Generalized Wavelet Transform Associated with Legendre Polynomials

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ABSTRACT
The convolution structure for the Legendre transform developed by Gegenbauer is exploited to define Legendre translation by means of which a new wavelet and wavelet transform involving Legendre Polynomials is defined. A general reconstruction formula is derived.

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33A40; 42C10

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Legendre function, Legendre transforms, Legendre convolution, Wavelet transforms.

1. INTRODUCTION
Special functions play an important role in the construction of wavelets. Pathak and Dixit [5] have constructed Bessel wavelets using Bessel functions. But the above construction of wavelets is on semi-infinite interval (0, ∞). Wavelets on finite intervals involving solution of certain Sturm-Liouville system have been studied by U. Depczynski [2]. In this paper we describe a new construction of wavelet on the bounded interval (-1, 1) ⊂R using Legendre function. We follow the notation and terminology used in [7].

Let X denote the space $L^p(-1,1),\ 1 \le p < \infty,\ \text{or}\ C[-1,1]$ endowed with the norms

$$\| f \|_p = \left[\frac{1}{2} \int_{-1}^{1} | f(x) |^p \, dx \right]^{1/p} < \infty, \quad 1 \le p < \infty, \tag{1.1}$$

$$\| f \|_C = \sup_{-1 \le x \le 1} | f(x) | . \tag{1.2}$$

An inner product on X is given by

$$< f, g > = \frac{1}{2} \int_{-1}^{1} f(x) \, \overline{g(x)} \, dx . \tag{1.3}$$

As usual we denote the Legendre polynomial of degree n ∈N0 by Pn(x), i.e.

$$P_n(x) = \left(2^n \, n!\right)^{-1} \left(\frac{d}{dx} \right)^n (x^2 - 1)^n ; \quad x \in [-1,1].$$

For these polynomials one has

(i) $| P_n(x) | \le P_n(1) = 1 \quad ; \ x \in [-1,1]$ (1.4)

(ii) $(1-x^2)\, P_n''(x) - 2x\, P_n'(x) + n(n+1)\, P_n(x) = 0$; (1.5)

(iii) $P_n'(1) = \frac{n(n+1)}{2}$ (1.6)

The Legendre transform of a function f ∈ X is defined by

$$L[f](k) = \hat{f}(k) = \frac{1}{2} \int_{-1}^{1} f(x) P_k(x) dx; \tag{1.7}$$

The operator L associates to each $f \in X$ sequence of real (complex) numbers $\left\{ \hat{f}(k) \right\}_{k=0}^{\infty}$, called the Fourier Legendre coefficients.

The inverse Legendre transform is given by

$$L[f]^{v}(x) = f(x) = \sum_{k=0}^{\infty} (2k+1)\, \hat{f}(k)\, P_k(x) . \tag{1.8}$$

Lemma 1.1. Assume f, g ∈ X, k ∈N0 and c ∈R, then

(i) $\left| L[f](k) \right| \le \| f \|_{X}$;

(ii) $L[f+g](k) = L[f](k) + L[g](k)$, $L[cf](k) = cL[f](k)$;

(iii) $L[f](k) = 0$ for all $k \in \mathbf{N}_0$ iff f(x) = 0 a.e ;

(iv) $L[P_k](j) = \begin{cases} \frac{1}{2k+1}, & k = j \\ 0 & , \quad k \ne j, \ (k, j) \in N_o \end{cases}$

Let us recall the function K(x,y,z) which plays role in our investigation

$$K(x,y,z) = \begin{cases} 1 - x^2 - y^2 - z^2 + 2xyz, & z_1 < z < z_2 \\ 0 & \text{otherwise}, \end{cases} \tag{1.9}$$

where $z_1 = xy - [(1-x^2)(1-y^2)]^{1/2}$ and $z_2 = xy + [(1-x^2)(1-y^2)]^{1/2}$.

Then the function K(x,y,z) possesses the following properties;

(i) K(x,y,z) is symmetric in all the three variables

(ii) $\int_{-1}^{1} K(x,y,z)dz = \pi$.

Also it has been shown in [8] that

$$P_k(x)\,P_k(y) = \frac{1}{\pi}\int_{-1}^{1} P_k(z)K(x,y,z)dz \quad (1.10)$$

Applying (1.8) to (1.10), we have

$$K(x,y,z) = \frac{\pi}{2}\sum_{k=0}^{\infty}(2k+1)\,P_k(x)\,P_k(y)\,P_k(z). \quad (1.11)$$

The generalized Legendre translation τ_y for $y \in [-1,1]$ of a function $f \in X$ is defined by

$$(\tau_y f)(x) = f(x,y) = \frac{1}{\pi}\int_{-1}^{1} f(z)\;K(x,y,z)dz \quad (1.12)$$

Using Hölder's inequality it can be shown that

$$\|\tau_y f\|_X \le \|f\|_X \quad (1.13)$$

and the map $y \to \tau_y f$ is a positive linear operator from X into itself.

As in [7], for functions f,g defined on [-1,1] thegeneralized Legendre convolution is given by

$$(f*g)\;(x) = \frac{1}{2}\int_{-1}^{1}(\tau_y f)(x)\;g(y)dy$$

$$= \frac{1}{2\pi}\int_{-1}^{1}\int_{-1}^{1} f(z)\,g(y)\,K(x,y,z)\,dy\,dz \quad (1.14)$$

Lemma 1.2. If $f \in X, g \in L^1(-1,1),$ then the convolution (f*g) (x) exists (a.e.) and belongs to X. Moreover,

$$\|f*g\|_X \le \|f\|_X\|g\|_1, \quad (1.15)$$

$$(f*g)^\wedge(k) = \hat{f}(k)\hat{g}(k) \quad (1.16)$$

For any $f \in L^2(-1,1)$ the following Parseval identity holds for Legendre transform,

$$\sum_k (2k+1)\,|\hat{f}(k)|^2 = \|f\|_2^2. \quad (1.17)$$

In this paper, motivated from the work on classical wavelet transforms (cf. [1], [3]) we define the generalized wavelet transform and study its properties. A general reconstruction formula is derived. A reconstruction formula, under a suitable stability condition is obtained. Furthermore, discrete LWT is investigated.

Using Legendre Wavelet, frame and Riesz basis are also studied. A few examples of LWT are given. Similar constructions of wavelets and wavelet transforms on semi-infinite interval can be found in [4] and [5].

2. GENARALAISED WAVELET TRANSFORM

For a function $\psi \in X,$ define the dilation Da by $D_a\psi(t) = \psi(at),\; 0 < a \le 1$. (2.1)

Using the Legendre translation and the above dilation, the wavelet $\psi_{b,a}(t)$ is defined as follows:

$$\psi_{b,a}(t) = \tau_b D_a \psi(t) = \tau_b \psi(at) \quad (2.2)$$

$$= \frac{1}{\pi}\int_{-1}^{1} K(b,t,z)\,\psi(az)dz, \quad \psi \in X, \quad (2.3)$$

where $-1 \le b \le 1$ and $0 < a \le 1.$ The integral is convergent by virtue of (1.13).

Now, using the wavelet $\psi_{b,a}$ the Legendre wavelet transform (LWT) is defined as follows:

$$(L_\psi f)(b,a) = \langle f(t),\; \psi_{b,a}(t)\rangle \quad (2.4)$$

$$= \frac{1}{2}\int_{-1}^{1} f(t)\overline{\psi_{b,a}(t)}\;dt \quad (2.5)$$

$$= \frac{1}{2\pi}\int_{-1}^{1}\int_{-1}^{1} f(t)\overline{\psi(az)}\,K(b,t,z)dz\;dt \quad (2.6)$$

provided the integral is convergent.

Since by (1.13) and (2.2) $\psi_{b,a} \in X$ whenever $\psi \in X,$ by Lemma 1.2, the integral (2.6) is convergent for $f \in L^1(-1,1).$

The admissibility condition for the Legendre wavelet is given by

$$A_\psi = \sum_{k=0}^{\infty}\frac{|\hat{\psi}(k)|^2}{k} < \infty \quad . \quad (2.7)$$

From (2.7) it follows that $\hat{\psi}(0) = 0.$ But

$$\hat{\psi}(k) = \frac{1}{2}\int_{-1}^{1}\psi(t)\;P_k(t)dt$$

Yields

$$2\hat{\psi}(0) = \int_{-1}^{1}\psi(t)\,P_0(t)dt = \int_{-1}^{1}\psi(t)dt = 0 \, .$$

Hence, ψ(t) changes sign in (-1,1) therefore it represents a wavelet.

Theorem 2.1. If $\psi \in X$ defines a Legendre wavelet and $\phi \in L^1(-1,1)$, then the convolution ($\psi * \phi$) defines a Legendre wavelet.

Proof. Let $\psi \in X$ and $\phi \in L^1(-1,1)$, so that $\hat{\phi}$ is a bounded function on (-1,1). By Lemma 1.2, ($\psi * \phi$) $\in X$. We have

$$A_{\psi*\phi} = \sum_k \frac{|(\psi * \phi)^\wedge(k)|^2}{k}$$

$$= \sum_k \frac{|\hat{\psi}(k)|^2 |\hat{\phi}(k)|^2}{k}$$

$$\leq \|\phi\|_1 \sum_k \frac{|\hat{\psi}(k)|^2}{k} < \infty .$$

Therefore, $(\psi * \phi)$ represents a Legendre wavelet.

Theorem2.2. Let $f \in L^1(-1,1)$ and $\psi \in X$ and $(L_\psi f)$ (b,a) be the continuous Legendre wavelet transform. Then, we have the following inequality

$$\|(L_\psi f)(b,a)\|_X \leq \|f\|_1 \|\psi\|_X .$$

Proof. The above inequality follows from (1.15).

3. A GENERAL RECONSTRUCTION FORMULA

In this section we derive a general reconstruction formula and show that the function f can be recovered from its Legendre wavelet transform. Using representation (2.6), we have

$$(L_\psi f)(b,a) = \frac{1}{2\pi} \int_{-1}^{1}\int_{-1}^{1} f(t)\, \overline{\psi}(az)\, K(b,t,z)\, dz\, dt$$

$$= \frac{1}{2\pi} \int_{-1}^{1}\int_{-1}^{1} f(t)\overline{\psi}(az)\, dz\, dt \left(\frac{\pi}{2} \sum_k (2k+1) P_k(b) P_k(t) P_k(z) \right)$$

$$= \sum_k (2k+1) P_k(b) \left(\frac{1}{2}\int_{-1}^{1} f(t) P_k(t) dt \right) \left(\frac{1}{2}\int_{-1}^{1} \overline{\psi}(az) P_k(z) dz \right)$$

$$= \sum_k (2k+1)\, P_k(b)\, \hat{f}(k) \overline{\hat{\psi}(a,k)}$$

where

$$\overline{\hat{\psi}(a,k)} = \frac{1}{2}\int_{-1}^{1} \overline{\psi(az)}\, P_k(z) dz. \qquad (3.1)$$

Therefore,

$$(L_\psi f)^\wedge(k,a) = \hat{f}(k)\, \overline{\hat{\psi}(a,k)};$$

so that

$$\frac{1}{2}\int_{-1}^{1} (L_\psi f)(b,a)\, P_k(b) db = \hat{f}(k)\, \overline{\hat{\psi}(a,k)}.$$

Multiplying both sides by $\hat{\psi}(a,k)$ and a weight function q(a) and integrating both sides with respect to a from 0 to 1, we have

$$\frac{1}{2}\int_0^1 q(a)\, \hat{\psi}(a,k) \left(\int_{-1}^{1} (L_\psi f)(b,a)\, P_k(b) db \right) da = \left[\int_0^1 q(a) |\hat{\psi}(a,k)|^2 \, da \right] \hat{f}(k). \qquad (3.2)$$

Assume that

$$Q(k) = \int_0^1 q(a) |\hat{\psi}(a,k|^2 \, da > 0. \qquad (3.3)$$

Using (3.3), (3.2) can be written as

$$\hat{f}(k) = \frac{1}{2Q(k)} \int_0^1 q(a)\hat{\psi}(a,k) \left(\int_{-1}^{1} (L_\psi f)(b,a) P_k(b) db \right) da$$

$$= \frac{1}{2Q(k)} \int_0^1 q(a) \left(\int_{-1}^{1} (L_\psi f)(b,a) \hat{\psi}(a,k) P_k(b) db \right) da \qquad (3.4)$$

We have from (2.3)

$$\psi_{b,a}(t) = \frac{1}{\pi}\int_{-1}^{1} K(b,t,z)\psi(az) dz$$

$$= \frac{1}{2}\int_{-1}^{1} \psi(az) \sum_k (2k+1)\, P_k(b)\, P_k(t)\, P_k(z) dz$$

$$= \frac{1}{2}\sum_k (2k+1)\, P_k(b)\, P_k(t) \int_{-1}^{1} \psi(az)\, P_k(z) dz$$

$$= \sum_k (2k+1)\, P_k(b)\, P_k(t)\, \hat{\psi}(a,k)$$

$$= \left(\hat{\psi}(a,k) P_k(b) \right)^{v} (t) . \qquad (3.5)$$

Using (3.5) in (3.4) we have

$$\hat{f}(k) = \frac{1}{2Q(k)} \int_0^1 q(a) \int_{-1}^{1} (L_\psi f)(b,a)\, \hat{\psi}_{b,a}(k) da db$$

Set $\hat{\psi}^{b,a}(k) = \hat{\psi}_{b,a}(k)/Q(k)$. (3.6)

Then $\hat{f}(k)=\frac{1}{2}\int_0^1 q(a)\int_{-1}^{1}(L_\psi f)(b,a)\,\hat{\psi}^{b,a}(k)\,da\,db$. (3.7)

Putting (3.7) in (1.8), we have

$$f(t)=\frac{1}{2}\sum_k (2k+1)P_k(t)\int_0^1\int_{-1}^1 q(a)(L_\psi f)(b,a)\,\hat{\psi}^{b,a}(k)\,da\,db$$

$$=\frac{1}{2}\int_0^1\int_{-1}^1 q(a)\,(L_\psi f)(b,a)\sum_k (2k+1)\,\hat{\psi}^{b,a}(k)P_k(t)\,da\,db$$

Therefore

$$f(t)=\frac{1}{2}\int_0^1\int_{-1}^1 q(a)\,(L_\psi f)(b,a)\psi^{b,a}(t)\,da\,db$$

(3.8)

4. THE DISCRETE TRANSFORM

The continuous Legendre wavelet transform of the function f in terms of two continuous parameters a and b can be converted into a semi-discrete Legendre wavelet transform by assuming that a = 2-m; m ∈ Z and $-1\le b\le 1.$

Now, we assume that $\psi\in L^2(-1,1)$ satisfies the so called "stability condition"

$$A\le\sum_{m=-\infty}^{\infty}|\hat{\psi}(2^{-m}k)|^2\le B \qquad (4.1)$$

for certain positive constants A and B, 0 < A $\le B<\infty.$

The function $\psi\in L^2(-1,1)$ satisfying (4.1) is called dyadic wavelet.

Using the definition (2.4), we define the semi-discrete Legendre wavelet transform of any $f\in L^2(-1,1)$ by

$$(L_m^\psi f)(b)=(L_\psi f)(b,\frac{1}{2^m})=\langle f(t),\psi_{b,2^{-m}}(t)\rangle \qquad (4.2)$$

$$=\frac{1}{2}\int_{-1}^{1} f(t)\tau_b\overline{\psi(2^{-m}t)}dt \qquad (4.3)$$

$$=(f*\overline{\psi}_m), \qquad (4.4)$$

where $\psi_m(z)=\psi(2^{-m}z), \qquad m\in Z.$

Now, using Parseval identity (1.17), (4.1) yields the following

$$A\|f\|_2^2\le\sum_{m=-\infty}^{\infty}\|L_m^\psi f\|_2^2\le B\|f\|_2^2,\quad f\in L^2(-1,1)$$

(4.5)

Theorem 4.1. Assume that the semi-discrete LWT of any $f\in L^2(-1,1)$ is defined by (5.2). Let us consider another wavelet □* defined by means of its Legendre transform.

$$\hat{\psi}^*(k)=\frac{\hat{\psi}(k)}{2\sum_{j=-\infty}^{\infty}|\hat{\psi}(2^{-j}k)|^2} \qquad (4.6)$$

Then

$$f(t)=\sum_{m=-\infty}^{\infty}\int_{-1}^{1}\left(L_m^\psi f\right)(b)\left(\hat{\psi}^*(2^{-m}k)P_k(t)\right)^{\vee}(b)db \qquad (4.7)$$

Proof: In view of (4.4), for any $f\in L^2(-1,1),$ we have

$$\sum_m\int_{-1}^{1}\left(L_m^\psi f\right)(b)\left(\hat{\psi}^*(2^{-m}k)P_k(t)\right)^{\vee}(b)db$$

$$=\sum_m\int_{-1}^{1}\left(L_m^\psi f\right)(b)\sum_k(2k+1)\hat{\psi}^*(2^{-m}k)P_k(t)P_k(b)db$$

$$=\sum_m\sum_k(2k+1)\hat{\psi}^*(2^{-m}k)\,P_k(t)\int_{-1}^{1}\left(L_m^\psi f\right)(b)P_k(b)db$$

$$=2\sum_m\sum_k(2k+1)\hat{\psi}^*(2^{-m}k)P_k(t)\left(L_m^\psi f\right)^{\wedge}(k)$$

$$=2\sum_m\sum_k(2k+1)P_k(t)\hat{f}(k)\overline{\hat{\psi}(2^{-m}k)}\,\hat{\psi}^*(2^{-m}k)$$

$$=2\sum_m\sum_k(2k+1)P_k(t)\hat{f}(k)\frac{\overline{\hat{\psi}(2^{-m}k)}\hat{\psi}(2^{-m}k)}{2\sum_j\left|\hat{\psi}(2^{-m}2^{-j}k\right|^2}$$

$$=\sum_k(2k+1)\hat{f}(k)P_k(t)$$

=f(t) .

The above theorem leads to the following definition of dyadic dual.

Definition 4.2. A function $\tilde{\psi}\in L^2(-1,1)$ is called a dyadic dual of a dyadic wavelet □, if every $f\in L^2(-1,1)$ can be expressed as

$$f(t)=\sum_m\int_{-1}^{1}\left(L_m^\psi f\right)(b)\left(\hat{\tilde{\psi}}(2^{-m}k)P_k(t)\right)^{\vee}(b)db. \qquad (4.8)$$

So far we have considered semi-discrete Legendre wavelet transform of any $f\in L^2(-1,1)$ discretizing only variable

a. Now, we discretize the translation parameter b also by restricting it to the discrete set of points

$$b_{m,n} = \frac{n}{2^m} b_0, \; m \in Z, \; n \in N0, \quad (4.9)$$

where $b_0 \in [-1,1]$ is a fixed constant. We write $\psi_{b_0;m,n}(t) = \psi_{b_{m,n};a_m}(t) = \psi(2^{-m}t, 2^{-m}n\; b_0)$ (4.10)

Then the discrete Legendre wavelet transform of any $f \in L^2(-1,1)$ can be expressed as

$$(L_\psi f)\;(b_{m,n;}a_m) = < f, \psi_{b_0;m,n} > \quad m \in Z, \quad n \in N0. \quad (4.11)$$

The "stability" condition for this reconstruction takes the form

$$A\| f \|_2^2 \le \sum_{\substack{m\in Z \\ n\in N_0}} \left| < f, \psi_{b_0;m,n} > \right|^2 \le B \| f \|_2^2, \; f \in L^2(-1,1) \quad , \quad (4.12)$$

where A and B are positive constants such that $0 < A \le B < \infty.$

Theorem 4.3. Assume that the discrete LWT of any $f \in L^2(-1,1)$ is defined by (4.11) and stability condition (4.12) holds. Let T be a linear operator on L2(-1,1) defined by

$$Tf = \sum_{\substack{m\in Z \\ n\in N_0}} < f, \psi_{b_0;m,n} > \psi_{b_0;m,n} \quad . \quad (4.13)$$

Then

$$f = \sum < f, \psi_{b_0;m,n} > \psi_{b_0}^{m,n} \quad (4.14)$$

where

$$\psi_{b_0}^{m,n} = T^{-1}\psi_{b_0;m,n}; \quad m \in Z, \; n \in N0$$

Proof. From the stability condition (4.12), it follows that the operator defined by (4.13) is a one-one bounded linear operator.

Set

$$g = Tf, \quad f \in L^2(-1,1)$$

Then, we have

$$< Tf, f > = \sum_{\substack{m\in Z \\ n\in N_0}} \left| < f, \psi_{b_0;m,n} > \right|^2 .$$

Therefore

$$A\| T^{-1}g \|_2^2 = A \| f \|_2^2 \le < Tf, f >$$
$$= < g, T^{-1}g > \le \| g \|_2 \| T^{-1}g \|_2$$

so that $\| T^{-1}g \|_2 \le \frac{1}{A} \| g \|_2$.

Hence, every $f \in L^2(-1,1)$ can be reconstructed from its discrete LWT given by (4.11). Thus

$$f = T^{-1}Tf = \sum_{\substack{m\in Z \\ n\in N_0}} < f, \psi_{b_0;m,n} > T^{-1}\psi_{b_0;m,n} \quad (4.15)$$

Finally, set

$$\psi_{b_0}^{m,n} = T^{-1}\psi_{b_0;m,n}; \quad m \in Z, \; n \in N0$$

Then, the reconstruction (4.15) can be expressed as $f = \sum_{\substack{m\in Z \\ n\in N_0}} < f, \psi_{b_0;m,n} > \psi_{b_0}^{m,n}$

which completes the proof of theorem 4.3.

5. FRAMES AND RIESZ BASIS IN L2(-1, 1)

In this section, using $\psi_{b_0;m,n}$ a frame is defined and Riesz basis of L2(-1,1) is studied.

Definition 5.1. A function $\psi \in L^2(-1,1)$ is said to generate a frame $\{\psi_{b_0;m,n}\}$ of $L^2(-1,1)$ with sampling rate b0 if (5.12) holds for some positive constants A and B. If A = B, then the frame is called a tight frame.

Definition 5.2. A function $\psi \in L^2(-1,1)$ is said to generate a Riesz basis of $\{\psi_{b_0;m,n}\}$ with sampling rate b0 if the following two properties are satisfied.

The linear span $< \psi_{b_0;m,n} : m \in Z >$ is dense in L2(-1,1) (5.1)

There exist positive constants A and B with $0<A \le B < \infty$ such that

$$A\| \{c_{m,n}\} \|_2^2 \le \left\| \sum_{\substack{m\in Z \\ n\in N_0}} c_{m,n}\psi_{bo;m,n} \right\|_2^2 \le B \left\| \{c_{m,n}\} \right\|_{\ell^2}^2$$

(5.2)

for all $\{c_{m,n}\} \in \ell^2 (N_0^2)$. Here A and B are called the Riesz bounds of $\{\psi_{b_0;m,n}\}$.

Theorem 5.3. Let $\psi \in L^2(-1,1),$ then the following statements are equivalent.

$\{\psi_{b_0;m,n}\}$ is a Riesz basis of L2(-1,1);

$\{\psi_{b_0;m,n}\}$ is a frame of L2(-1,1) and is also an ℓ^2 -linearly independent family in the sense that if $\sum \psi_{b_0;m,n} c_{m,n} = 0$ and $\{c_{m,n}\} \in \ell^2$, then cm,n = 0.

Furthermore, the Riesz bounds and frame bounds agree.

Proof. It follows from (5.2) that any Riesz basis is ℓ^2-linearly independent.

Let $\{\psi_{b_0;m,n}\}$ be a Riesz basis with Riesz bounds A and B, and consider the matrix operator

$$M = \lfloor \gamma_{r,s,m,n} \rfloor_{(r,s),(m,n)\in N_0 \times N_0}$$

where the entries are defined by

$$\gamma_{r,s,m,n} = \left\langle \psi_{b_0;r,s}, \psi_{b_0;m,n} \right\rangle . \quad (5.3)$$

Then from (5.2), we have

$$A \| \{c_{m,n}\} \|_{\ell^2}^2 \le \sum_{r,s,m,n} c_{r,s} \gamma_{r,s,m,n} c_{m,n} \le B \| \{c_{m,n}\} \|_2^2$$

so that M is positive definite. We denote the inverse of M by

$$M^{-1} = [\mu_{r,s,m,n}]_{(r,s),(m,n)\in N_0^2} \quad (5.4)$$

which means that both

$$\sum_{t,u} \mu_{r,s;t,u} \gamma_{t,u;m,n} = \delta_{r,m} \delta_{s,m;} \ r,s,m,n \in N0 \quad (5.5)$$

and

$$B^{-1} \left\| \{c_{m,n}\} \right\|_{\ell^2}^2 \le \sum_{r,s,m,n} c_{r,s} \ \mu_{r,s,m,n} \ c_{m,n} \le A^{-1} \left\| \{c_{m,n}\} \right\|_{\ell^2}^2 \quad (5.6)$$

are satisfied. This allows us to introduce

$$\psi^{r,s}(x) = \sum_{m,n} \mu_{r,s;m,n} \psi_{b_0;m,n}(x) . \quad (5.7)$$

Clearly, $\psi^{r,s} \in L^2(-1,1)$ and it follows from (5.3) and (5.5) that

$$\left\langle \psi^{r,s} ; \psi_{b_0;m,n} \right\rangle = \delta_{r,m} \delta_{s,n} ; \ r,s,m,n \in N$$

which means that {□r,s} is the basis of L2(-1,1), which is dual to $\{\psi_{b_0;m,n}\}$.

Furthermore, from (6.5) and (6.6); we conclude that $\left\langle \psi^{r,s}, \psi^{m,n} \right\rangle = \mu_{r,s,m,n}$

and the Riesz bounds of {□r,s} are B-1 and A-1.

In particular, for any $f \in L^2(-1,1)$ we may write

$$f(x) = \sum_{m,n} < f, \psi_{bo;m,n} > \psi^{m,n}(x)$$

and

$$B^{-1} \sum_{m,n} \left| < f, \psi_{bo;m,n} > \right|^2 \le \| f \|_2^2 \le A^{-1} \sum_{m,n} \left| < f, \psi_{bo;m,n} > \right|^2 \quad (5.8)$$

Since, (5.8) is equivalent to (4.12), therefore, statement (i) implies statement (ii). To prove the converse part, we recall **Theorem 4.3** and we have for any $g \in L^2(-1,1)$ and f = T-1g,

$$g(x) = \sum_{\substack{m\in Z \\ n\in N_0}} < f, \psi_{bo;m,n} > \psi_{bo;m,n} .$$

Also, by the ℓ^2 – linear independence of $\{\psi_{b_0;m,n}\}$, this representation is unique. From the Banach-Steinhaus and open mapping theorem it follows that $\{\psi_{b_0;m,n}\}$ is a Riesz basis of L2(-1,1).

6. REFERENCES

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