New View of Ideals on PU-Algebra

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ABSTRACT
In this manuscript, we introduce a new concept, which called PU-algebra X . We state and prove some theorems about fundamental properties of it. Moreover ,we give the concepts of a weak right self-maps, weak left self-maps and investigated some its properties. Further, we have proved that every associative PU-algebra is a group and every p-semisimple algebra is an abelian group. We define the centre of a PU-algebra X and show that it is a p-semisimple sub-algebra of X, which consequently implies that every PU-algebra contains a p-semisimple PU-algebra .Furthermore, we give the concepts of ideals (-ideals , i=1,2,3,4) in PU-algebra , classified they into classes correspond to various formula and we have proved that, they are coincide .
Mathematics Subject Classification: 06F35, 03G25, 08A30.

Keywords
PU-algebra, ideals of PU-algebra, G-part and P-radical of a PU-algebra, homomorphism of PU-algebra.

1. INTRODUCTION
In 1966, Imai and Iseki [2] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [1], Hu and Li introduced a wide class of abstract algebras: BCH-algebras. They are shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. In [7], Neggers and Kim introduced the notion of d-algebras, which is a generalization of BCK-algebras and investigated a relation between d-algebras and BCK-algebras. Neggers et al. introduced the notion of Q-algebras [8], which is a generalization of BCH/BCI/BCK-algebras. Recently, Kim [3] defined a BE-algebra.[5] Meng, defined the notion of CI-algebra as a generalization of a BE-algebra.[4] Megalai and Tamilarasi introduced the notion of a TM-algebra which is a generalization of BCK/BCI/BCH-algebras and several results are presented. In 2009 , C. Prabpayak and U. Leerawat [9,10] introduced algebraic structure which is called KU-algebras , and studied ideals and congruencies in KU-algebras .They gave the concept of homomorphisms of KU-algebras and investigated some related properties. Moreover they derived some straightforward consequences of the relations between quotient KU-algebras and isomorphisms and also investigated some of its properties. In this paper we will introduce a new algebraic structure called PU-algebra, which is a dual for TM-algebra and investigated severed basic properties. Moreover we derived new view of several ideals on PU-algebra and studied some properties of them.

2. PRELIMINARIES
Now we will recall some known concepts related to PU-algebra from the literature which will be helpful in further study of this article.

Definition 2.1[9].By a KU-algebra we mean an algebra (X, *, 0) of type (2, 0) with a single binary operation * that satisfies the following identities:

for any x, y, z $\in$ X,

(ku1): $(x * y) * [(y * z) * (x * z)] = 0$,

(ku2): $x * 0 = 0$,

(ku3): $0 * x = x$,

(ku4) : $x * y = 0 = y * x$ implies $x = y$.

Example 2.2: Let X = {0, 1, 2, 3, 4} in which * is defined by the following table:

*	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	1	0	3	3
3	0	0	2	0	2
4	0	0	0	0	0

It is easy to show that X is a KU - algebra.

Lemma 2.3 [6]. In a KU-algebra (X , * , 0) , the following hold :

(i) $x \leq y$ imply $y * z \leq x * z$.

(ii) $z * (y * x) = y * (z * x)$.

Definition 2.4. A PU-algebra is a non-empty set X with a constant $0 \in X$ and a binary operation $*$ satisfying the following conditions:

(I) $0 * x = x$,

(II) $(x * z) * (y * z) = y * x$ for any x, y, z $\in$ X.

On X we can define a binary relation " $\leq$ " by: $x \leq y$ if and only if $y * x = 0$.

Example 2.5. Let X = {0, 1, 2, 3, 4} in which $*$ is defined by

$*$	0	1	2	3	4
0	0	1	2	3	4
1	4	0	1	2	3
2	3	4	0	1	2
3	2	3	4	0	1
4	1	2	3	4	0

Using the algorithms in Appendix, we can prove that (X, $*$, 0) is a PU-algebra, but not a KU-algebra , since $1*0=4\neq 0$ .On the other hand, in Example 2.2., X is a KU-algebra, but is not a PU-algebra since $(2 * 1) * (3 * 1) = 1 * 0 \neq 3 * 2 = 2$, which means that PU-algebra and KU-algebra are deferent.

Example 2.6. ($\mathbb{R}$, $*$, 0) where $*$ is defined by $x * y = y - x$ for all $x, y \in \mathbb{R}$ is a PU-algebra.

Proposition 2.7. In PU-algebra (X, $*$, 0) the following hold for all $x, y, z \in X$:

(a) $x * x = 0$.

(b) $(x * z) * z = x$.

(c) $x * (y * z) = y * (x * z)$.

(d) $x * (y * x) = y * 0$.

(e) $(x * y) * 0 = y * x$.

(f) If $x \le y$, then $x * 0 = y * 0$.

(g) $(x * y) * 0 = (x * z) * (y * z)$.

(h) $x * y \le z$ if and only if $z * y \le x$.

(i) $x \le y$ if and only if $y * z \le x * z$.

(j) In PU-algebra (X, $*$, 0) , the following are equivalent:

(1) $x = y$,

(2) $x * z = y * z$,

(3) $z * x = z * y$.

(k) The right and the left cancellation laws hold in X.

Proof:

(a) Putting $x = y = 0$ in Definition 2.4. (II), we get $(0 * z) * (0 * z) = 0 * 0$. Then

$z * z = 0$ (by Definition 2.4. (I)).

(b) $(x * z) * z = (x * z) * (0 * z)$ (by Definition 2.4. (I))

$= 0 * x = x$ (by Definition 2.4. (I), (II))

(c) $x * (y * z) = [(x * z) * z] * (y * z)$ (from (b))

$= y * (x * z)$ (by Definition 2.4. (II)).

(d) $x * (y * x) = y * (x * x)$ (from Proposition 2.7 (c))

$= y * 0$ (from Proposition 2.7 (a)).

(e) $(x * y) * 0 = (x * y) * (y * y)$ (from Proposition 2.7 (a))

$= y * x$ (by Definition 2.4. (II)).

(f) $x \le y \Rightarrow y * x = 0$ (by the definition of PU-algebra)

$\Rightarrow x * 0 = x * (y * x) = y * 0$(from Proposition 2.7 (d)).

(g) $(x * y) * 0 = y * x$ (from Proposition 2.7 (e))

$= (x * z) * (y * z)$ (by Definition 2.4. (II)).

(h) $x * y \le z \Leftrightarrow z * (x * y) = 0 \Leftrightarrow x * (z * y) = 0$ (from Proposition 2.7 (c))

$\Leftrightarrow z * y \le x$.

(i) $x \le y \Leftrightarrow y * x = 0$ (by the definition of PU-algebra)

$\Leftrightarrow (x * z) * (y * z) = 0$ (by Definition 2.4. (II))

$\Leftrightarrow y * z \le x * z$.

(j) ((1) $\Rightarrow$ (3)): Clear.

((3) $\Rightarrow$ (2)): $z * x = z * y \Rightarrow (x * z) * 0 = (y * z) * 0$ (from Proposition 2.7 (e))

$\Rightarrow ((x * z) * 0) * 0 = ((y * z) * 0) * 0$

$\Rightarrow x * z = y * z$ (from Proposition 2.7 (b)).

((2) $\Rightarrow$ (1)): $x * z = y * z \Rightarrow (x * z) * z = (y * z) * z$

$\Rightarrow x = y$(from Proposition 2.7 (b)).

(k) Follows directly from (j).

Proposition 2.8. If (X, $*$, 0) is a PU-algebra, then for any $x, y, z \in X$,

(1) $(z * x) * (z * y) = x * y$,

(2) $(x * y) * z = (z * y) * x$.

Proof:

(1) By the definition of PU-algebra, we have that

$(z * x) * (z * y) = [(x * y) * (z * y)] * [0 * (z * y)] = 0 * (x * y) = x * y$.

(2) $(x * y) * z = [z * (x * y)] * 0$ (from Proposition 2.7 (e))

$= [x * (z * y)] * 0$ (from Proposition 2.7 (c))

$= (z * y) * x$ (from Proposition 2.7 (e)).

Lemma 2.9. If (X, $*$, 0) is a PU-algebra, then (X, $\le$) is a partially ordered set.

Proof: By Proposition 2.7. (a), we have that $x * x = 0$ i.e. $x \le x$.

Let $x \le y$, $y \le x$, then $x * y = 0 = y * x$. It follows that

$x = 0 * x$ (by Definition 2.4. (I))

$= (y * x) * (0 * x) = 0 * y = y$ (by Definition 2.4. (II) ,(I))

Let $x \le y$, $y \le z$ i.e. $y * x = 0 = z * y$. It follows that

$z * x = 0 * (z * x)$ (by Definition 2.4. (I))

$= (y * x) * (z * x) = z * y = 0$ (by Definition 2.4. (II))

i.e. $x \le z$. Therefore (X, $\le$) is a partially ordered set.

Remark 2.10. Every PU-algebra (X,$*$, 0) satisfying $(y * x) * x = y * x$ for all

$x, y \in X$ is a trivial algebra.

Proof: Putting $x = y$ in the equation $(y * x) * x = y * x$, we have $0 * x = 0$. By Definition 2.4. (I), $x = 0$. Hence X is a trivial algebra.

Proposition 2.11. If (X, $*$, 0) is a PU-algebra, then $(x * y) * (z * u) = (x * z) * (y * u)$ for all x, y, z and $u \in X$.

Proof: Let (X, $*$, 0) be a PU-algebra, then for all x, y, z and $u \in X$ we have that

$(x * y) * (z * u) = 0 * [(x * y) * (z * u)]$ (by Definition 2.4. (I))

$= [(y * u) * (y * u)] * [(x * y) * (z * u)]$ (from Proposition 2.7 (a))

= [[(x ∗ y) ∗ (z ∗ u)] ∗ (y ∗ u)] ∗ (y ∗ u) (from Proposition 2.8 (2))

= [[(y ∗ u) ∗ (z ∗ u)] ∗ (x ∗ y)] ∗ (y ∗ u) (from Proposition 2.8 (2))

= [(z ∗ y) ∗ (x ∗ y)] ∗ (y ∗ u) (by Definition 2.4. (II))

= (x ∗ z) ∗ (y ∗ u) (by Definition 2.4. (II)).

Corollary 2.12. If (X, ∗, 0) is a PU-algebra, then (x ∗ y) ∗ z = (x ∗ 0) ∗ (y ∗ z) for all x, y and z ∈ X.

Proof: Let (X, ∗, 0) be a PU-algebra, then for all x, y, z ∈ X we have that (x ∗ y) ∗ z = (x ∗ y) ∗ (0 ∗ z) (by Definition 2.4. (I))

= (x ∗ 0) ∗ (y ∗ z) (from Proposition 2.11).

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3. G-PART AND P-RADICAL OF A PU - ALGEBRA

Definition 3.1. Let X be a PU-algebra. For any subset S of X, we define G(S) = {x ∈ S: x ∗ 0 = x} , in particular if S = X, then we say that G(X) is the G-part of X. It is clear that if (X, ∗, 0) is a PU-algebra and if x ∈ G(X), then x = 0 ∗ x = x ∗ 0. For any PU-algebra X, the set B(X) = {x ∈ X: x ∗ 0 = 0} is called a P-radical of X. A PU-algebra X is said to be P-semisimple, if every element of X is minimal, i.e

B(X)={0}.The following property is obvious: G(X) ∩B(X) = {0}.we define $A(a\ ,b) = \{x \in X, b \le a * x\}$ *and the set* $\{x \in X, (x*0)*0 = x\}$ is called the center of X.

Proposition 3.2. Let (X, ∗, 0) be a PU-algebra and x, y, z ∈ X, then

(a) y ∈ G(X) ⇔ x ∗ (y ∗ x) = y.

(b) y ∈ B(X) ⇔ x ∗ (y ∗ x) = 0.

(C) x ∈ G(X) ⇔ x ∗ 0 ∈ G(X).

Proof: (a) By Proposition 2.7. (d), x ∗ (y ∗ x) = y ∗ 0 = y ⇔ y ∈ G(X).

(b) By Proposition 2.7. (d), x ∗ (y ∗ x) = y ∗ 0 = 0 ⇔ y ∈ B(X).

(c) x ∈ G(X) ⇔ x ∗ 0 = x (by the definition of G(X))

⇔ x ∗ 0 = (x ∗ 0) ∗ 0 (by Proposition 2.7 (b))

⇔ x ∗ 0 ∈ G(X).

Proposition 3.3. The following are equivalent in PU-algebra (X, ∗, 0) :

(1) x = y ∗ z,

(2) y = z ∗ x,

(3) z = x ∗ y for all x, y, z ∈ G(X).

Proof: (1) ⇒ (2): x = y ∗ z ⇒ z ∗ x = z ∗ (y ∗ z) = y ∗ 0 = y (by Proposition 2.7 (d) and the definition of G(X)).

(2) ⇒ (3): y = z ∗ x ⇒ x ∗ y = x ∗ (z ∗ x) = z ∗ 0 = z.

(3) ⇒ (1): z = x ∗ y ⇒ y ∗ z = y ∗ (x ∗ y) = x ∗ 0 = x.

Lemma 3.4. If G(X) = X, then X is P-semisimple.

Proof: Assume that G(X) = X. Then {0} = G(X) ∩ B(X) = X ∩ B(X) = B(X), and hence X is P-semisimple.

Definition 3.5. Let (X, ∗, 0) be a PU-algebra. For a fixed a ∈ X.

The map Ra: X → X given by Ra(y) = y ∗ a for all y ∈ X is called a right self-maps of X. Similarly the map La: X → X given by La(y) = a ∗ y for all y ∈ X is called a left self-maps of X.

Definition 3.6. Let (X, ∗, 0) be a PU-algebra. For a fixed a ∈ X.

The map Ta: X → X given by Ta(y) = (y ∗ a) ∗ (a ∗ 0) for all y ∈ X is called a weak right self-maps of X.

Similarly the map Ma: X → X given by Ma(y) = (a ∗ 0) ∗ (a ∗ y) for all y ∈ X is called a weak left self-maps of X.

Theorem 3.7. Let (X, ∗, 0) be a PU-algebra, then Lx = Mx o Lx if and only if (x ∗ 0) ∗ (x ∗ (x ∗ y)) = x ∗ y for all x, y ∈ X.

Proof: (⇒): Let (X, ∗, 0) be a PU-algebra and Lx = Mx o Lx for all x ∈ X. Then x ∗ y = Lx(y) = (Mx o Lx)(y) = Mx(Lx(y)) = Mx(x ∗ y) = (x ∗ 0) ∗ (x ∗ (x ∗ y)) for all x, y ∈ X.

(⇐): Let (X, ∗, 0) be a PU-algebra and (x ∗ 0) ∗ (x ∗ (x ∗ y)) = x ∗ y for all x, y ∈ X, then Lx(y) = x ∗ y = (x ∗ 0) ∗ (x ∗ Lx(y)) = Mx(Lx(y)) = (Mx o Lx)(y). Hence Lx = Mx o Lx.

Definition 3.8. A non-empty subset I of a PU-algebra (X, ∗, 0) is called a PU-sub algebra of X if x ∗ y ∈ I whenever x, y ∈ I.

Lemma 3.9. If (X, ∗, 0) is a PU-algebra, then:

(a) G(X) is a PU-sub algebra of X.

(b) B(X) is a PU-sub algebra of X.

Proof: (a) Assume that (X, ∗, 0) is a PU-algebra and x, y ∈ G(X), i.e. x ∗ 0 = x, y ∗ 0 = y. Then by Proposition 2.7. (g), (x ∗ y) ∗ 0 = (x ∗ 0) ∗ (y ∗ 0) = x ∗ y. Hence x ∗ y ∈ G(X). Therefore G(X) is a PU-sub algebra of X.

(b) Assume that (X, ∗, 0) is a PU-algebra and x, y ∈ B(X), i.e. x ∗ 0 = 0 = y ∗ 0. Then by Proposition 2.7. (g), (x ∗ y) ∗ 0 = (x ∗ 0) ∗ (y ∗ 0) = 0 ∗ 0 = 0. Hence x ∗ y ∈ B(X). Therefore B(X) is a PU-sub algebra of X.

Lemma 3.10. If (X, ∗, 0) is a PU-algebra, then

(a) x ∗ (y ∗ z) = (x ∗ y) ∗ z for all x ∈ G(X) and y, z ∈ X.

(b) x ∗ y = y ∗ x for all x, y ∈ G(X).

Proof:

(a) By the definition of G(X) and Proposition 2.11, we have

$x * (y * z) = (x * 0) * (y * z) = (x * y) * (0 * z) = (x * y) * z$ (by Definition 2.4.(I)).

(b) By Definition 2.4. (I) and the definition of G(X), we have

$x * y = (0 * x) * (y * 0)$

$= (0 * y) * (x * 0)$ (by Proposition 2.11)

$= y * x$ (by Definition 2.4. (I) and the definition of G(X)).

Theorem 3.11. If (X, ∗, 0) is PU-algebra, then G(X) is an abelian group.

Proof: Let (X, ∗, 0) be a PU-algebra. Then for all $x \in G(X)$ we have

$x = 0 * x = x * 0$. By Proposition 2.7. (a), we have $x * x = 0$ for all $x \in G(X)$. By Lemma 3.10. (b), we have $x * y = y * x$ for all $x, y \in G(X)$. Finally by Lemma 3.10. (a), we have $x * (y * z) = (x * y) * z$ for all x, y and $z \in G(X)$. Therefore G(X) is an abelian group.

In Example 2.5., (X, ∗, 0) is a PU-algebra, but associatively does not hold, since $1 * (2 * 1) = 2 * 0 = 3 \neq 0 = 1 * 1 = (1 * 2) * 1$.

Theorem 3.12. If (X, ∗, 0) is associative PU-algebra, then G(X) = X and B(X) = {0}.

Proof: If (X, ∗, 0) is associative PU-algebra, then clearly $G(X) \subseteq X$. If $x \in X$, then $x * 0 = x * (x * x) = (x * x) * x = 0 * x = x$, and it follows that $x \in G(X)$. Hence $X \subseteq G(X)$. Thus G(X) = X. For the second part, clearly $\{0\} \subseteq B(X)$. If $x \in B(X)$, then $x = 0 * x = (x * x) * x = x * (x * x) = x * 0 = 0$ and $B(X) \subseteq \{0\}$. Thus B(X) = {0}.

Theorem 3.13. Every associative PU-algebra (X, ∗, 0) is a group.

Proof: Putting x = y = z in the associative law $(x * y) * z = x * (y * z)$ and using Definition 2.4. (I) and Proposition 2.7 (a), we obtain $0 * x = x * 0 = x$. This means that 0 is the identity of X. Also by Proposition 2.7 (a), every element x of X has an inverse. Therefore (X, ∗) is a group.

4. NEW VIEW OF IDEALS ON PU-ALGEBRA

Definition 4.1[9]. A non-empty subset I of a PU-algebra (X, ∗, 0) is called an ideal of X if for any $x, y \in X$,

(i) $0 \in I$,

(ii) $x * y, x \in I$ imply $y \in I$.

Definition 4.2[9]. A non empty subset I of a PU-algebra X is called a KU-ideal of X if it satisfies the following conditions:

(1) $0 \in I$,

(2) $x * (y * z) \in I$, $y \in I$ imply $x * z \in I$, for all $x, y, z \in X$.

Theorem 4.3. Let (X, ∗, 0) be a PU-algebra and let I be a non-empty subset of X. Then I is an ideal of X if and only if I is a KU-ideal of X.

Proof: (⇒): Suppose that I is an ideal of X. It is clear that $0 \in I$. Let $x * (y * z) \in I$ and $y \in I$, it follows by Proposition 2.7.(c) that $y * (x * z) \in I$. Since I is an ideal of X, then $x * z \in I$. Hence I is a KU-ideal of X.

(⇐): Suppose that I is a KU-ideal of X. It is clear that $0 \in I$. Put x = 0 in the definition of KU-ideal we have that $0 * (y * z) \in I$, $y \in I$ imply $0 * z \in I$. By using the definition of PU-algebra, we have $0 * (y * z) = y * z$ and $0 * z = z$, i.e. $y * z \in I$, $y \in I$ imply $z \in I$. Therefore I is an ideal of X.

Example 4.4. Let X = {0, a, b, c} in which ∗ is defined by the following table:

∗	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Using the algorithms in Appendix , we can prove that (X, ∗, 0) is a PU-algebra. It is easy to show that I1 = {0, a}, I2 = {0, b}, I3 = {0, c} are KU-ideals of X.

Definition 4.5. A non-empty subset I of a PU-algebra (X, ∗, 0) is called a PU1-ideal of X if it satisfies the following conditions:

(i) $0 \in I$,

(ii) $y * x, x * z \in I$ imply $y * z \in I$, for all $x, y, z \in X$.

Theorem 4.6. Let (X, ∗, 0) be a PU-algebra and let I be a non-empty subset of X. Then I is an ideal of X if and only if I is a PU1-ideal of X.

Proof: (⇒): Suppose that I is an ideal of X. It is clear that $0 \in I$. Let $y * x, x * z \in I$. Since $y * x = (x * z) * (y * z)$ (by Definition 2.4. (II)), then we have $(x * z) * (y * z) \in I$ and $x * z \in I$. It follows by the definition of ideal that $y * z \in I$. Therefore I is a PU1-ideal of X.

(⇐): Suppose that I is a PU1-ideal of X. It is clear that $0 \in I$. Put y = 0 in the definition of PU1-ideal and by using the definition of PU-algebra, we get $x \in I$, $x * z \in I$ imply $z \in I$. Therefore I is an ideal of X.

Definition 4.7. A non-empty subset I of a PU-algebra (X, ∗, 0) is called a PU2-ideal of X if for any $x, y, z \in X$,

(i) $0 \in I$,

(ii) $(x * y) * z \in I$, $z * y \in I$ imply $x \in I$.

Theorem 4.8. Let (X, ∗, 0) be a PU-algebra and let I be a non-empty subset of X. Then I is an ideal of X if and only if I is a PU2-ideal of X.

Proof: (⇒): It is clear that $0 \in I$. Let $(x * y) * z \in I$, $z * y \in I$. Since (X, ∗, 0) is PU-algebra, then $(z * y) * x = (x * y) * z \in I$, it follows by the definition of an ideal of PU-algebra that $x \in I$. Hence I is a PU2-ideal of X.

(⇐): It is clear that $0 \in I$. Let $x * y \in I$, $x \in I$. It follows by the definition of PU-algebra and its properties that $x * y = (y * x) * 0 \in I$ and $x = 0 * x \in I$. Since I is a PU2-ideal of a PU-algebra, then $y \in I$. Hence I is an ideal of X.

Definition 4.9. A non-empty subset I of a PU-algebra (X, ∗, 0) is called a PU3-ideal of X if,

(i) 0 ∈ I,

(ii) (a * (b * x)) * x ∈ I, for all a, b ∈ I and x ∈ X.

Theorem 4.10. Let (X, ∗, 0) be a PU-algebra and let I be a non-empty subset of X. Then I is a PU3-ideal of X if and only if I is a PU1-ideal of X.

Proof: Let I be a PU3-ideal of X, obviously 0 ∈ I. Let x * y, y * z ∈ I. Now applying (Definition 2.4. (I), (II)), we get

$$x*z = 0*(x*z) = \left\{\overbrace{(x*y)*((y*z)*(x*z))}^{0}\right\}*(x*z) =$$

$$\left\{\overbrace{(x*y)}^{a}*\overbrace{((y*z)*(x*z))}^{b}\right\}*(x*z) = a*(b*(x*z))*(x*z) \in I$$

Hence I is a PU1-ideal of X.

Conversely. If I is a PU1-ideal of X, it is clear that 0 ∈ I and (by Theorem 4.6) I is an ideal of X.

To prove (ii) (of Definition 4.9), observe that (a * (b * x)) * (a * (b * x)) = 0 ∈ I, for a, b ∈ I and x ∈ X . By Proposition 2.7.(c), we have a ∗ ((a ∗ (b ∗ x)) ∗ (b ∗ x)) ∈ I. Since I is an ideal and a ∈ I, it follows that ((a ∗ (b ∗ x)) ∗ (b ∗ x)) ∈ I. By Proposition 2.7.(c), we have b ∗ ((a ∗ (b ∗ x)) ∗ x) ∈ I. Since I is an ideal and b ∈ I, it follows that (a ∗ (b ∗ x)) ∗ x ∈ I. Therefore I is a PU3-ideal of X.

Lemma 4.11 : If I is a PU3-ideal of a PU-algebra X ,then for every a ∈ I and x ∈ X,

(a * x) * x ∈ I

Proof: Clear.

Corollary 4.12 : If a ∈ I and x ≤ a , then x ∈ I .

Proof: The condition x ≤ a in PU-algebra mean a * x = 0 and by Lemma 4.11, we get x = 0 * x = (a * x) * x ∈ I.

Definition 4.13. A non-empty subset I of a PU-algebra (X, ∗, 0) is called a PU4-ideal of X if,

0 ∈ I,

(ii) (a * 0) * b ∈ I, for all a, b ∈ I.

Lemma 4.14. If (X, ∗, 0) is a PU-algebra, then (x * (y * z)) * z = (y * 0) * x for all x, y, z ∈ X.

Proof: Let (X, ∗, 0) be a PU-algebra and let x, y, z ∈ X, then we have that (x * (y * z)) * z = (z * (y * z)) * x (by Proposition 2.8 (2))

= (y * (z * z)) * x (by Proposition 2.7 (c))

= (y * 0) * x (by Proposition 2.7 (a)).

Theorem 4.15. Let (X, ∗, 0) be a PU-algebra and let I be a non-empty subset of X. Then I is a PU3-ideal of X if and only if I is a PU4-ideal of X.

Proof: Follows directly by using Lemma 4.14.

The following result is a direct consequence of Theorems (4.3, 4.6, 4.8, 4.10 and 4.15)

Theorem 4.16. If X is PU-algebra, then the following are equivalent:

(1) I is an ideal of X.	(2) I is a KU-ideal of X.
(3) I is a PU1-ideal of X.	(4) I is a PU2-ideal of X.
(5) I is a PU3-ideal of X.	(6) I is a PU4-ideal of X.

Lemma 4.17. Let (X, ∗, 0) be a PU-algebra and $\{A_i\}_{i\in I}$ be a family of PU1-ideals of X, then $\bigcap_{i\in I} A_i$ is also PU1-ideal of X.

Proof: Let x, y and z ∈ X be such that y ∗ x, x ∗ z ∈ $\bigcap_{i\in I} A_i$. Then y ∗ x, x ∗ z ∈ Ai for all i ∈ I. But Ai is a PU1-ideal of X for all i ∈ I. Then y ∗ z ∈ Ai for all i ∈ I, and hence we get y ∗ z ∈ $\bigcap_{i\in I} A_i$. Therefore $\bigcap_{i\in I} A_i$ is also PU1-ideal of X.

Remark 4.18. Let (X, ∗, 0) be a PU-algebra.

1) If $\{A_i\}_{i\in I}$ is a family of KU-ideals of X, then $\bigcap_{i\in I} A_i$ is also KU-ideal of X.

2) If $\{A_i\}_{i\in I}$ is a family of PU1-ideals of X, then $\bigcap_{i\in I} A_i$ is also PU1-ideal of X.

3) If $\{A_i\}_{i\in I}$ is a family of PU2-ideals of X, then $\bigcap_{i\in I} A_i$ is also PU2-ideal of X.

4) If $\{A_i\}_{i\in I}$ is a family of PU3-ideals of X, then $\bigcap_{i\in I} A_i$ is also PU3-ideal of X.

5) If $\{A_i\}_{i\in I}$ is a family of PU4-ideals of X, then $\bigcap_{i\in I} A_i$ is also PU4-ideal of X.

Proposition 4.19. If (X, ∗, 0) is a PU-algebra, then

(a) G(X) is a PU1-ideal of X.

(b) B(X) is a PU1-ideal of X.

Proof: (a) Clearly 0 ∈ G(X). Let x ∗ y ∈ G(X), x ∈ G(X). Then We have that

y ∗ 0 = x ∗ (y ∗ x) (by Proposition 2.7. (d))

= x ∗ ((x ∗ y) ∗ 0) (by Proposition 2.7. (e))

= x ∗ (x ∗ y) (by the definition of G(X)).

Since G(X) is a PU-sub algebra of X, then y ∗ 0 ∈ G(X). Hence by Proposition 2.7. (b), we have that y ∗ 0 = (y ∗ 0) ∗ 0 = y, then y ∈ G(X), thus G(X) is an ideal of X. Therefore by Theorem 4.6., we have that G(X) is a PU1-ideal of X.

(b) Clearly 0 ∈ B(X). Let x ∗ y ∈ B(X), x ∈ B(X). we have that

y ∗ 0 = x ∗ (y ∗ x) = x ∗ ((x ∗ y) ∗ 0) (by Proposition 2.7. (d),(e))

= x ∗ 0 = 0 (by the definition of B(X)).

Then $y \in B(X)$, and thus B(X) is an ideal of X. Therefore by Theorem 4.6., we have that B(X) is a PU1-ideal of X.

5. HOMOMORPHISMS OF PU-ALGEBRA

Definition 5.1. Let $(X, *, 0)$ and $(X\backslash, *\backslash, 0\backslash)$ be PU-algebras. A map $f: X \to X\backslash$ is called a homomorphism if $f(x * y) = f(x) *\backslash f(y)$ for all $x, y \in X$.

Theorem 5.2. Let $(X, *, 0)$ and $(X\backslash, *\backslash, 0\backslash)$ be PU-algebras, and $f: X \to X\backslash$ be a homomorphism, then

(1) $f(0) = 0\backslash$.

(2) If S is a PU-sub algebra of X, then f(S) is a PU-sub algebra of X\.

(3) If S is a PU-sub algebra of X\, then f -1(S) is a PU-sub algebra of X.

(4) If $x \leq y$, then $f(x) \leq f(y)$.

(5) If B is a PU1-ideal of X\, then f -1(B) is a PU1-ideal of X.

(6) ker f is a PU1-ideal of X.

Proof: (1) $f(0) = f(0 * 0) = f(0) *\backslash f(0) = 0\backslash$ (by Definition 2.4. (I), Definition 5.1. and Proposition 2.7. (a)).

(2) Let $x\backslash, y\backslash \in f(S)$. It follows that $x\backslash = f(x)$, $y\backslash = f(y)$ for some $x, y \in S$. It follows by Definition 5.1., that $x\backslash *\backslash y\backslash = f(x) *\backslash f(y) = f(x * y)$. Since S is a PU-sub algebra of X, then $x * y \in S$ and hence $x\backslash *\backslash y\backslash = f(x * y) \in f(S)$ which complete the proof.

(3) Let $x, y \in f^{-1}(S)$. It follows that $f(x), f(y) \in S$. Since S is a PU-sub algebra of X\ and f is a homomorphism, then $f(x) *\backslash f(y) = f(x * y) \in S$. It follows that $x * y \in f^{-1}(S)$. Hence f -1(S) is a PU-sub algebra of X.

(4) Since $x \leq y$, then $y * x = 0$. It follows that $f(y * x) = f(0) = 0\backslash$. Since f is a homomorphism, then $f(y) *\backslash f(x) = 0\backslash$. Therefore $f(x) \leq f(y)$.

(5) Since B is a PU1-ideal of X\, then $0\backslash \in B$ (i.e. $f(0) \in B$). It follows that $0 \in f^{-1}(B)$. Let $x, y, z \in X$ be such that $y * x \in f^{-1}(B)$, $x * z \in f^{-1}(B)$. It follows that $f(y * x) \in B$, $f(x * z) \in B$. Since f is a homomorphism, then $f(y) *\backslash f(x) \in B$, $f(x) *\backslash f(z) \in B$. Since B is a PU1-ideal of X\, then $f(y) *\backslash f(z) \in B$. Since f is a homomorphism, then $f(y * z) \in B$. It follows that $y * z \in f^{-1}(B)$. Therefore f -1(B) is a PU1-ideal of X.

(6) It is clear that $0 \in \ker f$. Let $x, y, z \in X$ be such that $y * x$, $x * z \in \ker f$, then $f(y * x) = 0\backslash$, $f(x * z) = 0\backslash$. Since $(X, *, 0)$ is PU-algebra, then $y * x = (x * z) * (y * z)$. Since f is a homomorphism, then we have $f(y * x) = f(x * z) *\backslash f(y * z) = 0\backslash$. It follows that $0\backslash *\backslash f(y * z) = 0\backslash$, hence $f(y * z) = 0\backslash$ (i.e. $y * z \in \ker f$). Therefore ker f is a PU1-ideal of X.

6. CONCLUSION

In this manuscript, we introduce a new concept, which called PU-algebra X .

We state and prove some theorems about fundamental properties of it. Moreover ,we give the concepts of a weak right self-maps, weak left self-maps and investigated some its properties. Further, we have proved that every associative PU-algebra is a group and every p-semisimple algebra is an abelian group. We define the centre of a PU-algebra X and show that it is a p-semisimple sub-algebra of X, which consequently implies that every PU-algebra contains a p-semisimple PU-algebra.

We posed the following problem, is the set Hom(X) of all PU-homomorphisms of X into itself, is a PU- algebra? We can proved that it is not always a PU-algebra. However, it may be established that Hom(X) is a PU-algebra, if X is an associative PU-algebra. But an associative PU-algebra is again a p-semisimple algebra. Thus homological study of PU-algebras did not develop for PU-algebras in general. The future purpose of this paper is to study the set of all left-regular self- maps of a positive implicative PU-algebra X, we can show that it forms a positive implicative PU-algebra. But no such effort was made for PU-algebras ,We form weakly positive implicative PU-algebras in terms of its Right Self-maps and Weak Right Self-maps. Further, some properties of Weak Right Self- maps, Weak Left Self-maps and Weak Left-Regular Self-maps can be studied. It can also shown that the set of all Weak Left-Regular Self-maps of a weakly positive implicative PU-algebra X, is a weakly positive implicative PU-algebra. Thus homological study has been made in the class of weakly positive implicative PU-algebras a class which contains the class of p-semisimple PU-algebras, the class of associative PU-algebra, the class of weakly implicative PU-algebras and weakly positive implicative PU-algebras. As is well known , the concept of ideal I plays an important role in PU-algebras X and a lot of results on ideals can be obtained .We have classified ideals into the following classes as followes : Ideals have elements of X , ideals have elements of X and I and Ideals have elements of I .We know that every ideal is not necessarily a sub-algebra. Thus a question arises -what type of ideals are sub-algebras? We hope in the further work can answer these open questions .

7. ACKNOWLEDGMENTS

The authors are greatly appreciate the referees for their valuable comments and suggestions for improving the paper.

Algorithms for PU-algebra

Input (X: set with 0 element, : Binary operation)

Output ("X is a PU-algebra or not")

If X = ϕ then;
Go to (1.)
End if
If $0 \notin X$ then go to (1.);
EndIf
Stop: = false
i = 1;
While $i \leq |X|$ and not (Stop) do
If 0 * xi ≠ xi, then
Stop: = true
End if
j = 1;
While $j \leq |X|$, and not (Stop) do
k = 1;
While $k \leq |X|$ and not (stop) do
If (xi * xk)* (xj * xk) ≠ xj * xi, then
Stop: = true
End if
End while
End if
End while
If stop then

Output ("X is a PU-algebra")
Else
(1.) Output ("X is not a PU-algebra")
End if
End.
Algorithms for PU-ideal in PU-algebra
Input (X: PU-algebra, I: subset of X)
Output ("I is a PU-ideal of X or not")
If I = ϕ then
Go to (1.);
End if
If 0 $\notin$ I then
Go to (1.);
End if
Stop: = false
i = 1;
While $i \leq |X|$ and not (stop) do
j = 1
While $j \leq |X|$ and not (stop) do
k = 1
While $k \leq |X|$ and not (stop) do
If xj * xi $\in$ I, and xi * xk $\in$ I then
If xj* xk $\notin$ I then
Stop: = false
End if
End while
End while
End while
If stop then
Output ("I is a PU-ideal of X")
Else
(1.) Output ("I is not ("I is a PU-ideal of X")
End if
End.

8. REFERENCES

[1] Q.P. Hu, X. Li, On BCH-algebras, Mathematics Seminar Notes 11 (1983) 313–320.

[2] Y. Imai, K. Iseki, On axiom systems of propositional calculi, XIV Proceedings of Japan Academy 42 (1966) 19–22.

[3] H.S. Kim, Y.H. Kim, On BE-algebras, Scientiae Mathematicae Japonica Online, e-2006, 1299–1302.

[4] K. Megalai and A. Tamilarasi, Classification of TM-algebra, IJCA Special Issue on "Computer Aided Soft Computing Techniques for Imaging and Biomedical Applications" CASCT. 2010.

[5] B.L. Meng, CI-algebra, Scientiae Mathematicae Japonica Online, e-2009, 695-701.

[6] S.M. Mostafa, M.A. Abd-Elnaby and M.M.M. Yousef, Fuzzy ideals of KU-Algebras, International Mathematical Forum, 6(63) (2011) 3139-3149.

[7] J. Neggers, H.S. Kim, On d-algebras, Mathematica Slovaca 49 (1999) 19–26.

[8] J. Neggers, S.S. Ahn, H.S. Kim, On Q-algebras, International Journal of Mathematics & Mathematical Sciences 27 (2001) 749–757.

[9] C. Prabpayak and U. Leerawat, On ideas and congurences in KU-algebras, Scientia Magna Journal, 5(1) (2009) 54-57.

[10] C. Prabpayak and U. Leerawat, On isomorphisms of KU-algebras, Scientia Magna Journal, 5(3) (2009) 25-31