On the Stability of Quadratic Functional Equation

Sushma
Department of Mathematics,
Kanya Mahavidyalaya Kharkhoda (India)

ABSTRACT
In this paper the stability of quadratic functional equation, $f(xy)+f(xy^{-1})=2f(x)+2f(y)$ on class of groups is obtained and also prove that quadratic functional equation may not be stable in any abelian group.

Keywords
Quadratic functional equation, pseudo-quadratic mapping, Banach space, quasi-quadratic mapping

2000 Mathematics Subject Classification
Primary 54E40; Secondary 39B82, 46S50, 46S40

1. INTRODUCTION
In 1940, Ulam[16] raised the following question concerning the stability of group homomorphism “Under what conditions does there is an additive mapping near an approximately additive mapping between a group and a metric group ?”. In 1941, D.H.Hyers[10] answered the stability problem of Ulam under the assumption that the groups are Banach spaces. In 1950 Aoki[1] generalized the Hyers theorem for additive mappings. In 1978 Th.M.Rassias[14] provided a generalized version of the theorem of Hyers which permitted the Cauchy difference to become bounded. Since then, the stability problems of various functional equations have been extensively investigated by a number of authors [1-3,6,9-11].The functional equation $f(x+y)+f(x-y)=2f(x)+2f(y)$ is called a quadratic functional equation. In particular, every solution of quadratic functional equation is said to be a quadratic mapping. A generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof[15] for mappings f: X $\rightarrow$Y where X is a real normed space and Y is Banach space . Cholewa [4] noticed that the theorem of Skof[15] is still true if the relevant domain X is replaced by an Abelian group. Czerwik[5] and S.Y.Jang, J.R.Lee, C.Park and D.Y.Shin[12] proved the generalized Hyers-Ulam stability of the quadratic functional equation in Normed and Fuzzy Banach spaces respectively.

In this paper we study the stability of quadratic functional equation

$f(xy)+f(xy^{-1}) = 2f(x)+2f(y)$ on groups.

Suppose that G is an arbitrary group and E is an arbitrary real Banach space. We denote 1 the identity element of G.

Definition 2.1: We will say that a function $f : G \to E$ is a (G; E) – Quadratic function if for any $x, y \in G$ we have

$$f(xy)+f(xy^{-1})-2f(x)-2f(y) = 0 \quad (2.1)$$

We denote the set of all (G; E) – Quadratic function by $Q(G\,;E)$

Denote by $Q_0(G\,;E)$ the subset of $Q(G\,;E)$ consisting of functions f such that f(1)=0. Obviously $Q_0(G\,;E)$ is a subspace of $Q(G\,;E)$ and $Q(G\,;E) = Q_0(G\,;E) \oplus E$.

Definition 2.2: A function $f : G \to E$ is a $(G\,;E)$ - quasi – Quadratic function if there is $C > 0$ such that for any $x,\ y \in G$ we have

$$\left\| f(xy)+f(xy^{-1})-2f(x)-2f(y) \right\| \le c \quad (2.2)$$

It is clear that the set of $(G\,;E)$ - quasi – Quadratic functions is a linear space.

Denote it by $KQ(G\,;E)$. In (2.2), we put x = 1,

$$\left\| f(y)+f(y^{-1})-2f(y)-2f(1) \right\| < c$$

$$\left\| f(y^{-1})-f(y) \right\| < c+2\left\| f(1) \right\| = c_1 \quad (2.3)$$

Where $c_1 = c+2\left\| f(1) \right\|$

Now put y = x in (2.2), we get

$$\left\| f(x^2)+f(1)-2f(x)-2f(x) \right\| < c$$

$$\left\| f(x^2) -4f(x) \right\| < c+\left\| f(1) \right\| = c_2 \quad (2.4)$$

where $c_2 = c+\left\| f(1) \right\|$

Put $y = x^2$ in (2.2), we get

$$\left\| f(x^3)+f(x^{-1})-2f(x)-2f(x^2) \right\| < c$$

$$\Rightarrow \left\| f(x^3)-9f(x) \right\| < c +c_1+2c_2 = c_3 \quad (2.5)$$

where $c_3 = c+c_1+2c_2$

Put $y = x^3$ in (2.2), we obtain

$\left\| f(x^4)-16f(x) \right\| < c+4c_1+c_2+2c_3 = c_4$ Take $c_4 = c+4c_1+c_2+2c_3$

Let c be as in (2.2) and define the set C as follows

$C = \{ c_m \; ; \; m \in N \}$,where $c_1 = c + 2\| f(1) \|$, $c_2 = c + \| f(1) \|$,

$c_3 = c + c_1 + 2c_2$ and

$c_m = c + (m-2)^2 c_1 + c_{m-2} + 2c_{m-1}$ if $m > 3$.

Lemma 2.3: Let $f \in KQ(G \; ; \; E)$ such that

$$\| f(xy) + f(xy^{-1}) - 2f(x) - 2f(y) \| \le c$$

Then for any $x \in G$ and any $m \in N$ the following relation holds

$$\| f(x^m) - m^2 f(x) \| \le c_m \qquad (2.6)$$

Proof: The proof is by induction on m. For m = 3, the lemma is established. Suppose that lemma is true for m. Let us verified it for m + 1. Put $y = x^m$ in (2.2), we have

$$\| f(x^{m+1}) + f(x^{-m+1}) - 2f(x) - 2f(x^m) \| < c$$

$$\| f(x^{m+1}) + f(x^{-m+1}) - (m-1)^2 f(x^{-1}) + (m-1)^2 f(x^{-1}) - 2f(x) - 2f(x^m) \| < c$$

$$\Rightarrow \| f(x^{m+1}) + (m-1)^2 f(x^{-1}) - 2f(x) - 2f(x^m) \| < c + \| f(x^{-1})^{m-1} + (m-1)^2 f(x^{-1}) \|$$

By assumption of induction

$$\| f(x^{m+1}) + (m-1)^2 f(x^{-1}) - 2f(x) - 2f(x^m) \| < c + c_{m-1}$$

$$\Rightarrow \| f(x^{m+1}) + (m-1)^2 f(x^{-1}) - (m-1)^2 f(x) + (m-1)^2 f(x) - 2f(x) - 2f(x^m) \| < c + c_{m-1}$$

$$\Rightarrow \| f(x^{m+1}) + (m-1)^2 f(x) - 2f(x) - 2f(x^m) \| < c + c_{m-1} + (m-1)^2 c_1$$

$$\Rightarrow \| f(x^{m+1}) + (m-1)^2 f(x) - 2f(x) - 2[f(x^m) - m^2 f(x) + m^2 f(x)] \| < c + c_{m-1} + (m-1)^2 c_1$$

$$\Rightarrow \| f(x^{m+1}) + (m-1)^2 f(x) - 2f(x) - 2m^2 f(x) \| < c + c_{m-1} + (m-1)^2 c_1 + 2c_m$$

$$\Rightarrow \| f(x^{m+1}) + f(x)[m^2 + 1 - 2m - 2 - 2m^2] \| < c + c_{m-1} + (m-1)^2 c_1 + 2c_m \Rightarrow$$

$$\| f(x^{m+1}) - (m+1)^2 f(x) \| < c + (m-1)^2 c_1 + c_{m-1} + 2c_m$$

Now the lemma is proved.

Lemma 2.4: Let $f \in KQ(G \; ; \; E)$. For any $m > 1, \; k \in N$ and $x \in G$, we have

$$\| f(x^{m^k}) - (m^k)^2 f(x) \| \le c_m \left(1 + m^2 + (m^2)^2 + ... + (m^2)^{k-1}\right) \qquad (2.7)$$

and

$$\left\| \frac{1}{(m^k)^2} f(x^{m^k}) - f(x) \right\| \le c_m \qquad (2.8)$$

Proof: The proof is based on induction on k. If k = 1, then (2.7) follows from (2.6). Suppose that (2.7) is true for k, we will show it is true for k + 1. Replace x by x^m in (2.7). $\Rightarrow$

$$\| f(x^m)^{mk} - (m^k)^2 f(x^m) \| < c_m (1 + m^2 + ... + (m^2)^{k-1})$$

$$\Rightarrow \| f(x^{m^{k+1}}) - (m^k)^2 f(x^m) \| < c_m (1 + m^2 + ... + (m^{k-1})^2)$$

$$\Rightarrow \| f(x^{m^{k+1}}) - (m^k)^2 f(x^m) - (m^{k+1})^2 f(x) + (m^{k+1})^2 f(x) \| < c_m (1 + m^2 + ... + (m^{k-1})^2)$$

$$\| f(x^{m^{k+1}}) - (m^{k+1})^2 f(x) \| \Rightarrow < c_m (1 + m^2 + ... + (m^{k-1})^2) + \| (m^k)^2 f(x^m) - (m^{k+1})^2 f(x) \|$$

$$\| f(x^{m^{k+1}}) - (m^{k+1})^2 f(x) \| \Rightarrow < c_m (1 + m^2 + ... + (m^{k-1})^2) + (m^k)^2 c_m$$

$$\Rightarrow \| f(x^{m^{k+1}}) - (m^{k+1})^2 f(x) \| < c_m (1 + m^2 + ... + (m^2)^{k-1} + (m^2)^k)$$

Further implies

$$(m^{k+1})^2 \left\| \frac{1}{(m^{k+1})^2} f(x^{m^{k+1}}) - f(x) \right\| < c_m (1 + m^2 + ... + (m^2)^{k-1} + (m^2)^k)$$

$$\left\| \frac{1}{(m^{k+1})^2} f(x^{m^{k+1}}) - f(x) \right\| \Rightarrow \le \frac{c_m}{(m^2)^{k+1}} (1 + m^2 + ... + (m^2)^{k-1} + (m^2)^k) \le c_m$$

This completes the proof of the lemma.

From (2.8) it follows that the set $\left\{ \frac{1}{(m^k)^2} f(x^{m^k}) \; ; \; k \in N \right\}$ is bounded

Replacing x by x^{m^n} in (2.8), we get

$$\left\| \frac{1}{(m^k)^2} f(x^{m^{n+k}}) - f(x^{m^n}) \right\| \le c_m$$

$$\Rightarrow \left\| \frac{1}{(m^{n+k})^2} f(x^{m^{k+1}}) - \frac{1}{(m^n)^2} f(x) \right\|$$

$$\le \frac{c_m}{(m^n)^2} \to 0 \ as\, n \to \infty$$

This implies that the sequence $\left\{ \frac{1}{(m^k)^2} f(x^{m^k}) \ ; \ k \in N \right\}$ is a Cauchy sequence. Since the space E is Banach. So, the above sequence has a limit in E and we denote it by $\phi_m(x)$. Thus

$$\phi_m(x) = \lim_{k \to \infty} \frac{1}{(m^k)^2} \ f(x^{m^k})$$

From (2.8), we have

$$\| \phi_m(x) - f(x) \| \le c_m \quad \forall \quad x \in G \ (2.9)$$

Lemma 2.5: Let $f \in KQ(G \ ; \ E)$ such that

$$\| f(xy) + f(xy^{-1}) - 2f(x) - 2f(y) \| \le c \ \ \forall \ \ x, y \in G$$

Then for any $m \in N$, we have $\phi_m \in KQ(G \ ; \ E)$.

Proof: $\| \phi_m(xy) + \phi_m(xy^{-1}) - 2\phi_m(x) - 2\phi_m(y) \|$

$$\left\| \begin{array}{l} \phi_m(xy) + f(xy) + \phi_m(xy^{-1}) - f(xy^{-1}) - 2\phi_m(x) + 2f(x) \\ - 2\phi_m(y) + 2f(y) + f(xy) + f(xy^{-1}) - 2f(x) - 2f(y) \end{array} \right\|$$

$$\le \|\phi_m(xy) - f(xy)\| + \|\phi_m(xy^{-1}) - f(xy^{-1})\| + 2\|\phi_m(x) - f(x)\| + 2\|\phi_m(y) - f(y)\| + \|f(xy) + `f(xy^{-1}) - 2f(x) - 2f(y)\| \Rightarrow$$

$$\le 6c_m + c$$

$$\| \phi_m(xy) + \phi_m(xy^{-1}) - 2\phi_m(x) - 2\phi_m(y) \| < c' \quad ,$$

where $c' = 6c_m + c$

Hence the proof of the lemma for any $x \in G$ we have the relation

$$\phi_m(x^{m^k}) = (m^k)^2 \phi_m(x) \qquad (2.10)$$

In fact

$$\phi_m(x^{m^k}) = \lim_{l \to \infty} \frac{1}{(m^l)^2} \ f\left((x^{m^k})^{m^l} \right)$$

$$= \lim_{l \to \infty} \frac{(m^l)^2}{(m^{l+k})^2} \ f(x^{m^{k+l}})$$

$$= (m^k)^2 \ \lim_{p \to \infty} \frac{1}{(m^p)^2} \ f(x^{m^p}) = (m^k)^2 \phi_m(x)$$

.**Lemma 2.6:** If $f \in KQ(G \ ; \ E)$, then $\phi_2 \equiv \phi_m$ for any $m \ge 2$.

Proof: By above lemma, $\phi_2, \phi_{m,} \in KQ(G \ ; \ E)$.

Hence $g(x) = \lim_{k \to \infty} \frac{1}{(m^k)^2} \phi_2(x^{m^k})$ is well defined and $g \in KQ(G \ ; \ E)$

Also $g(x^{m^k}) = (m^k)^2 \ g(x)$ and $g(x^{2^k}) = (2^k)^2 \ g(x)$ for any $x \in G$ and any $k \in N$

also from (2.9), there exists $d_1, d_2 \in R_+$ such

$$\| \phi_2(x) - g(x) \| \le d_1 \ \text{and} \ \| \phi_m(x) - g(x) \| \le d_2 \qquad (2.11)$$

Hence $g \equiv \phi_2$ and $g \equiv \phi_m$, we obtain $\phi_2 \equiv \phi_m$.

Definition 2.7: By $(G \ ; \ E)$ - pseudo – Quadratic function we will mean a $(G \ ; \ E)$-quasi–Quadratic function f such that $f(x^n) = n^2 \ f(x)$ for any $x \in G$ and any $n \in Z$ $(G \ ; \ E)$ - pseudo–Quadratic function is denoted by $PQ(G \ ; \ E)$.

Lemma 2.8: For any $f \in KQ(G \ ; \ E)$ the function $g(x) = \lim_{k \to \infty} \frac{1}{(2^k)^2} \ f(x^{2^k})$ is well – defined and is $(G \ ; \ E)$ - pseudo – Quadratic function such that for any $x \in G$, $\| \ g(x) - \ f(x) \| \le c_2$.

Proof: By the previous lemma, g is $a(G \ ; \ E)$ - Quasi – Quadratic function. Now by above lemma, we have $g(x^m) = \phi_m(x^m) = m^2 \phi_m(x) = m^2 \ g(x)$

Thus $\phi_m(x) = g(x)$ and hence $\phi_2(x) = g(x)$ by above lemma. From equality

$g = \phi_2$, we have $\| g(x) - f(x) \|$

$= \| \phi_2(x) - f(x) \| \le c_2$.

Lemma 2.9: If $f \in PQ(G; \ E)$ then

(i) If $y \in G$ is an element of finite order then $f(y) = 0$.

(ii) If f is bounded function on G, then $f \equiv 0$.

Proof: Let order of $y \in G$ is t then

$$y^t = 1$$

$$\Rightarrow \quad f(y^t) = f(1) = 0$$

$$t^2 f(y) = 0$$

$$\Rightarrow \quad f(y) = 0.$$

Similarly we verify assertion 2.

We denoted by $B(G\,;E)$ the space of all bounded functions on a group G that takes values in E.

Theorem 2.10: For an arbitrary group G the following decomposition holds:

$$KQ(G\,;E) = PQ(G\,;E) \oplus B(G\,;E).$$

Proof: It is clear $PQ(G\,;E)$ and $B(G\,;E)$ are subspaces of $KQ(G\,;E)$, and $PQ(G\,;E) \cap B(G\,;E) = \{0\}$. Hence the subspace of $KQ(G\,;E)$ generated by $PQ(G\,;E)$ and $B(G\,;E)$ is their direct sum.

That is $PQ(G\,;E) \oplus B(G\,;E) \subseteq KQ(G\,;E)$.

Now let $f \in KQ(G\,;E)$, then $g \in PQ(G\,;E)$ and $g - f \in B(G\,;E)$

```````````` and

$$f = g + (g - f) \in PQ(G\,;E) + B(G\,;E)$$

So, $KQ(G\,;E) = PQ(G\,;E) \oplus B(G\,;E)$.

**Remark2.11:** Let G={1,-1,i,-i} be an abelian group and the quadratic functional equation is not stable in G i.e, $f(xy) + f(xy^{-1}) = 2f(x) + 2f(y)$ does not satisfy for x=-1, y=-1.

**Definition2.12**: We shall say that equation (2.1) is stable for the pair (G;E) if for any f: G$\longrightarrow$E satisfying functional inequality

$$\left\| f(xy) + f(xy^{-1}) - 2f(x) - 2f(y) \right\| \le c$$

$$\forall \ x, y \in G$$

for some c > 0, there is a solution $\overline{f}$ of the functional equation (2.1) such that the function $\overline{f}$ (x)-f(x) belongs to B(G;E).

It is clear that equation (2.1) is stable on G iff PQ (G; E) =$Q_0$ (G; E). We will say that a (G; E) – pseudo – quadratic function f is nontrivial if f $\notin Q_0$ (G; E)

**Theorem2.13:** Let $E_1$, $E_2$ be a branch space over real. Then equation (2.1) is stable for the pair (G; $E_1$) if and only if it is stable for the pair (G;$E_2$).

**Proof**: $\longrightarrow$Let E be a branch space and R be the set of real . Suppose that equation (2.1) is stable for the pair (G, E). Suppose that (2.1) is not stable for the pair (G; R), then there is a non trivial real –valued pseudo – quadratic function f on G. Now let e $\in$ E and $\|e\|$=1. Consider the function $\phi$ :G$\longrightarrow$ E given by the formula $\phi(x)$ =f(x).e. It clears that $\phi$ is a nontrivial pseudo –quadratic E-valued function, and we obtain a contradiction.

Now suppose that equation (2.1) is stable for the pair (G;R), that is , PQ(G;R)=$Q_0$(G;R). Denote by $E^*$ the space of linear bounded functional on E endowed by functional norm topology. It is clear that for any $\psi \in$ PQ (G;H) and $\lambda \in H^*$ the function $\lambda 0 \psi$ belongs to the space PQ(G;R). Indeed, let for some c>0 and any x,y $\in$ G we have

$$\left\| \psi(xy) + \psi(xy^{-1}) - 2\psi(x) - 2\psi(y) \right\| \le c.$$

Hence

$$\left| \lambda o\psi(xy) + \lambda o\psi(xy^{-1}) - \lambda o\psi(2x) - \lambda o\psi(2y) \right| =$$

$$\left| \lambda\left[\psi(xy) + \psi(xy^{-1}) - 2\psi(x) - 2\psi(y)\right] \right| \le c\|\lambda\|$$

Obviously, $\lambda o\psi(x^n) = n^2 \lambda o\psi(x)$ any x $\in$ G ad for any n $\in$ N. Hence the function $\lambda o\psi$ belong to the space PQ(G,R). Let f: G$\longrightarrow$H be a nontrivial pseudo- Quadratic mapping. Then x,y $\in$ G such that $f(xy) + f(xy^{-1}) - 2f(x) - 2f(y) \ne 0$. Hahn-Banach theorem implies that there is a $l \in H^*$ such that $l\left(f(xy) + f(xy^{-1}) - 2f(x) - 2f(y)\right) \ne 0$, and we have $lof$ is a nontrivial pseudo- Quadratic real valued function on G. This contradiction proves the theorem.

**Corollary2.14:** If a group G has nontrivial pseudo- character, then equation (2.1) is not stable on G.

**Proof:** Let $\phi$ be a nontrivial pseudo- character of G. Suppose that there is $\overline{f} \in Q_0$ (G) such that the function $\phi - \overline{f}$ is bounded. Then there is c>0 such that $\left|\phi(x) - \overline{f}(x)\right| \le c$ for any x $\in$ G. Hence for any n $\in$ N we have $c \ge \left|\phi(x^n) - \overline{f}(x^n)\right| = n^2 \left|\phi(x) - \overline{f}(x)\right|$ and we see that the latter is possible if $\phi(x) = \overline{f}(x)$. So, $\phi \in PQ(G) \cap Q_0(G)$. Hence, $f \in X(G)$ and this contradiction with the assumption about f.

## 2. REFERENCES

[1] Aoki,T., (1950) “On stability of the linear transformation in Banach spaces,” Journal of the Mathematical Society of Japan, 2,64-66

[2] Chang, S. and Kim,H.M., (2002), On the Hyer-Ulam stability of a quadratic functional equations, J.Ineq. Appl. Math., 33, 1-12.

[3] Chang,S., Lee,E.H. and Kim,H.M., (2003)On the Hyer-Ulam Rassias stability of a quadratic functional equations, Math. Ineq. Appl. Math., 6(1), 87-95.

[4] Cholewa,P.W (1984) Remarks on the stability of functional equations, Aequationes Math., 27, 76-86.
````````````

[5] Czerwik,S., (1992),On the stability of a quadratic mappings in normed spaces, Abh. Math.Sem. Univ. Hamburg.,62, 59-64.

[6] Gavruta,P., (1994) A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, J.Math. Anal. Appl., 184,431-436.

[7] Gruber,P.M., (1978), Stability of isometries, Trans. Amer. Math. Soc. 245, 263-277.

[8] Hyers,D.H. and Rassias,T.M., (1992) Approximate homomorphism, Aequationes Math., 44,125-153.

[9] Hyers,D.H., Isac, G., and Rassias, Th.M., (1998), Stability of functional equations in several variables, Birkhauser Basel.

[10] Hyers,D.H., (1941) On the stability of the linear functional equation, Proc. Nat. Acad. Sci. U.S.A., 27, 222-223.

[11] Hyers,D.H., Isac, G. and Rassias,Th.M., (1998) On the asymptotically of Hyer-Ulam stability of mappings, Proc. Amer. Math. Soc., 126),425-430.

[12] Jang,S.Y., Lee,J.R., Park,C. and Shin,D.Y., (2009)Fuzzy stability of Jensen-type Quadratic functional equations, Abstract and Applied Analysis, pp-17.

[13] Mirzavaziri,M. and Moslehian,M.S., A fixed point approach to stability of Quadratic equation, arXivimath/0512007v1 [math.FA] 1 Dec 2005.

[14] Rassias,T.M., (1978) On stability of linear mapping in Banach spaces, Proc. Amer. Math. Soc. 72, 297-300.

[15] SKOF,F., (1983) Proprieta localie approssimazione di operatori, Rend. Sem. Mat. Fis. Milano, 53, 113-129

[16] Ulam,S.M., (1960) A collection of Mathematical Problems, Interscience tracts in Pure and Applied Mathematics, U.S.A 16.

[17] Ulam,S.M., 1963, Problems in Modern Mathematics, John Wiley & Sons, New York, USA,.