# A Common Coupled Fixed Point Theorem in Complex Valued Metric Space

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## ABSTRACT
The aim of this paper is to establish a unique common coupled fixed point theorem for two mappings satisfying a rational inequality in complex valued metric space.

## General Terms
47H10, 54H25

## Keywords
Coupled fixed point, complex valued metric space, Cauchy sequence, Convergent sequence, complete complex valued metric space.

## 1. INTRODUCTION AND PRELIMINARIES
Azam, Fisher and Khan [6] introduced the concept of complex-valued metric spaces and established the existence of common fixed point theorems for a pair of contractive type mappings involving rational expressions. These results were further extended and generalized by Rouzkard and Imdad [10]. Klin-Eam and Suanoom [9] extended the concept of complex valued metric spaces and generalized the results of Azam et al. [1] and Rouzkard and Imdad [10]. Subsequently, several authors studied the existence and uniqueness of fixed points in complex valued spaces [1-5, 7,8,11,12].

Consistent with Azam et al. [6], the following definitions and notations will be needed in the sequel.

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$z_1 \precsim z_2$ if and only if $\operatorname{Re}(z_1) \le \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) \le \operatorname{Im}(z_2)$

If follows that $z_1 \precsim z_2$ if and only if one of the following conditions is satisfied:

(i) $\operatorname{Re}(z_1) = \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) < \operatorname{Im}(z_2)$,

(ii) $\operatorname{Re}(z_1) < \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) = \operatorname{Im}(z_2)$,

(iii) $\operatorname{Re}(z_1) < \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) < \operatorname{Im}(z_2)$,

(iv) $\operatorname{Re}(z_1) = \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) = \operatorname{Im}(z_2)$,

In particular, we will write $z_1 \precnsim z_2$ if $z_1 \ne z_2$ and one of (i), (ii), and (iii) is satisfied and we will write $z_1 \prec z_2$ if only (iii) satisfied.

**Definition 1.1.** Let $X$ be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbb{C}$ satisfies:

(i) $0 \precsim d(x,y)$ for all $x, y \in X$; and $d(x,y) = 0$ if and only if $x = y$,

(ii) $d(x,y) = d(y,x)$ for all $x, y \in X$,

(iii) $d(x,y) \precsim d(x,z) + d(z,y)$ for all $x, y, z \in X$.

Then $d$ is called a complex valued metric on $X$ and $(X,d)$ is called a complex valued metric space.

**Lemma 1.1.** Let $(X,d)$ be a complex-valued metric space and let $\{x_n\}$ be a sequence in $X$. Then $\{x_n\}$ converges to $x$ iff $|d(x_n, x)| \to 0$ as $n \to \infty$.

**Lemma 1.2.** Let $(X,d)$ be a complex-valued metric space and let $\{x_n\}$ is a Cauchy sequence iff $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$.

**Definition 1.2.** An element $(x,y) \in X \times X$ is called a coupled fixed point of the mapping $T : X \times X \to X$ if

$$T(x,y) = x \text{ and } T(y,x) = y.$$

**Definition 1.3.** Let $(X,d)$ be a complex-valued metric space. If every Cauchy sequence is convergent in $(X,d)$, then $(X,d)$ is called a complete complex-valued metric space.

## 2. MAIN RESULTS
**Theorem 2.1.** Let $(X,d)$ be a complete complex-valued metric space and $S, T : X^2 \to X$ are mapping satisfying

$$d(S(x_1,x_2), T(y_1,y_2)) \le \alpha \frac{[d(x_1, S(x_1,x_2)) d(x_1, T(y_1,y_2)) + d(x_1, T(y_1,y_2)) d(y_1, S(x_1,x_2))]}{d(x_1, T(y_1,y_2)) + d(y_1, S(x_1,x_2))}$$

for all $x_1, x_2, y_1, y_2 \in X$, where $0 \le \alpha < 1$.

Then $S$ and $T$ have a unique common coupled fixed point.

**Proof.** For any arbitrary points $x_0, y_0 \in X$, construct the sequence $\{x_n\}, \{y_n\}$ in $X$ such that

$x_{n+1} = S(x_n, y_n)$, $x_{n+2} = T(x_{n+1}, y_{n+1})$ and $y_{n+1} = S(y_n, x_n)$, $y_{n+2} = T(y_{n+1}, x_{n+1})$, for all $n = 0,1,2,...$

Now

$$\begin{aligned} & d(x_{n+1},x_{n+2}) \\ & = d(S(x_n,y_n),T(x_{n+1},y_{n+1})) \\ & \le \alpha \frac{\begin{pmatrix} [d(x_n,S(x_n,y_n))d(x_n,T(x_{n+1},y_{n+1})) \\ +d(x_{n+1},T(x_{n+1},y_{n+1}))d(x_{n+1},S(x_n,y_n))] \end{pmatrix}}{d(x_n,T(x_{n+1},y_{n+1}))+d(x_{n+1},S(x_n,y_n))} \\ & = \alpha \frac{[d(x_n,x_{n+1})d(x_n,x_{n+2})+d(x_{n+1},x_{n+2})d(x_{n+1},x_{n+1})]}{d(x_n,x_{n+2})+d(x_{n+1},x_{n+1})} \\ & = \alpha\, d(x_n,x_{n+1}) \quad \text{for all } n \ge 0 \end{aligned}$$

Hence $d(x_{n+1},x_{n+2}) \le \alpha\, d(x_n,x_{n+1}) \le \ldots \le \alpha^{n+1} d(x_0,x_1)$

Now for $m > n$, we have

$$\begin{aligned} d(x_m,x_n) & \le d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+\ldots+d(x_{m-1},x_n) \\ & \le \alpha^n d(x_0,x_1)+\alpha^{n+1}d(x_0,x_1)+\ldots+\alpha^{m-1}d(x_0,x_1) \\ & \le \frac{\alpha^n}{1-\alpha} d(x_0,x_1) \end{aligned}$$

implies

$$|d(x_n,x_n)| \le \frac{\alpha^n}{1-\alpha} |d(x_0,x_1)|$$

which implies that $|d(x_n,x_m)| \to 0$ as $n,m \to \infty$.

Hence $\{x_n\}$ is Cauchy sequence.

Also

$$\begin{aligned} & d(y_{n+1},y_{n+2}) \\ & = d(S(y_n,x_n),T(y_{n+1},x_{n+1})) \\ & \le \alpha \frac{\begin{pmatrix} [d(y_n,S(y_n,x_n))d(y_{n+1},T(y_{n+1},x_{n+1})) \\ +d(y_{n+1},T(y_{n+1},x_{n+1}))d(y_{n+1},S(y_n,x_n))] \end{pmatrix}}{d(y_n,T(y_{n+1},x_{n+1}))+d(y_{n+1},S(y_n,x_n))} \\ & = \alpha \frac{[d(y_n,y_{n+1})d(y_{n+1},y_{n+2})+d(y_{n+1},y_{n+2})d(y_{n+1},y_{n+1})]}{d(y_n,y_{n+2})+d(y_{n+1},y_{n+1})} \\ & \le \alpha\, d(y_n,y_{n+1}) \end{aligned}$$

implies

$$d(y_{n+1},y_{n+2}) \le \alpha\, d(y_n,y_{n+1}) \quad \text{for all } n \ge 0$$

Hence

$$\begin{aligned} d(y_{n+1},y_{n+2}) & \le \alpha\, d(y_n,y_{n+1}) \\ & \le \alpha^2 (y_{n-1},y_n) \le \ldots \le \alpha^{n+1} d(y_0,y_1) \end{aligned}$$

For $m > n$, we have

$$\begin{aligned} d(y_m,y_n) & \le d(y_n,y_{n+1})+d(y_{n+1},y_{n+2})+\ldots+d(y_{m-1},y_m) \\ & \le \alpha^n d(y_0,y_1)+\alpha^{n+1}d(y_0,y_1)+\ldots+\alpha^{m-1}d(y_0,y_1) \\ & \le \frac{\alpha^n}{1-\alpha} d(y_0,y_1) \end{aligned}$$

implies

$$|d(y_m,y_n)| \le \frac{\alpha^n}{1-\alpha} |d(y_0,y_1)|$$

implies

$$|d(y_m,y_n)| \to 0 \ \ as \ \ n,m \to \infty$$

implies

$\{y_n\}$ is also a Cauchy sequence.

So, $\{x_n\}$, $\{y_n\}$ are Cauchy sequences in $X$. But $X$ is complete so $\exists\ x,y \in X$ such that $\{x_n\}$, $\{y_n\}$ converges to $x,y$ respectively.

Now we will show that $S(x,y)=x$. If not then $\exists\ z \in c$ such that

$$d(S(x,y),x)=z \ \ where\ |z|>0.$$

Now

$$\begin{aligned} z & \le d(x,x_{n+2})+d(x_{n+2},S(x,y)) \\ & = d(x,x_{n+2})+d(S(x,y),T(x_{n+1},y_{n+1})) \\ & \le d(x,x_{n+2})+\alpha \frac{\begin{pmatrix} [d(x,S(x,y))d(x,T(x_{n+1},y_{n+1})) \\ +d(x_{n+1},T(x_{n+1},y_{n+1}))d(x_{n+1},S(x,y))] \end{pmatrix}}{d(x,T(x_{n+1},y_{n+1}))+d(x_{n+1},S(x,y))} \end{aligned}$$

implies

$$|z| \le |d(x,x_{n+2})| + \alpha \frac{\begin{pmatrix} [|z|\,|d(x,x_{n+2})| \\ +|d(x_{n+1},x_{n+2}))|\,|d(x_{n+1},S(x,y))|] \end{pmatrix}}{|d(x,x_{n+2})|+|d(x_{n+1},S(x,y))|}.$$

On taking limit as $n \to \infty$ we get $|z| \le 0$, a contradiction.

So $|z|=0$. Hence $S(x,y)=x$.

Also if possible, let $d(S(y,x),y)=t$ where $|t|>0$. Now

$$\begin{aligned} t & \le d(y,y_{n+2})+d(y_{n+2},S(y,x)) \\ & = d(y,y_{n+2})+d(S(y,x),T(y_{n+1},x_{n+1})) \\ & \le d(y,y_{n+2})+\alpha \frac{\begin{pmatrix} [d(y,S(y,x))d(y,T(y_{n+1},x_{n+1})) \\ +d(y_{n+1},T(y_{n+1},x_{n+1}))d(y_{n+1},S(y,x))] \end{pmatrix}}{d(y,T(y_{n+1},x_{n+1}))+d(y_{n+1},S(y,x))} \\ & = d(y,y_{n+2})+\alpha \frac{[t\,d(y,y_{n+2})+d(y_{n+1},y_{n+2})d(y_{n+1},S(y,x))]}{d(y,y_{n+2})+d(y_{n+1},S(y,x))} \end{aligned}$$

implies

$$|t| \le |d(y,y_{n+2})| + \alpha \frac{\begin{pmatrix} [|t|\,|d(y,y_{n+2})| \\ +|d(y_{n+1},y_{n+2}))|\,|d(y_{n+1},S(y,x))|] \end{pmatrix}}{|d(y,y_{n+2})|+|d(y_{n+1},S(y,x))|}.$$

On taking limit as $n \to \infty$ we get $|t| \le 0$, a contradiction.

So $t=0$. Hence $S(y,x)=y$.

Similarly $T(x,y)=x$ and $T(y,x)=y$.

Hence $(x,y)$ is the common coupled fixed point of $S$ and $T$

**For uniqueness.** Let $(u,v)$ is another common fixed point of $S$ and $T$. Then

$$d(x,u) = (S(x,y),T(u,v)) \leq \alpha \frac{\begin{pmatrix}[d(x,S(x,y))d(x,T(u,v)) \\ +d(u,T(u,v))d(u,S(x,y))]\end{pmatrix}}{d(x,T(u,v))+d(u,S(x,y))}$$

implies

$$d(x,u) \leq 0$$

implies

$$x = u$$

Similarly $y = v$.

So, $(x, y)$ is the unique common coupled fixed point for $S$ and $T$.

## 3. REFERENCES

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