PH/PH/1 Bulk Arrival and Bulk Service Queue

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ABSTRACT
This paper studies two stochastic bulk arrivals and bulk services PH/PH/1 queue Models (A) and (B) with k_1 *and* k_2 as the number of phases of PH arrival and PH service distributions respectively. The system has infinite storing capacity and the arrival and service sizes are finite valued random variables. Matrix partitioning method is used to study the models. In Model (A) the maximum of the arrival sizes is greater than the maximum of the service sizes and the infinitesimal generator is partitioned as blocks of k_1k_2 times the maximum of the arrival sizes for analysis. In Model (B) the maximum of the arrival sizes is less than the maximum of the service sizes. The generator is partitioned using blocks of k_1k_2 times the maximum of the service sizes. Block circulant matrix structure is noticed in the basic system generator. The stationary queue length probabilities, its expected values, its variances and probabilities of empty levels are derived for the two models using matrix geometric methods. Numerical examples are presented for illustration.

General Terms
Bulk Arrivals, Bulk Service, Block Circulant Matrix, Markov Chain and Phase Type Distribution

Keywords
Block Sizes, Stationary Probability, Infinitesimal Generator and Matrix Geometric Approach.

1. INTRODUCTION
In this paper two bulk arrival and bulk service PH/PH/1 queues have been studied using matrix geometric methods. Retrial queues are studied by Aissani.A and Artalejo.J.R [1] and Ayyappan, Subramanian and Gopal Sekar [2]. Numerical methods on matrix methods are presented by Bini, Latouche and Meini [3]. Multi server model has been of interest in Chakravarthy and Neuts [4]. Birth and death model has been analyzed by Gaver, Jacobs and Latouche [5]. Analytic methods are focused in Latouche and Ramaswami [6] and for matrix geometric methods one may refer Neuts [7]. The models considered here are general compared to existing models. Here random number of arrivals and random number of services are considered at a time whereas a fixed number of customers arrive or are served at any arrival or service epochs in existing queue models. Fixed numbers of customers are cleared by a service in the models of Neuts and Nadarajan [8]. In real life situations when a machine manufactures a fixed number of products in every production schedule, the defective items are rejected in all production lots, making the production lot is only of random size and not a fixed one always. Situations of random bulk services are seen often in software based industries where finished software projects waiting for marketing are sold in bulk sizes when there is economic boom and the business may be insignificant when there is economic recession. In industrial productions, bulk types are very common. Manufactured products arrive in various bulk sizes for sale in markets and the products are sold in various bulk sizes depending on market requirements. Noam Paz and Uri Yechali [9] have studied M/M/1 queue with disaster. Usually bulk arrival models have M/G/1 upper-Heisenberg block matrix structure. The decomposition of a Toeplitz sub matrix of the infinitesimal generator is required to find the stationary probability vector as done in William J. Stewart [10] and even in such models the recurrence relation method to find the stationary probabilities is stopped at certain level in most general cases indicating limitations of such approach. For M/M/1 bulk queues with random environment models one may refer Rama Ganesan, Ramshankar and Ramanarayanan [11]. In this paper the partitioning of the matrix is carried out in a way that the stationary probability vector exhibits a matrix geometric structure for PH/PH/1 bulk queues where the arrivals and service sizes are finite. Two models (A) and (B) on PH/PH/1 bulk queue systems with infinite storage space for customers are studied here using the block partitioning method. In the models considered here, the maximum arrival sizes and the maximum service sizes are different. Model (A) presents the case when M, the maximum of arrival sizes is bigger than N, the maximum of the sales sizes. In Model (B), its dual case N is bigger than M, is treated. In general in Queue models, the state space of the system has the first co-ordinate indicating the number of customers in the system but here the customers in the system are grouped and considered as members of M sized blocks of customers when M >N and N sized blocks of customers when N > M for finding the rate matrix. Using the maximum of the bulk arrival size or the maximum of the bulk service size and grouping the customers as members of the blocks for the partitioning the infinitesimal generator is a new approach in this area. The matrices appearing as the basic system generators in these two models due to block partitioned structure are seen as a block circulant matrices. The stationary probability of the number of customers waiting for service, the expectation and the variance and the probability of empty queue are derived for these models. Numerical cases are presented to illustrate them.

2. MODEL (A): MAXIMUM ARRIVAL SIZE M > MAXIMUM SERVICE SIZE N.

2.1Assumptions
i) The time between consecutive epochs of bulk arrivals of customers has phase type distribution ($\underline{\alpha}$, T) where T is a matrix of order k_1 with absorbing rate $T_0 = -Te$ to the absorbing state k_1+1 from where the arrival process moves instantaneously to a starting state as per the starting vector $\underline{\alpha}$ = $(\alpha_1, \alpha_2., ..., \ \alpha_{k_1})$ and $\sum_{i=1}^{k_1} \alpha_i = 1$. Let φ be the invariant probability vector of the generator matrix $(T + T_0\underline{\alpha})$.
ii) When the absorption occurs in the PH arrival process due to transition from a state i to state $k_1 + 1$, χ_i number of customers arrive with probabilities P (χ_i= j) = p_j^i for 1 ≤ j ≤ M_i and $\sum_{j=1}^{M_i} p_j^i$=1where M_i is the maximum size for 1≤ i≤k_1.

iii) The time between consecutive epochs of bulk services of customers has phase type distribution ($\underline{\beta}$, S) where S is a matrix of order k_2 with absorbing rate $S_0 = -Se$ to the absorbing state k_2+1 from where the service process moves instantaneously to a starting state as per the staring vector $\underline{\beta}$ = $(\beta_1, \beta_2., ..., \beta_{k_2})$ and $\sum_{i=1}^{k_2} \beta_i$ = 1. Let ϕ be the invariant probability vector of the generator matrix $(S + S_0\underline{\beta})$.
iv) Customers of bulk size ψ_i are served at epochs when the absorption occurs due to a transition from state i to state k_2+1, with probabilities P (ψ_i = j) = q_j^i for 1≤ j ≤ N_i and $\sum_{j=1}^{N_i} q_j^i$ = 1, when more than N_i customers are waiting for service where N_i is the maximum for 1 ≤ i ≤ k_1. When n customers n < N_i are waiting for service, then j customers are served with probability q_j^i, for 1≤ j ≤ n-1 and n customers are served with probability $\sum_{j=n}^{N_i} q_j^i$ for 1 ≤i ≤ k_2.
v)The maximum arrival size M=$\max_{1 \le i \le k_1} M_i$ is greater than the maximum service size N=$\max_{1 \le j \le k_2} N_j$.

2.2Analysis

The state of the system of the continuous time Markov chain X(t) under consideration is presented as follows. X(t) = {(0,i) : for 1 ≤ i ≤ k_1)} U {(0, k, i, j) ; for 1 ≤ k≤ M-1; 1 ≤ i ≤ k_1; 1 ≤ j ≤ k_2} U{(n, k, i, j): for 0 ≤ k ≤ M-1; 1 ≤ i ≤ k_1; 1 ≤ j ≤ k_2 and n ≥ 0}. (1)
The chain is in the state (0, i) when the number of customers in the queue is 0, and the arrival phase is i for 1 ≤ i ≤ k_1. The chain is in the state (0, k, i, j) when the number of customers is k for 1 ≤ k ≤ M-1, arrival phase is i for 1 ≤ i ≤ k_1 and the service phase is j for 1 ≤ j ≤ k_2. The chain is in the state (n, k, i, j) when the number of customers in the queue is n M + k, for 0 ≤ k ≤ M-1 and 1 ≤ n < ∞, arrival phase is i for 1 ≤ i ≤ k_1 and the service phase is j for 1 ≤ j ≤ k_2. When the number of customers waiting in the system is r, then r is identified with (n, k) where r on division by M gives n as the quotient and k as the remainder. Let the survivor probabilities of arrivals χ_i and of services ψ_j be respectively
P(χ_i>m)= P_m^i =1-$\sum_{n=1}^{m} p_n^i$,for 1 ≤ m ≤ M_i -1and1≤ i ≤k_1 (2)
P(ψ_j>m)= Q_m^j=1-$\sum_{n=1}^{m} q_n^j$, for 1 ≤ m ≤ N_i -1and 1≤ j≤ k_2 (3)
with $P_0^i = 1$, for all i, 1≤ i ≤k_1 and $Q_0^i = 1$ for all i ,1≤i≤ k_2. The chain X(t) describing model has the infinitesimal generator Q_A of infinite order which can be presented in block partitioned form given below.

$$Q_A = \begin{bmatrix} B_1 & B_0 & 0 & 0 & . & . & . & \cdots \\ B_2 & A_1 & A_0 & 0 & . & . & . & \cdots \\ 0 & A_2 & A_1 & A_0 & 0 & . & . & \cdots \\ 0 & 0 & A_2 & A_1 & A_0 & 0 & . & \cdots \\ 0 & 0 & 0 & A_2 & A_1 & A_0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \quad (4)$$

In (4) the states of the matrices are listed lexicographically as $\underline{0}, \underline{1}, \underline{2}, \underline{3}$, ... For partition purpose the zero states in the first two sets of (1) are combined. The vector $\underline{0}$ is of type 1 x $[k_1 + k_1k_2$(M-1)] and the vector $\underline{n}$ is of type 1 x (k_1k_2M). $\underline{0}$=((0,1),(0,2),(0,3)…(0,k_1),(0,1,1,1),(01,1,2)….(0,1,1, k_2), (0,1,2,1),…,(0,1,k_1, k_2),(0,2,1,1),..,(0,2,k_1, k_2),(0,3,1,1)…. (0,3,k_1, k_2),…,(0,M-1,1,1)… (0,M-1,k_1k_2)) and for n ≥ 1 $\underline{n}$ = ((n,0,1,1), (n,0,1, 2) … (n, 0,1, k_2), (n, 0, 2, 1), (n,0,2,2),…(n,0,2,k_2),(n,0,3,1)…(n,0,k_1, k_2),(n,1,1,1)…. (n,1,k_1, k_2),(n,2,1,1)…(n,2, k_1, k_2)……..(n,M-1,1,1), (n,M-1,1,2)…(n,M-1,1,k_2),(n,M-1,2,1)…(n,M-1,k_1, k_2)).
The matricesB_1 and A_1 have negative diagonal elements, they are of orders $k_1 + k_1k_2$(M-1) and k_1k_2M respectively and their off diagonal elements are non- negative. The matrices A_0 $and A_2$ have nonnegative elements and are of order k_1k_2M. The matrices B_0 and B_2 have non-negative elements and are of type $[k_1 + k_1k_2$(M-1)] x (k_1k_2M) and (k_1k_2M) x $[k_1 + k_1k_2$(M-1)] and they are given below. Let ⊕ and⊗ denote the Kronecker sum and Kronecker products respectively. Let $Q_1^{'}$=T⊕S = (T⊗I_{k_2}) + (I_{k_1}⊗S) (5) where I indicates the identity matrices of orders given in the suffixes and $Q_1^{'}$ is of order k_1k_2. Let T_0 = $(t_0^1, t_0^2, ... t_0^{k_1})'$ be the column vector of absorption rates in PH arrival process. Let S_0 = $(s_0^1, s_0^2, ... s_0^{k_2})'$ be the column vector of absorption rates concerning the PH service distribution. Let T_{0j} = $(t_0^1 p_j^1, t_0^2 p_j^2, ..., t_0^{k_1} p_j^{k_1})'$ for 1 ≤ j ≤ M
$S_{0j} = \left(s_0^1 q_j^1, s_0^2 q_j^2, ..., s_0^{k_2} q_j^{k_2}\right)'$ for1 ≤ j ≤ N and

$$\Lambda_j = [T_{0j}\underline{\alpha}\,] \otimes I_{k_2} \;\; for\ 1 \le j \le M, \quad (6)$$

$$U_j = I_{k_1} \otimes \left[S_{0j}\underline{\beta}\right] for\ 1 \le j \le N\ with\ orders\ \ k_1k_2, \ (7)$$

Λ'_j =[T_{0j} ⊗ $\underline{\alpha} \otimes \underline{\beta}$] , $for 1 \le j \le M$ with order k_1x (k_1k_2)
$V_j = I_{k_1} \otimes [(s_0^1 Q_j^1, s_0^2 Q_j^2, ..., s_0^{k_2} Q_j^{k_2})']$, for 1 ≤ j ≤ N
$U = I_{k_1} \otimes \left[\left(s_0^1, s_0^2, ..., s_0^{k_2}\right)'\right]$ with orders k_1k_2xk_1.
The matrix B_0 is same as that of A_0 when Λ_M in $A_0$is replaced byΛ'_M. The matrix B_2 is same as that of $A_2$when the first block column with 0 is considered as k_1 columns block instead of $k_1k_2$columns block of A_2.

$$A_0 = \begin{bmatrix} \Lambda_M & 0 & \cdots & 0 & 0 & 0 \\ \Lambda_{M-1} & \Lambda_M & \cdots & 0 & 0 & 0 \\ \Lambda_{M-2} & \Lambda_{M-1} & \cdots & 0 & 0 & 0 \\ \Lambda_{M-3} & \Lambda_{M-2} & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \Lambda_3 & \Lambda_4 & \cdots & \Lambda_M & 0 & 0 \\ \Lambda_2 & \Lambda_3 & \cdots & \Lambda_{M-1} & \Lambda_M & 0 \\ \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{M-2} & \Lambda_{M-1} & \Lambda_M \end{bmatrix} (8)$$

$$A_2 = \begin{bmatrix} 0 & \cdots & 0 & U_N & U_{N-1} & \cdots & U_2 & U_1 \\ 0 & \cdots & 0 & 0 & U_N & \cdots & U_3 & U_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & U_N & U_{N-1} \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & U_N \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} (9)$$

$$A_1 = \begin{bmatrix} Q_1^{'} & \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{M-N-2} & \Lambda_{M-N-1} & \Lambda_{M-N} & \cdots & \Lambda_{M-2} & \Lambda_{M-1} \\ U_1 & Q_1^{'} & \Lambda_1 & \cdots & \Lambda_{M-N-3} & \Lambda_{M-N-2} & \Lambda_{M-N-1} & \cdots & \Lambda_{M-3} & \Lambda_{M-2} \\ U_2 & U_1 & Q_1^{'} & \cdots & \Lambda_{M-N-4} & \Lambda_{M-N-3} & \Lambda_{M-N-2} & \cdots & \Lambda_{M-4} & \Lambda_{M-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ U_N & U_{N-1} & U_{N-2} & \cdots & Q_1^{'} & \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{M-N-2} & \Lambda_{M-N-1} \\ 0 & U_N & U_{N-1} & \cdots & U_1 & Q_1^{'} & \Lambda_1 & \cdots & \Lambda_{M-N-3} & \Lambda_{M-N-2} \\ 0 & 0 & U_N & \cdots & U_2 & U_1 & Q_1^{'} & \cdots & \Lambda_{M-N-4} & \Lambda_{M-N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & U_N & U_{N-1} & U_{N-2} & \cdots & Q_1^{'} & \Lambda_1 \\ 0 & 0 & 0 & \cdots & 0 & U_N & U_{N-1} & \cdots & U_1 & Q_1^{'} \end{bmatrix} \quad (10)$$

$$B_1 \begin{bmatrix} T & \Lambda'_1 & \Lambda'_2 & \cdots & \Lambda'_{M-N-2} & \Lambda'_{M-N-1} & \Lambda'_{M-N} & \cdots & \Lambda'_{M-2} & \Lambda'_{M-1} \\ U & Q_1^{'} & \Lambda_1 & \cdots & \Lambda_{M-N-3} & \Lambda_{M-N-2} & \Lambda_{M-N-1} & \cdots & \Lambda_{M-3} & \Lambda_{M-2} \\ V_1 & U_1 & Q_1^{'} & \cdots & \Lambda_{M-N-4} & \Lambda_{M-N-3} & \Lambda_{M-N-2} & \cdots & \Lambda_{M-4} & \Lambda_{M-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ V_{N-1} & U_{N-1} & U_{N-2} & \cdots & Q_1^{'} & \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{M-N-2} & \Lambda_{M-N-1} \\ 0 & U_N & U_{N-1} & \cdots & U_1 & Q_1^{'} & \Lambda_1 & \cdots & \Lambda_{M-N-3} & \Lambda_{M-N-2} \\ 0 & 0 & U_N & \cdots & U_2 & U_1 & Q_1^{'} & \cdots & \Lambda_{M-N-4} & \Lambda_{M-N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & U_N & U_{N-1} & U_{N-2} & \cdots & Q_1^{'} & \Lambda_1 \\ 0 & 0 & 0 & \cdots & 0 & U_N & U_{N-1} & \cdots & U_1 & Q_1^{'} \end{bmatrix} (11)$$

$$Q_A^{''} = \begin{bmatrix} Q_1^{'}+\Lambda_M & \Lambda_1 & \cdots & \Lambda_{M-N-2} & \Lambda_{M-N-1} & \Lambda_{M-N}+U_N & \cdots & \Lambda_{M-2}+U_2 & \Lambda_{M-1}+U_1 \\ \Lambda_{M-1}+U_1 & Q_1^{'}+\Lambda_M & \cdots & \Lambda_{M-N-3} & \Lambda_{M-N-2} & \Lambda_{M-N-1} & \cdots & \Lambda_{M-3}+U_3 & \Lambda_{M-2}+U_2 \\ \Lambda_{M-2}+U_2 & \Lambda_{M-1}+U_1 & \cdots & \Lambda_{M-N-4} & \Lambda_{M-N-3} & \Lambda_{M-N-2} & \cdots & \Lambda_{M-4}+U_3 & \Lambda_{M-3}+U_3 \\ \vdots & \vdots & ⁝ & \vdots & \vdots & \vdots & ⁝ & \vdots & \vdots \\ \Lambda_{M-N+2}+U_{N-2} & . & \cdots & . & . & . & \cdots & \Lambda_{M-N}+U_N & \Lambda_{M-N+1}+U_{N-1} \\ \Lambda_{M-N+1}+U_{N-1} & . & \cdots & . & . & . & . & \Lambda_{M-N-1} & \Lambda_{M-N}+U_N \\ \Lambda_{M-N}+U_N & . & \cdots & Q_1^{'}+\Lambda_M & \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{M-N-2} & \Lambda_{M-N-1} \\ \Lambda_{M-N-1} & \Lambda_{M-N}+U_N & \cdots & \Lambda_{M-1}+U_1 & Q_1^{'}+\Lambda_M & \Lambda_1 & \cdots & \Lambda_{M-N-3} & \Lambda_{M-N-2} \\ \Lambda_{M-N-2} & \Lambda_{M-N-1} & \cdots & \Lambda_{M-2}+U_2 & \Lambda_{M-1}+U_1 & Q_1^{'}+\Lambda_M & \cdots & \Lambda_{M-N-4} & \Lambda_{M-N-3} \\ \vdots & \vdots & ⁝ & \vdots & \vdots & \vdots & ⁝ & \vdots & \vdots \\ \Lambda_2 & \Lambda_3 & \cdots & \Lambda_{M-N}+U_N & \Lambda_{M-N+1}+U_{N-1} & \Lambda_{M-N+2}+U_{N-2} & \cdots & Q_1^{'}+\Lambda_M & \Lambda_1 \\ \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{M-N-1} & \Lambda_{M-N}+U_N & \Lambda_{M-N+1}+U_{N-1} & \cdots & \Lambda_{M-1}+U_1 & Q_1^{'}+\Lambda_M \end{bmatrix} \quad (12)$$

The basic generator of the bulk queue which is concerned with only the arrival and service is a matrix of order k_1k_2M given above in (12) where $Q_A^{''}$=$A_0 + A_1 + A_2$ (13)
Its probability vector w gives, $wQ_A^{''}$ =0 and w. e = 1 (14)
It is well known that a square matrix in which each row (after the first) has the elements of the previous row shifted cyclically one place right, is called a circulant matrix. It is very interesting to note that the matrix $Q_A^{''} = A_0 + A_1 + A_2$ is a block circulant matrix where each block matrix is rotated one block to the right relative to the preceding block partition. In (13),the first block-row of type k x Mk is, $W = (Q_1^{'} + \Lambda_M, \Lambda_1, \Lambda_2,\ \ldots, \Lambda_{M-N-2}, \Lambda_{M-N-1}, \Lambda_{M-N} + U_N,\ \ldots,\ \Lambda_{M-2} + U_2,\ \Lambda_{M-1} + U_1)$ which gives as the sum of the blocks $(Q_1^{'} + \Lambda_M) + \Lambda_1 + \Lambda_2 + \ldots + \Lambda_{M-N-2} + \Lambda_{M-N-1} + \Lambda_{M-N} + U_N + \ldots + \Lambda_{M-2} + U_2 + \Lambda_{M-1} + U_1 =$ (T+$T_0\underline{\alpha}$) $\oplus$(S+$S_0\underline{\beta}$) = (T+$T_0\underline{\alpha}$)$\otimes I_{k_2} + I_{k_1}\otimes(S + S_0\underline{\beta})$ whose stationary vector of $\varphi\otimes\phi$. This gives $\varphi\otimes\phi(Q_1^{'} + \Lambda_M) + \varphi\otimes\phi\sum_{i=1}^{M-N-1}\Lambda_i +$ $\varphi\otimes\phi\sum_{i=1}^{N}(\Lambda_{M-i} + U_i)$ = 0. So $(\varphi\otimes\phi, \varphi\otimes\phi \ldots, \varphi\otimes\phi)$. W = 0 = $(\varphi\otimes\phi, \varphi\otimes\phi \ldots,\varphi\otimes\phi)$ W'. Since all blocks, in any block-row are seen somewhere in each and every column block structure (the matrix is block circulant), the above equation shows the left eigen vector of the matrix $Q_A^{''}$ is $(\varphi\otimes\phi,\varphi\otimes\phi,\ldots, \varphi\otimes\phi)$.Using (14), this gives probability vector w=$\left(\frac{\varphi\otimes\phi}{M}, \frac{\varphi\otimes\phi}{M}, \frac{\varphi\otimes\phi}{M}, \ldots\ldots, \frac{\varphi\otimes\phi}{M}\right)$ (15)
Neuts [7], gives the stability condition as, w $A_0\, e < w\, A_2\, e$ where w is given by (15). Taking the sum cross diagonally in the A_0 *and* A_2 matrices, it can be seen that w $A_0\, e = \frac{1}{M}\ \varphi\otimes\phi(\sum_{n=1}^{M} n\Lambda_n)e = \frac{1}{M}\ (\sum_{n=1}^{M} n(\varphi\otimes\phi\Lambda_n)e$ $=\frac{1}{M}\ (\sum_{n=1}^{M} n(\varphi\otimes\phi)(T_{0n}\underline{\alpha} \otimes I_{k_2})e$ $=\frac{1}{M}\sum_{n=1}^{M} n\,\varphi\,[T_{0n}\underline{\alpha}\,]e \otimes \phi e$ $=\frac{1}{M}\sum_{n=1}^{M} n\sum_{j=1}^{k_1}\varphi_j\ t_0^j p_n^i =$ $\frac{1}{M}\sum_{j=1}^{k_1}\varphi_j\ t_0^j E(\chi_j) <$ $w\, A_2\, e = \frac{1}{M}\ \varphi\otimes\phi(\sum_{n=1}^{N} nU_n)e =$ $\frac{1}{M}\ (\sum_{n=1}^{M} n(\varphi\otimes\phi U_n)e =$ $\frac{1}{M}\ (\sum_{n=1}^{M} n(\varphi\otimes\phi)(I_{k_1}\otimes S_{0n}\underline{\beta})e = \frac{1}{M}\sum_{n=1}^{M} n\,\varphi\, e \otimes \phi S_{0n}\underline{\beta} e$ $=\frac{1}{M}\sum_{n=1}^{M} n\sum_{j=1}^{k_2}\phi_j\ s_0^j q_n^j = \frac{1}{M}\sum_{j=1}^{k_2}\phi_j\ s_0^j E(\psi_j)$ where $\varphi_i$and ϕ_j are components of φ and ϕ respectively for 1≤ i ≤k_1 and 1≤ j≤ k_2. So the inequality for the steady state reduces to
$\sum_{i=1}^{k_1}\varphi_i\ t_0^i E(\chi_i) < \sum_{i=1}^{k_2}\phi_i\ s_0^i E(\psi_i)$ (16)
This is the stability condition for PH/PH/1 bulk queue with random sizes of arrivals and of services where maximum arrival size in all arrival phases is greater than the maximum service size in all service phases. When (16) is satisfied, the stationary distribution of the queue length exists Neuts[7]. Let π(0, i) for 1 ≤ i ≤ k_1; π(0, k, i, j) for1 ≤ i≤ k_1, 1≤ j ≤ k_2, 1 ≤ k ≤ M-1; π (n, k, i, j), for 1≤ i ≤ k_1,1 ≤ j ≤ k_2,0 ≤ k ≤ M-1 and 1 ≤ n < ∞ be the stationary probability of the states in (1). Let π_0 = (π(0, 1), π(0, 2),…, π(0, k_1), π(0, 1, 1, 1), π(0, 1, 1, 2)…π(0, M-1, k_1, k_2)) be of type 1 x [$k_1 + k_1k_2$(M-1)]. Let π_n= (π(n, 0, 1,1), π(n, 0,1, 2) … π(n, 0, 1,k_1), π(n, 0, 2, 1), π(n,0, 2, 2),…π(n,0,k_1, k_2), π(n,1,1,1),…….π(n, M-1, 1,1), π(n, M-1,1,2)…π(n, M-1,k_1k_2) be of type 1 x k_1k_2M for n ≥ 1. The stationary probability vector $\pi = (\pi_0, \pi_1, \pi_3, \ldots)$ satisfies the equations πQ_A=0 and πe=1. (17)
From (17), it can be seen $\pi_0 B_1 + \pi_1 B_2$=0. (18)
$\pi_0 B_0$+$\pi_1 A_1$+$\pi_2 A_2$ = 0 (19)
$\pi_{n-1}A_0$+$\pi_n A_1$+$\pi_{n+1}A_2$ = 0, for n ≥ 2. (20)

Introducing the rate matrix R as the minimal non-negative solution of the non-linear matrix equation
A_0+RA_1+R^2A_2=0, (21)
it can be proved (Neuts [7]) that π_n satisfies the following.
$\pi_n = \pi_1 R^{n-1}$ for n ≥ 2. (22)
Using (18), π_0 satisfies $\pi_0 = \pi_1 B_2(-B_1)^{-1}$ (23)
So using (19) and (23) and (22) the vector π_1 can be calculated up to multiplicative constant since π_1 satisfies the equation π_1 $[B_2(-B_1)^{-1}B_0 + A_1 + RA_2]$ =0. (24)
Using (17) and (23) it can be seen that $\pi_1[B_2(-B_1)^{-1}$e+(I-R)$^{-1}e]$ = 1. (25)
Replacing the first column of the matrix multiplier of π_1 in equation (24), by the column vector multiplier of π_1 in (25), a matrix which is invertible may be obtained. The first row of the inverse of that same matrix is π_1 and this gives along with (22) and (23) all the stationary probabilities of the system. The matrix R is iterated starting with $R(0) = 0$; and finding $R(n+1)$=$-A_0A_1^{-1}$–$R^2(n)A_2A_1^{-1}$, n ≥ 0. The iteration may be terminated to get a solution of R at a norm level where $\|R(n+1) - R(n)\|$ < ε.

2.3 Performance Measures of the System

(i) The probability of the queue length S = r, P(S=r) can be seen as follows. Let n ≥ 0 and k for 0 ≤ k ≤ M-1 be non-negative integers such that r = n M + k. From (22), (23), (24) and (25) P(S=r) = $\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\pi\,(n,k,i,j\,)$, where r = n M + k.
(ii) The probability that the queue length is zero P(S=0)=$\sum_{i=1}^{k_1}\pi\,(0,i)$
(iii) The expected queue level E(S), can be calculated.
E(S)=$\sum_{i=1}^{k_1} 0\,\pi(0,i)$+$\sum_{K=1}^{M-1}\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\pi(0,k,i,j)k$
+$\sum_{n=1}^{\infty}\sum_{k=0}^{M-1}\ \sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\pi\,(n,k,i,j)\ (Mn+k)$
=$\sum_{k=1}^{M-1}k\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\pi(0,k,i,j) + \sum_{n=1}^{\infty}\pi_n$.(Mn,…,Mn, Mn+1,… ,Mn+1, Mn+2,…,Mn+2,…Mn+M-1,… ,Mn+M-1)
=$\sum_{k=1}^{M-1}k\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\pi(0,k,i,j)$ +M$\sum_{n=1}^{\infty}n\pi_n\,e$
+$\pi_1(\,I-R)^{-1}\xi$. Here ξ is of type k_1k_2M x1 column vector ξ=$(0,\dots 0,1,\dots,1,2,\dots,2,\dots,M-1,\dots,M-1)'$ in which consecutively k_1k_2 times 0,1,2,3.., M-1 appear. Let it be called ξ' when 0 appears $k_1$times and others in that order appear$k_1k_2$times.
E(S)= π_0ξ' + $\pi_1(\,I-R)^{-1}\xi + M\pi_1(I-R\,)^{-2}e$ (25)
(iv) Variance of S can be derived. Let η be column vector η=$[0,\dots,0,1^2,\dots 1^2\ 2^2,\dots,2^2,\dots\ (M-1)^2,\dots,(M-1)^2]'$ of type k_1k_2M x1 in which consecutively k_1k_2 times squares of 0,1,2,3.., M-1 appear. Let it be called η' when 0 appears $k_1$times and others in the same manner as in η appear k_1k_2 times. Then it can be seen that the second moment,
E(S^2) = $\sum_{i=1}^{k_1}0\,\pi(0,i)$ +$\sum_{K=1}^{M-1}\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\pi(0,k,i,j)k^2$
+$\sum_{n=1}^{\infty}\sum_{k=0}^{M-1}\ \sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\pi\,(n,k,i,j)\ (Mn+k)^2$
=π_0η' + $M^2[\sum_{n=1}^{\infty}n(n-1)\pi_n\,e + \sum_{n=1}^{\infty}n\pi_n\,e]$ + $\sum_{n=1}^{\infty}\pi_n\eta$ + 2M$\sum_{n=1}^{\infty}n\,\pi_n\,\xi$.
So, E(S^2)= π_0η' + $M^2[\pi_1(I-R)^{-3}2R\,e + \pi_1(I-\ R)^{-2}e]$ + $\pi_1(I-R)^{-1}\eta + 2M\pi_1(I-R)^{-2}\xi$. (26)
VAR(S)=E(S^2) − $[E(S)]^2$may be written from (26) and(25).

3. MODEL (B) MAXIMUM ARRIVAL SIZE M < MAXIMUM SERVICE SIZE N

The dual case of Model (A), namely the case, M < N is treated here. (When M =N both models are applicable and one can use any one of them.) The assumption (v) of Model (A) is changed and all its other assumptions are retained.

3.1Assumption

v). The maximum arrival size M=$\max_{1\le i\le k_1}M_i$ is less than the maximum service size N=$\max_{1\le j\le k_2}N_j$.

3.2Analysis

Since this model is dual, the analysis is same as that of Model (A). The differences are noted below. The state space of the chain is as follows presented in a similar way.
X (t) ={(0,i) : for 1 ≤ i ≤ k_1)} U {(0, k, i, j) ; for 1 ≤ k≤ N-1; 1 ≤ i ≤ k_1; 1 ≤ j ≤ k_2} U {(n, k, i, j): for 0 ≤ k ≤ N-1 for 1 ≤ i ≤ k_1, 1 ≤ j ≤ k_2, 1 ≤ n < ∞}. (27)
The chain is in the state (0, i) when the number of customers in the queue is 0, and the arrival phase is i for 1 ≤ i ≤ k_1. The chain is in the state (0, k, i, j) when the number of customers is k for 1 ≤ k ≤ N-1, arrival phase is i for 1 ≤ i ≤ k_1 and the service phase is j for 1 ≤ j ≤ k_2. The chain is in the state (n, k, i, j) when the number of customers in the queue is n N + k, for 0 ≤ k ≤ N-1 and 1 ≤ n < ∞, arrival phase is i for 1 ≤ i ≤ k_1 and the service phase is j for 1 ≤ j ≤ k_2. When the number of customers in the system is r, then r is identified with (n, k) where r on division by N gives n as the quotient and k as the remainder. The infinitesimal generator Q_B of the model has the same block partitioned structure given in (4) for Model (A) but the inner matrices are of different.

$$Q_B=\begin{bmatrix} B'_1 & B'_0 & 0 & 0 & . & . & . & \cdots \\ B'_2 & A'_1 & A'_0 & 0 & . & . & . & \cdots \\ 0 & A'_2 & A'_1 & A'_0 & 0 & . & . & \cdots \\ 0 & 0 & A'_2 & A'_1 & A'_0 & 0 & . & \cdots \\ 0 & 0 & 0 & A'_2 & A'_1 & A'_0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \qquad (28)$$

In (28) the states of the matrices are listed lexicographically as $\underline{0}, \underline{1}, \underline{2}, \underline{3}, \dots \underline{n}, \dots$. For partition purpose the zero states in the first two sets of (27) are combined. All states are listed as follows. The vector $\underline{0}$ is of type 1 x $[k_1 + k_1k_2$(N-1)] and the vector $\underline{n}$ is of type 1 x (k_1k_2N). They are as follows. $\underline{0}$=((0,1),(0,2),(0,3)…(0,k_1),(0,1,1,1),(0,1,1,2),…(0,N-1,1,1) …(0, N-1, k_1,k_2)) and $\underline{n}$ = ((n, 0, 1,1), (n, 0,1, 2) … (n,0,1, k_2),(n,0,2,1),…(n,0,2,k_2),(n,0,3,1)…(n,0,k_1, k_2),(n,1,1,1),…(n, N-1, 1,1),(n, N-1, 1,2)…… (n, N-1, 1,k_2),(n, N-1, 2,1) ……(n, N-1,k_1, k_2)), for n ≥ 1. The matrices $B'_1 and\ A'_1$ are of orders $k_1 + k_1k_2$(N-1) and k_1k_2N respectively. They have negative diagonal elements and their off diagonal elements are non- negative. The matrices $A'_0\ and A'_2$ have nonnegative elements and are square matrices of order k_1k_2N. The matrices $B'_0\ and\ B'_2$ have non negative elements and are of type $[k_1 + k_1k_2$(N-1)] x (k_1k_2N) and (k_1k_2N) x $[k_1 + k_1k_2$(N-1)].Using Model (A) for definitions of $\Lambda_j\ and \Lambda'_j, for\ 1 \le j \le M, and\ \ U_j, V_j\ \ for\ 1 \le j \le N,$ and U and letting Q'_1=T⊕S, the partitioning matrices are defined as follows. The matrix $B'_0$is same as that of A'_0 with first zero block row is of order $k_1xk_1k_2N$. The matrix $B'_2$is same as that of A'_2 except the first column block which is of type k_1k_2x k_1 and is $(U'_N, 0, \dots .0)'$ where U'_n=$I_{k_1}\otimes S_{0N}$

$$A'_0=\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\ \Lambda_M & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\\ \Lambda_{M-1} & \Lambda_M & \cdots & 0 & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \Lambda_2 & \Lambda_3 & \cdots & \Lambda_M & 0 & 0 & \cdots & 0\\ \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{M-1} & \Lambda_M & 0 & \cdots & 0 \end{bmatrix} (29)$$

$$A'_2=\begin{bmatrix} U_N & U_{N-1} & U_{N-2} & \cdots & U_3 & U_2 & U_1\\ 0 & U_N & U_{N-1} & \cdots & U_4 & U_3 & U_2\\ 0 & 0 & U_N & \cdots & U_5 & U_4 & U_3\\ 0 & 0 & 0 & \ddots & U_6 & U_5 & U_4\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & U_N & U_{N-1} & U_{N-2}\\ 0 & 0 & 0 & \cdots & 0 & U_N & U_{N-1}\\ 0 & 0 & 0 & \cdots & 0 & 0 & U_N \end{bmatrix} (30)$$

$$A'_1=\begin{bmatrix} Q_1^{'} & \Lambda_1 & \Lambda_2 & \cdots & \Lambda_M & 0 & 0 & \cdots & 0 & 0\\ U_1 & Q_1^{'} & \Lambda_1 & \cdots & \Lambda_{M-1} & \Lambda_M & 0 & \cdots & 0 & 0\\ U_2 & U_1 & Q_1^{'} & \cdots & \Lambda_{M-2} & \Lambda_{M-1} & \Lambda_M & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ U_{N-M-1} & U_{N-M-2} & U_{N-M-3} & \cdots & Q_1^{'} & \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{M-1} & \Lambda_M\\ U_{N-M} & U_{N-M-1} & U_{N-M-2} & \cdots & U_1 & Q_1^{'} & \Lambda_1 & \cdots & \Lambda_{M-2} & \Lambda_{M-1}\\ U_{N-M+1} & U_{N-M} & U_{N-M-1} & \cdots & U_2 & U_1 & Q_1^{'} & \cdots & \Lambda_{M-3} & \Lambda_{M-2}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ U_{N-2} & U_{N-3} & U_{N-4} & \cdots & U_{N-M-2} & U_{N-M-3} & U_{N-M-2} & \cdots & Q_1^{'} & \Lambda_1\\ U_{N-1} & U_{N-2} & U_{N-3} & \cdots & U_{N-M-1} & U_{N-M-2} & U_{N-M-1} & \cdots & U_1 & Q_1^{'} \end{bmatrix} (31)$$

$$B'_1=\begin{bmatrix} T & \Lambda'_1 & \Lambda'_2 & \cdots & \Lambda'_M & 0 & 0 & \cdots & 0 & 0\\ U & Q_1^{'} & \Lambda_1 & \cdots & \Lambda_{M-1} & \Lambda_M & 0 & \cdots & 0 & 0\\ V_1 & U_1 & Q_1^{'} & \cdots & \Lambda_{M-2} & \Lambda_{M-1} & \Lambda_M & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ V_{N-M-2} & U_{N-M-2} & U_{N-M-3} & \cdots & Q_1^{'} & \Lambda_1 & \Lambda_2 & \cdots & \Lambda_{M-1} & \Lambda_M\\ V_{N-M-1} & U_{N-M-1} & U_{N-M-2} & \cdots & U_1 & Q_1^{'} & \Lambda_1 & \cdots & \Lambda_{M-2} & \Lambda_{M-1}\\ V_{N-M} & U_{N-M} & U_{N-M-1} & \cdots & U_2 & U_1 & Q_1^{'} & \cdots & \Lambda_{M-3} & \Lambda_{M-2}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ V_{N-3} & U_{N-3} & U_{N-4} & \cdots & U_{N-M-2} & U_{N-M-3} & U_{N-M-2} & \cdots & Q_1^{'} & \Lambda_1\\ V_{N-2} & U_{N-2} & U_{N-3} & \cdots & U_{N-M-1} & U_{N-M-2} & U_{N-M-1} & \cdots & U_1 & Q_1^{'} \end{bmatrix} (32)$$

$$Q_B^{''}=$$
$$\begin{bmatrix} Q_1^{'}+U_N & \Lambda_1+U_{N-1} & \cdots & \Lambda_{M-1}+U_{N-M+1} & \Lambda_M+U_{N-M} & U_{N-M-1} & \cdots & U_2 & U_1\\ U_1 & Q_1^{'}+U_N & \cdots & \Lambda_{M-2}+U_{N-M+2}` & \Lambda_{M-1}+U_{N-M+1} & \Lambda_M+U_{N-M} & \cdots & U_3 & U_2\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ U_{N-M-2} & U_{N-M-3} & \cdots & Q_1^{'}+U_N & \Lambda_1+U_{N-1} & \Lambda_2+U_{N-2} & \cdots & \Lambda_M+U_{N-M} & U_{N-M-1}\\ U_{N-M-1} & U_{N-M-2} & \cdots & U_1 & Q_1^{'}+U_N & \Lambda_1+U_{N-1} & \cdots & \Lambda_{M-1}+U_{N-M+1} & \Lambda_M+U_{N-M}\\ \Lambda_M+U_{N-M} & U_{N-M-1} & \cdots & U_2 & U_1 & Q_1^{'}+U_N & \cdots & \Lambda_{M-2}+U_{N-M+2} & \Lambda_{M-1}+U_{N-M+1}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \Lambda_2+U_{N-2} & \Lambda_3+U_{N-3} & \cdots & U_{N-M-1} & U_{N-M-2} & U_{N-M-3} & \cdots & Q_1^{'}+U_N & \Lambda_1+U_{N-1}\\ \Lambda_1+U_{N-1} & \Lambda_2+U_{N-2} & \cdots & \Lambda_M+U_{N-M} & U_{N-M-1} & U_{N-M-2} & \cdots & U_1 & Q_1^{'}+U_N \end{bmatrix} (33)$$

The basic generator which is concerned with only the arrival and service is $Q_B^{''} = A'_0 + A'_1 + A'_2$. This is also circulant. Using similar arguments given for Model (A) it can be seen that its probability vector is $\left(\frac{\varphi\otimes\phi}{N}, \frac{\varphi\otimes\phi}{N}, \frac{\varphi\otimes\phi}{N}, \ldots\ldots, \frac{\varphi\otimes\phi}{N}\right)$ and the stability condition remains the same. Following the arguments given for Model (A), one can find the stationary probability vector for Model (B) also in matrix geometric form. All performance measures including expectation of customers waiting for service and its variance for Model (B) have the form as in Model (A) except M is replaced by N

4. NUMERICAL ILLUSTRATIONS

For numerical illustration it is considered that the arrival time PH distribution has representation

T= $\begin{bmatrix} -3 & 1 & 1\\ 1 & -4 & 1\\ 2 & 1 & -5 \end{bmatrix}$ and $\underline{\alpha}$ =(.3, .4, .3) and the service time PH distribution has representation S= $\begin{bmatrix} -3 & 1\\ 1 & -4 \end{bmatrix}$ and $\underline{\beta}$ =(.4, .6). Six examples are studied. The maximum arrival size and maximum service size are fixed as M=4 and N=2 in two examples. Two examples each treat the cases M=N=4 and M=2 and N=4. The order of the rate matrix R is 24 since it is the product $k_1k_2M\ ork_1k_2N$ depending on M >N or N > M. The probabilities of bulk arrival sizes and bulk service sizes are varied in the examples. The bulk arrival probabilities of sizes 1, 2, 3 and 4 in the examples 1 and 3 are (.5, .2, .2, .1), (.5, .3, .2, 0) and (.5, .4, .1, 0) in arrival phases1, 2 and 3 respectively. In examples 2 and 4 they are (.5, .2, .2, .1), (.8, .2, 0, 0) and (.5, .4, .1, 0) in arrival phases 1, 2, 3 respectively. In example 5 they are (.5, .5, 0, 0), (.6, .4, 0, 0) and (.4, .6, 0,

0) and in example 6 they are (.6, .4, 0, 0), (.7, .3, 0, 0) and (.5, .5, 0, 0) in phases 1, 2, and 3 respectively. The bulk service probabilities of sizes 1, 2, 3 and 4 in the examples 1 and 2 are (.2, .8, 0, 0) and (.2, .8, 0, 0); in examples 3 and 4 are (.5, .4, 0, .1), and (.2, .8, 0, 0); and in examples (5) and (6) are (.1, .8, 0, .1) and (.2, .8, 0, 0) respectively in PH service phases 1 and 2. The iteration for the rate matrix R is performed for the same 20 number of times in all the six examples. When the arrival rates decrease, the probability of empty queue increases, the norm value of the convergence of the rate matrix for the same 20 iterations decreases, expected values of queue length and its variances also decrease. The situation is seen same for all the six models. The arrival rates, service rates, probabilities of empty queue, probabilities of queue length is 1; 2; 3; in between 0 and 3; in between 4 and 7; in between 8 and 11, in between 12 and 15; and greater than 15 are given in table1. Twenty iterations are performed and the difference norms are presented in the table along with the expected queue lengths and the variances for the six examples. The decrease in the arrival rate decreases the expected queue length and variances. Figure1 presents the variations seen in E (S) and π_0e for various maximum bulk sizes graphically. In Figure 2 the several of queue lengths probabilities for the six cases are exhibited

Table1.Results obtained for six examples

	M=4,N=2	M=4,N=2	M=N=4	M=N=4	M=2,N=4	M=2,N=4
Arrival rate	0.676819923	0.595402299	0.676819923	0.595402299	0.585057471	0.545785441
Service. rate	1.125000000	1.125000000	1.100000000	1.100000000	1.200000000	1.200000000
P(Q=0)	0.343363192	0.403076054	0.325905977	0.386498364	0.438311803	0.461638676
P(Q=1)	0.122523701	0.150694353	0.118911487	0.148288409	0.138481187	0.158525026
P(Q=2)	0.107999639	0.118735406	0.106399987	0.118346458	0.146464243	0.140139153
P(Q=3)	0.092853695	0.088989566	0.092338243	0.090120570	0.081938119	0.080552227
π0e	0.666740226	0.761495379	0.643555694	0.743253800	0.805195352	0.840855082
π1e	0.204961558	0.172011845	0.210841275	0.180196702	0.151642922	0.130042415
π2e	0.078850403	0.047865755	0.086060837	0.053624185	0.033508869	0.023744555
π3e	0.030386571	0.013407105	0.035189014	0.016057352	0.007492223	0.004370976
P(Q>15)	0.019061241	0.005219917	0.024353180	0.006867962	0.002160634	0.000986973
Norm	0.000016888	0.000000529	0.000028460	0.000001239	0.000000000	0.000000000
E(S)	3.187053542	2.242943673	3.456788020	2.409022886	1.880878826	1.617146983
VAR(S)	16.88847935	9.300504294	19.278161260	10.409189010	6.732235862	5.216230045

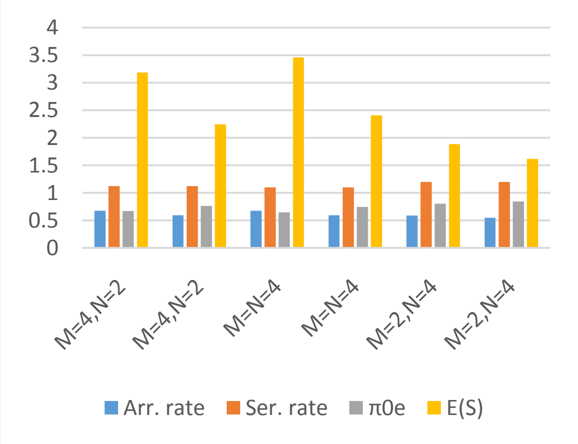

Figure1. The values E(S) and π_0e.

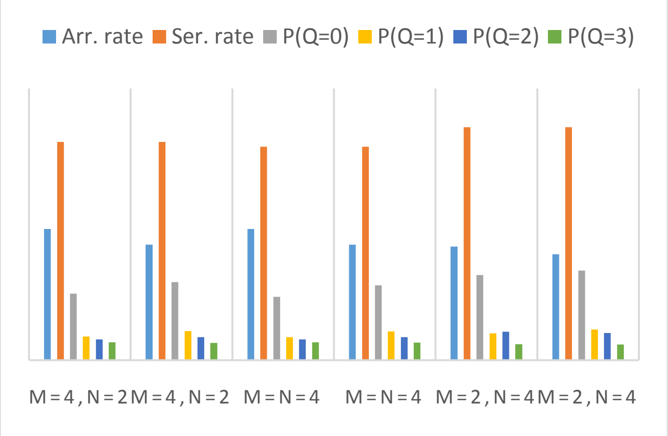

Figure2. The probabilities of queue lengths.

5. CONCLUSION

Two PH/PH/1 bulk arrivals and bulk service queues have been treated by identifying the maximum of the arrival and service sizes and grouping the customers as members of blocks of such maximum sizes. Matrix geometric results have been obtained by partitioning the infinitesimal generator by grouping of customers and PH phases together. The basic system generators of the queues are block circulant matrices which are explicitly presenting the stability condition in standard forms. Numerical results for varying bulk queue models are presented and discussed. Effects of variation of rates on expected queue length and on probabilities of queue lengths are exhibited. The decrease in arrival rates (so also increase in service rates) makes the convergence of R matrix faster which can be seen in the decrease of norm values. The variances also decrease. The PH/PH/1 queue with bulk arrival and bulk service has number of applications. The PH distributions include Exponential, Erlang, Hyper Exponential, and Coxian distributions as special cases and the PH distribution is also a best approximation for a general distribution. Further the PH/PH/1 queue is a most general form almost equivalent to G/G/1 queue. The bulk arrival models because they have non zero elements or blocks above the super diagonals in infinitesimal generators, they require for studies the decomposition methods with which queue length probabilities of the system are written in a recursive manner. Their applications are much limited compared to matrix geometric results. From the results obtained here, provided the maximum arrival and service sizes are not infinity, the most general model of the PH/PH/1 bulk arrivals and bulk services queue admits matrix geometric solution.

Further studies with block circulant basic generator system may produce interesting and useful results in inventory theory and finite storage models like dam theory. It is also noticed here that once the maximum arrival or service size increases, the order of the rate matrix increases proportionally. However the matrix geometric structure is retained and rates of convergence is not much affected. Randomly varying environments causing changes in the sizes of the PH phases may produce further results if studied with suitable partition techniques.

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