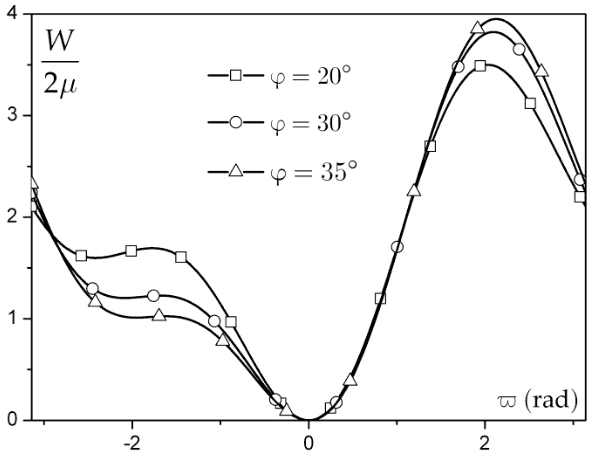

Figure 4. SOW on limit surface versus strain rate for standard model for different values of the friction angle φ.

The same simulations have been renewed for non-associated model for different pairs of the functions (φ,ψ) and then for various values of Poisson's ratio ν. The results sketched in figures 5 and 6 show clearly that the SOW loses its positivity on a ϖ range. The value of the SOW for ϖ = 0 is always zero, but it is no longer the minimum.

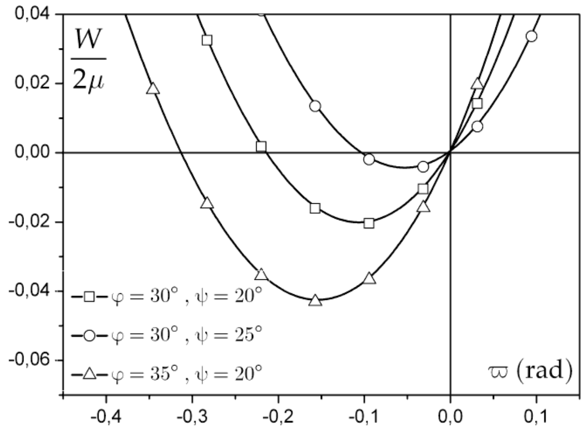

Figure 5. SOW on limit surface versus strain rate for non-associated model for different values of the angles (φ,ψ).

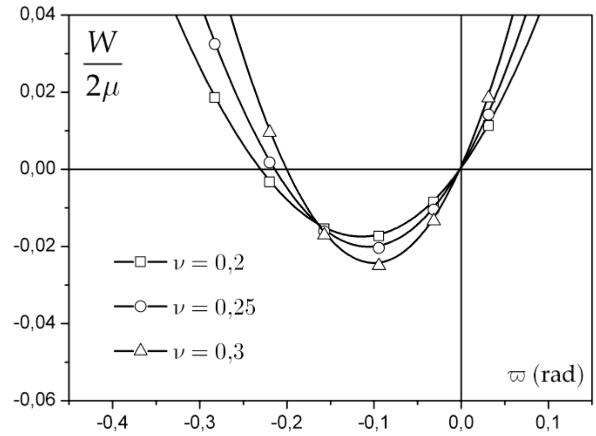

Figure 6. SOW on limit surface versus strain rate for non-associated model for different values of Poisson ratio ν.

The SOW is negative on a ϖ range for $r = \|\underline{b}'\| = 1$ (i.e. on the limit surface), which implies that it will continue to be so for the values of $r = \|\underline{b}'\| < 1$ (i.e. inside the limit surface) and for certain values of strain rate. Figure 7 clearly shows the areas on the plane (ϖ, r) where the SOW losses its positivity.

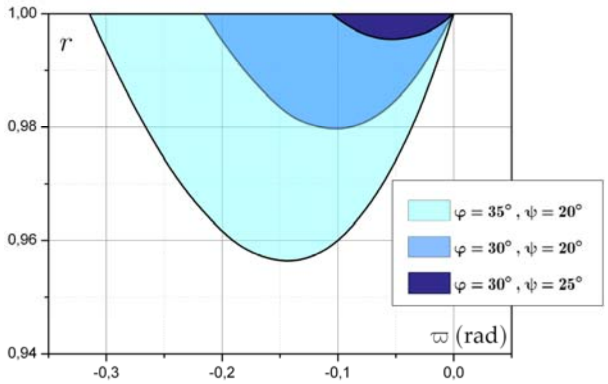

Figure 7. Area of SOW negativeness for non-associated model for different values of the functions (φ,ψ).

– **Situation 2 :** $\lambda = -2g$ ($g_i = g$)

This implies $a_3 = 0$ and $a_4^{\ 2} + a_5^{\ 2} + a_6^{\ 2} = 1 - a_1^{\ 2} - a_2^{\ 2}$, a_1 and a_2 are determined from the first two equations:

$$\begin{cases} \mathsf{L}a_1 + \mathsf{M}a_2 = \mathsf{P} \\ \mathsf{M}a_1 + \mathsf{N}a_2 = \mathsf{R} \end{cases}$$

$$\mathsf{L} = 2(R.\cos\theta - \mathcal{G}),\ \ \mathsf{M} = (1-R).\sin\theta,\ \ \mathsf{N} = 2(\cos\theta - \mathcal{G}),\ g = \mathcal{G}\mu,$$

$$\mathsf{P} = -b_1'.R.\cos\theta - b_2'.R.\sin\theta, \quad \mathsf{R} = +b_1'.R.\sin\theta - b_2'.R.\cos\theta$$

Hence, the minimum value of the SOW:

$$W = 2\mu\left(\mathcal{G} - \frac{1}{2}\cdot\frac{\mathsf{NP}^2 + \mathsf{LR}^2 - 2\mathsf{MPR}}{\mathsf{LN} - \mathsf{M}^2}\right) \qquad (29)$$

Figures 8 and 9 show the evolution of the minimum value of TSO with respect to $\mathcal{G}$ parameter ($\mathcal{G} = 1$ for the isotopic model) measuring the anisotropy for the standard model for different values of Poisson's ratio ν and different values of the friction function φ (resp.) for the state of stress on the limit surface. We observe that even if the model is associated the SOW may become negative for values of $\mathcal{G}$ less than almost 0.5. This result meets the results obtained for the Von Mises [3] and Mohr-Coulomb [4] mini-Cloe model.

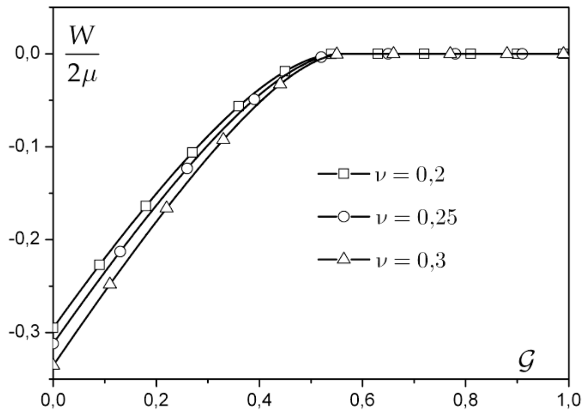

Figure 8. Minimum value of the SOW with respect of anisotropy for the associated model for different values of Poisson ratio ν.

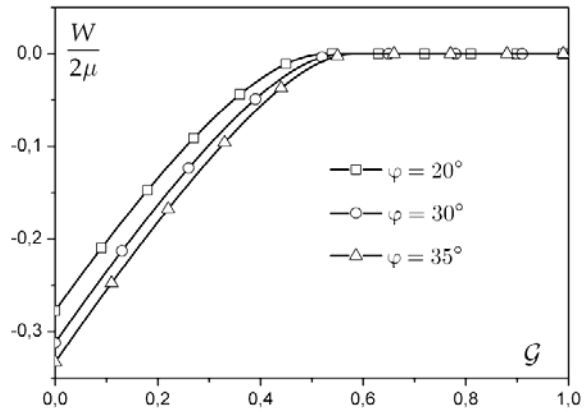

Figure 9. Minimum value of the SOW with respect of anisotropy for the associated model for different values of the friction angle φ.

The same simulations are conducted for non-associated case. The results sketched in figures 10 and 11 show that even for an isotropic model ($\mathcal{G}$ =1) the minimum value of the SOW is negative. In addition, the SOW starts to become more negative from a value $\mathcal{G}$ of nearly 0.5. Again, these results are similar to those established for the Mohr-Coulomb mini-CLoE [4].

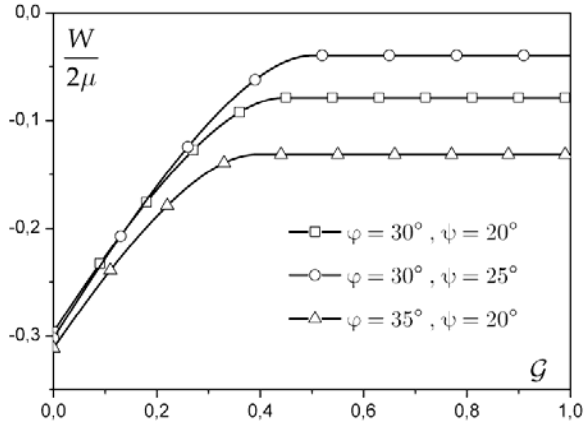

Figure 10. Minimum value of the SOW with respect of anisotropy for non-associated model for different values of the angles (φ,ψ).

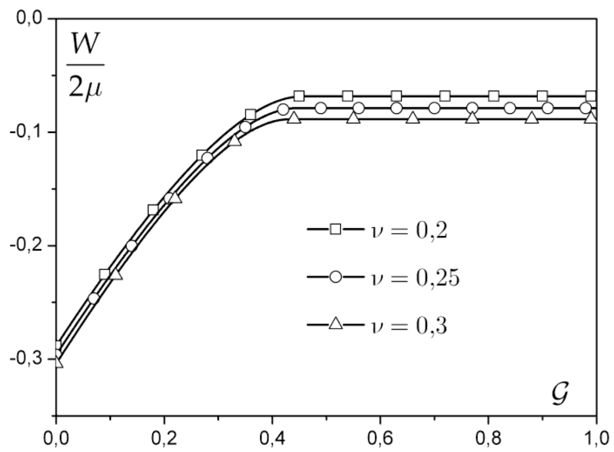

Figure 11. Minimum value of the SOW with respect of anisotropy for non-associated model for different values of Poisson ratio ν.

– **Situation 3 :** $\lambda = -2\mu$ and $\omega \neq 0$ ($\mathcal{G} \neq 1$)

This implies $a_4 = a_5 = a_6 = 0$ and $a_3^2 = 1 - a_1^2 - a_2^2$, a_1 and a_2 are determined from the first two equations. The expression of the minimum value of the SOW takes a form similar to the previous case:

$$W = 2\mu\left(1 - \frac{1}{2}\cdot\frac{\mathrm{NP}^2 + \mathrm{LR}^2 - 2\mathrm{MPR}}{\mathrm{LN} - \mathrm{M}^2}\right) \qquad (30)$$

For the associated model the minimum value of the SOW is zero and for the non-associated one it is negative, reflecting the onset of strain localization band.

– **Situation 4 :** $\lambda = -2\mu = -2g$ ($\omega = 0$)

$a_3^2 + a_4^2 + a_5^2 + a_6^2 = 1 - a_1^2 - a_2^2$, a_1 and a_2 are determined from the first two equations. The expression of the minimum value of the SOW is again the same as the two previous cases. Therefore, the same conclusions are reconducted:

$$W = 2\mu\left(1 - \frac{1}{2}\cdot\frac{\mathrm{NP}^2 + \mathrm{LR}^2 - 2\mathrm{MPR}}{\mathrm{LN} - \mathrm{M}^2}\right) \qquad (31)$$

4. CONCLUSION

The results of the study of the loss of uniqueness of the boundary value problem involving Drücker-Prager mini-Cloe model showed that for an isotropic associated model, SOW keeps its positivity within the limit surface and therefore the uniqueness of the solution is preserved and any bifurcation is excluded. Once an anisotropy is introduced in the model, areas of the negativity of the SOW appear. For non-associated model, the SOW loses its positivity for a range of strain rates even for the isotropic case and anisotropy makes the model more susceptible to the occurrence of bands of strain localization.

These conclusions are the same as those made for the Von Mises mini-Cloe model [3] and Mohr-Coulomb mini-Cloe model [4] and those well-known for classical nonlinear elastic-laws (Bigoni and Hueckel [11], Désoyer and Cormery [13] Loret and Rizzi [14]).

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