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Study of Some Properties of Squares of Whole Numbers

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ABSTRACT
In this paper, some important properties of squares of whole numbers are reported. An algorithm is presented to show the results that involve the differences between squares of whole numbers and multiples of five closest to the corresponding squares. Also, such kind of difference has a sequence, 0, 1, -1, -1, 1, which is followed for every five consecutive whole numbers starting from 0. The algorithm thus developed, has been demonstrated with some numerical examples, and graphical deductions.

General Terms
Number Theory, Algorithms

Keywords
Whole number, The Division Algorithm, Square of a whole number, MATLAB

1. INTRODUCTION
Squares of whole numbers and its properties have been considered to be of prime importance in Number Theory. The relevance of squares of numbers in Pythagorean Triangles dates back to time long before Pythagoras [1] [2] [3]. The G. A. Plympton Collection at Columbia University houses a Babylonian Clay Tablet which reveals that the Babylonians were well-aware of the theorem more than 3500 years ago [1]. Around 2000 B. C., the Egyptians and ancient Indians also used the primitive 3-4-5 Pythagorean triangles. The association of Fibonacci and Lucas Numbers with the Pythagorean Triangles has diversified the application of squares of whole numbers [4]. Diophantus' lemma on the sum of two squares, Euler's (1743) lemma on the sums of four squares [1] and Lagrange's (1770) theorem on expression of a whole number as the sum of four squares [1] are the notable fundamental works in Number Theory which deal with the properties of squares of whole numbers. The Fermat's Last Theorem [1] [2] and Beal's Conjecture [1] further discuss the use of squares of whole numbers. An interesting relationship between Pell's Equation [1] [5] [6] and Pythagorean Triples [1] [7] x-y-z such that x and y are consecutive integers, leads to intuitive possibilities of more properties of squares of whole numbers.

Rest of the paper is composed as follows: section 2 briefly describes the definitions and preliminaries of whole number and square of numbers. Section 3 describes the Division Algorithm. Section 4 deals with the proposed study in the present paper— arrangement of whole numbers in a table, and difference between multiples of 5 closest to squares of consecutive numbers. In section 5, a new algorithm has been developed to demonstrate our proposed approach. The algorithm has been verified with some numerical examples. Section 6 discusses the results of the proposed study with the help of some examples and plots. Finally, the paper is concluded in Section 7.

2. DEFINITIONS AND PRELIMINARIES
2.1 Whole Number
The set of positive integers, together with 0 constitutes the set of whole numbers, defined by

$W = \{0,1,2,3,4,......\}$ [1]

The whole numbers collectively comprise of the natural numbers and the number '0'. However, this is a convention or a notion and not a general agreement. The number '0' has been considered as a natural number in many cases [3] [8] [9].

2.2 Square of a whole number
A square of a whole number is another whole number obtained by multiplying the given number with itself.

3. THE DIVISION ALGORITHM
Divisibility [2] [5] [10] [11] is one of the basic concepts in number theory. If a and b be two integers, then b is a divisor of a, and a is a multiple of b if there exists an integer c such that

$a = cb$

If b divides a, then we denote the operation as $b \mid a$.

Theorem (Division Algorithm) [2] [3] [10] [11]: Let a and b be integers with $b > 0$. There exist unique integers, r and s such that

$a = rb + s$

and

$0 \le s \le b$.

The integer r is called the quotient and the integer s is called the remainder in the division of a by b.

4. PROPOSED STUDY
4.1 Arrangement of Whole Numbers in a Table
Let $x \in W$ such that all x are arranged in a table (Table 1) in rows of 5 columns. Then we have the following conclusions:

(a) The position of the whole number x in the table is given by (p, q) where $p = \lfloor x/5 \rfloor$ and q = x mod 5.

(b) The difference between $Y = x^2$ and Z which is multiple of 5 closest to Y, is either -1 or 0 or +1.

(c) Y – Z in each row follows an exact sequence which is 0, 1, -1, -1, 1. We have further two conclusions from this:

i. Since Y – Z = 0 for q = 0, so the multiple of 5 closest to the square of the number located in the first column of any row is the square of the number itself.

ii. The algebraic sum of Y – Z vanishes in each row.

4.2 Difference between Multiples of 5 Closest to Squares of Consecutive Numbers

The difference between the consecutive values of Z can again be arranged in a matrix of 5 columns as shown in Table 2. Here, Z (p, q) refers to the multiple of 5 closest to $Y = x^2$ where the position of x is defined by (p, q).

Let r denote any row number. Then, the conclusions derived from Table 2 for the rth row are:

(a) The number in the first column of Table 2 is given by

$$n_{1r} = 10r \tag{1}$$

(b) The numbers in the next three columns are equal and found to be $5(2r+1)$ each. The sum of the values in the three columns is

$$n_{2r} = 15(2r+1) \tag{2}$$

(c) The number in the fifth and last column of any row r is given as

$$n_{3r} = 10(r+1) \tag{3}$$

Then the sum of all the numbers in the rth row is given by

$$n_r = n_{1r} + n_{2r} + n_{3r} = 10r + 15(2r+1) + 10(r+1)$$

or,

$$n_r = 50r + 25 = 25(2r+1) \tag{4}$$

4.3 Finding the Square of any Whole Number

In order to find the square of a whole number, x, it is required to traverse in the row in which x is located. According to the method of arrangement explained in Section 4.1, $p = \lfloor x/5 \rfloor$ denotes the row in which the whole number x is located. By utilizing the conclusions of Table 1 and Table 2, the square can be determined using the three steps of traversal as discussed below:

(a) Determining the square of a whole number x whose position is (p, 0): The first step to reach the required square in the pth row in the matrix is to compute the square of the whole number located in the first column, that is, (p,0). Using (4), the square of a whole number located in the first column of the pth row is given by:

$$S_1 = \sum_{r=0}^{p-1} n_r \tag{5}$$

where $p = \lfloor x/5 \rfloor$**.** For a whole number whose q = 0, its square is given by (5). However for others, the following two steps need to be followed.

(b) Determining the difference Z (p, q) – S_{1r} for q = 1, 2, 3, and 4: From conclusions in Section 2, each square is related to the multiple of 5 closest to it. As the pattern of difference, Y – Z, between the square of a whole number and the multiples of 5 closest to it in a row is known, so, the square of a number can be found out once we have the multiple of 5 closest to it. In order to do so for a whole number x, we calculate the difference Z (p, q) –S_1 where $p = \lfloor x/5 \rfloor$ and $q = x \bmod 5$.

$$S_2 = 10p + (10p+5)(q-1) \tag{6}$$

(c) Square of x: In a row in Table 1, the difference between a square and the multiple of 5 closest to it follows a definite pattern (L): 1, -1, -1, 1 for values of q equal to 1, 2, 3, 4 respectively. Therefore, the square of a whole number, x is given by:

$$Y = \begin{cases} S_1 & q = 0 \\ S_1 + S_2 + L & q \neq 0 \end{cases} \tag{7}$$

TABLE 1: (A) to (F). Squares of some whole numbers to reflect the properties of their squares

The various symbols have the following meanings with respect to the whole number, x:

$\mathbf{p = \lfloor x/5 \rfloor}$

q = x mod 5

Y = $\mathbf{x^2}$

Z = multiple of 5 closest to Y

x	0	1	2	3	4
p	0	0	0	0	0
q	0	1	2	3	4
Y	0	1	4	9	16

Z	0	0	5	10	15
Y– Z	0	1	-1	-1	1

(A)

x	5	6	7	8	9
p	1	1	1	1	1
q	0	1	2	3	4
Y	25	36	49	64	81
Z	25	35	50	65	80
Y – Z	0	1	-1	-1	1

(B)

x	10	11	12	13	14
p	2	2	2	2	2
q	0	1	2	3	4
Y	100	121	144	169	196
Z	100	120	145	170	195
Y - Z	0	1	-1	-1	1

(C)

x	15	16	17	18	19
p	3	3	3	3	3
q	0	1	2	3	4
Y	225	256	289	324	381
Z	225	255	290	325	380
Y - Z	0	1	-1	-1	1

(D)

x	20	21	22	23	24
p	4	4	4	4	4
q	0	1	2	3	4
Y	400	441	484	529	576
Z	400	440	485	530	575
Y – Z	0	1	-1	-1	1

(E)

x	25	26	27	28	29
p	5	5	5	5	5
q	0	1	2	3	4
Y	625	676	729	784	841
Z	625	675	730	785	840
Y - Z	0	1	-1	-1	1

(F)

TABLE 2: Difference between consecutive values of Z derived from Table 1

p	**Z(p,1) – Z(p,0)**	**Z(p,2) – Z(p,1)**	**Z(p,3) – Z(p,2)**	**Z(p,4) – Z(p,3)**	**Z(p+1,0) – Z(p,4)**
0	0	5	5	5	10
1	10	15	15	15	20
2	20	25	25	25	30
3	30	35	35	35	40
4	40	45	45	45	50
5	50	55	55	55	60

Here, Z (p, q) refers to the multiple of 5 closest to x^2 where the row number is $p = \lfloor x/5 \rfloor$ and column number is q = x mod 5

5. ALGORITHM OF THE PROPOSED APPROACH

STEP 1: Consider a whole number, x.

STEP 2: Determine the position of x in the table where whole numbers are arranged in rows of 5; assign the row number to p and column number to q.

$$p \leftarrow \lfloor x/5 \rfloor \text{ and } q \leftarrow x \bmod 5$$

STEP 3: Compute the first square of the row in which x is located and assign it to S_1

$$S_1 \leftarrow \sum_{r=0}^{p-1} 25(2r+1)$$

STEP 4: Compute the difference between S_1 and the value of the multiple of 5 closest to $Y = x^2$ and assign it to S_2.

If q = 0

$$S_2 \leftarrow 0$$

else

$$S_2 \leftarrow [10p + \{(10p+5)(q-1)\}]$$

STEP 5: If q = 0

$$Y \leftarrow S_1 + S_2 + 0$$

else if q = 1

$$Y \leftarrow S_1 + S_2 + 1$$

else if q = 2

$$Y \leftarrow S_1 + S_2 - 1$$

else if q = 3

$$Y \leftarrow S_1 + S_2 - 1$$

else if q = 4

$$Y \leftarrow S_1 + S_2 + 1$$

Display Y as square of x.

6. RESULTS AND DISCUSSION

6.1 Findings

Some numerical examples have been tabulated in Table 3, showing the values of S_1, S_2, and Y. The results have been derived using the mathematical tool, MATLAB. The examples of whole number have been considered in such a manner that they reflect the properties of the squares for all possible values of q (0, 1, 2, 3, 4).

TABLE 3: Numerical examples of the proposed algorithm

x	**S_1**	**S_2**	**Y**
3	0	8	9
7	25	25	49

44	1600	335	1936
86	7225	170	7396
579	330625	4615	335241
895	801025	0	801025
6232	38812900	24925	38837824
9988	99700225	59920	99760144
56871	3234196900	113740	3234310641
86280	7444238400	0	7444238400

6.2 Variation of S_1 with Whole Number, x

The variation of the parameter S_1 mentioned in equation (5) with 30 whole numbers ranging from 0 to 29 has been shown in Figure 1. As evident, S_1 being the square of the first number located in p^{th} row (where $p = \lfloor x/5 \rfloor$), it remains same for a set of 5 numbers with the value of q ranging from 0 to 4.

6.3 Variation of S_2 with Whole Number, x

A variation of S_2 mentioned in equation (6), with 30 whole numbers ranging from 0 to 29 has been depicted in Figure 2. For a set of whole numbers falling in the same row, S_2 increases linearly. Observing equations (6) and (7), we have used $S_2 = 0$ for those whole numbers which lie in the first column of a row (q = 0). The slope of the locus of S_2 increases as we move from a lower value of p to a higher one.

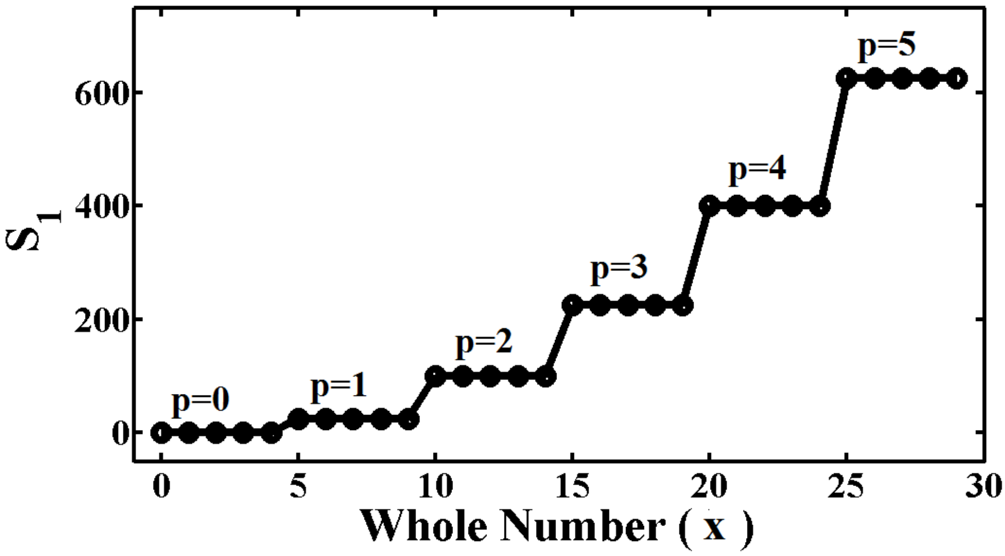

Fig 1: Plot of S_1 versus whole numbers mentioned in Table 1 and Table 23

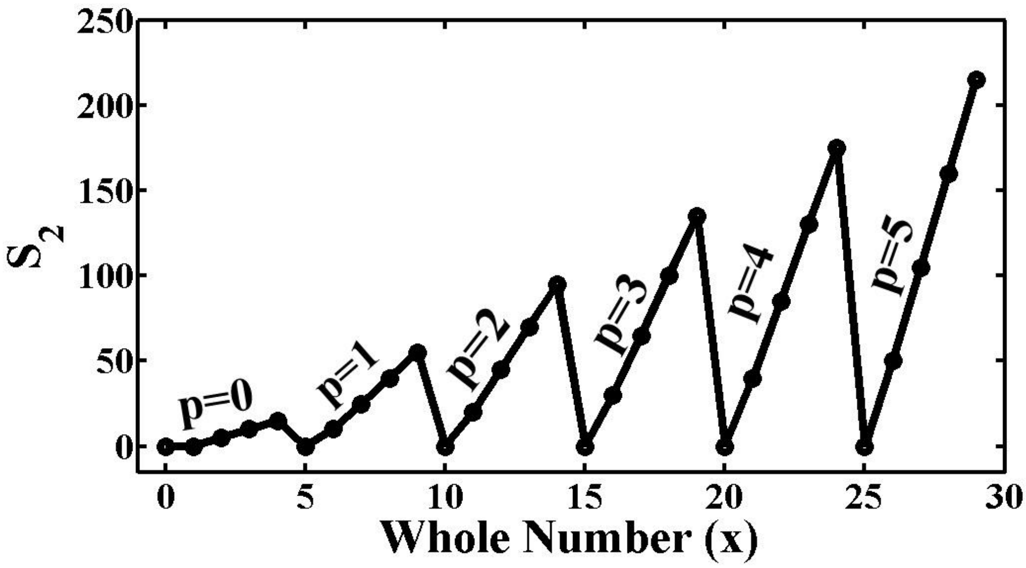

Fig 2: Plot of S_2 versus whole numbers mentioned in Table 1 and Table 2

7. CONCLUSION

In this paper attention has been made to the study of squares of whole numbers and their properties.

a. The difference of square of a whole number and multiple of 5 closest to the square of whole number is studied. It has been observed that this difference gives the result either -1 or 0 or +1 in case of any whole number.

b. Further, these differences for each and every 5 whole numbers follow a particular repetitive pattern, 0, 1, -1, -1, 1, whose sum vanishes.

In case of representing the whole numbers in a tabular form, suppose (p, q) is the position of the whole number where $p = \lfloor x/5 \rfloor$ and q = x mod 5.

c. When q = 0, the above difference vanishes, that is, square of any whole number is equal to the multiple of 5 closest to the square of the whole number.

The results of the algorithm developed in this paper establish the properties of the squares of whole numbers reported in this paper. This work suggests that whole numbers can be arranged in a matrix of 5 columns where each row possesses similar properties, and the position of each number can be defined. This may prove to be a major connecting link between Number Theory and Matrix Algebra. Moreover, detailed analyses of the new intermediate parameters, S_1 and S_2, are essential in order to discover more properties of whole numbers and their squares.

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