Construction of Q–Fuzzy Left H–Ideals Interms of HEMI Rings

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ABSTRACT
In this paper, the Biswas's idea of Q–fuzzy subgroups to left h-ideals of hemi rings is applied. We introduce the notion of Q–fuzzy subgroups to left h-ideals in hemi rings and investigate some of related properties. Relationship between Q–fuzzy left h–ideals, and Q–fuzzy left h–ideals of hemi ring are also given.

Index Terms
Q-fuzzy subgroup, Hemi rings, Left h-ideals, characteristic, normal Q-fuzzy h-ideals

1. INTRODUCTION
Fuzzy group is introduced by Zadeh [1965], and fuzzy group is introduced by Rosenfeld [1971]. Ideals of hemi rings play a central role in the structure theory and are very useful for many purposes. However, they do not in general coincide with the usual ring ideals. Many results in rings apparently have no analogues in hemi rings using only ideals. Henriksen defined in [1958] a more restricted class of ideals in semi rings, which is called the class of k- ideals, with the property that if the semi ring R is a ring then a complex in R is a k- ideal if and only if it is a ring ideal. Another more restricted, but very important class of ideals, called h- ideals, has been given and investigated by Izuka [1959] and La Torre [1965]. Other important results [Biswas, 1990] connected with fuzzy ideals in hemi rings were obtained in [Jun, 2004].The concept of Q- fuzzy subgroups can be obtained in the papers of Solairaju and Nagarajan [2008, 2009a, 2009b, 2009c, 2010]. In this paper, we introduce the notion of Q-fuzzy left h-ideals in terms of hemi rings and investigate their properties.

2. PRELIMINARIES
In this section, we review some elementary aspects that are necessary for this paper.

Definition 2.1: An algebra (R, +, ·) is said to be a semi ring if it satisfies the following conditions: (R, +) is a semi group; (R, ·) is a semi group; and a.(b + c) = ab + ac and (b + c)a = ba + ca for all a, b, c ∈ R.

Definition 2.2: A semi-ring (R, +, .) is called a *hemi-ring* if (H_1) addition + is commutative and (H_2) there exists an element 0∈R such that 0 is the identity of (R,+) and the zero element of (R, ·) [0.a = a.0 = 0 for all a ∈ R]. A subset I of a semi ring R is called a left ideal of R if I is closed under addition and RI ⊆ I. A left ideal of R is called a *left K-ideal* of R if y, z ∈ I and x ∈ R , x + y = z implies x ∈ I. A *left h – ideal* of a hemi ring R is defined to be a left ideal A of R such that (x + a + z = b + z → x ∈ A, for all (x, z ∈ R), (for all a, b ∈ A)). *Right h- ideals* are defined similarity.

Definition 2.3: A mapping f : $R_1 \to R_2$ is said to be hemi ring homomorphism of R_1 is to R_2 if f(x + y) = f(x) + f(y) and f(xy) = f(x).f(y) for all x,y ∈ R.

Definition 2.4: A mapping μ: X → [0,1] where X is an arbitrary non – empty set is a fuzzy set is X. For any fuzzy set μ is X and any X ∈ [0, 1] the set L(μ: α) = { x ∈ X : μ(x) ⊆ α} is defined (called lower level cut of μ).

Definition 2.5: Let Q and G be a set and a group respectively. A mapping μ: G × Q → [0, 1] is called a Q – fuzzy set.

Definition 2.6: A fuzzy subset is of a semi ring R is said to be Q - fuzzy left h – ideal of R if (i). μ (x + y, q) ≥ min{ μ (x, q), μ(y, q)} for all x, y ∈ R, and q ∈ Q; (ii) μ (xy, q) ≥ μ(y, q) for all x, y ∈ R; q ∈ Q. Note that if μ is a Q - fuzzy left h–ideal if a hemi ring R, then μ(0, q) ≥ μ(x, q) for all x ∈ R, and q ∈ Q.

Definition 2.7: A fuzzy subset μ of a hemi ring R is said to be a Q-fuzzy left h – ideal of R if (1). μ(x + y ,q) ≥ min{ μ(x, q), μ(y, q)} for all x, y ∈ R, and q ∈ Q; (2). μ(xy, q) ≥ μ(y,q) for all x, y ∈ R, q ∈ Q; (3). x + a + z = b + z ⟶ μ(x, q) ≥ min{ μ (a, q), μ (b, q)}.

Example: Let R = {0, 1, 2, 3, 4 } be a hemi ring with zero multiplication and addition defined by the following table

	0	1	2	3	4
0	0	1	2	3	4
1	1	1	4	4	4
2	2	4	4	4	4
3	3	4	4	4	4
4	4	4	4	4	4

A fuzzy set μ: R → [0,1] is defined by letting μ (0) = t_1 and μ (x) = t_2 for all x ≠ 0, $t_1 \subset t_2$. By routine computations, it also easily checks that μ is a fuzzy left h–ideal of hemi ring R.

Properties of Q–fuzzy left h - ideals

Proposition 3.1 let R be a hemi ring and μ be a Q – fuzzy set in R. Then μ is Q- fuzzy left h–ideal in R if and only if μ^c is a Q-fuzzy left h-ideal in R.

Proof: Let μ be an Q –fuzzy left h – ideal in R. For x, y ∈ **R**, it becomes that

$\mu^c(x + y,q) = 1- \mu(x + y,q)$

$\leq 1 - \min \{ \mu(x ,q), \mu(y,q)\}$

$= \max \{ 1 - \mu(x ,q), 1- \mu(y,q)\}$

$= \max \{ \mu^c(x ,q), \mu^c(y,q)\}$

$\mu^c(xy,q) = 1- \mu(xy,q) \leq 1- \mu(y,q) = \mu^c(y,q)$

Let x,z,a,b ∈ R be such that x+a+z = b + z

Then $\mu^c(x ,q) = 1- \mu(x ,q)$

$\leq 1 - \min \{ \mu(a ,q), \mu(b,q)\}$

$= \max \{ 1 - \mu(a ,q), 1- \mu(b,q)\}$

$= \max \{ \mu^c(a ,q), \mu^c(b,q)\}$.

Hence μ^c is a Q- fuzzy left h – ideal of R .

Conversely, μ^c is a Q - fuzzy left h – ideal of R. For x, y ∈ R, and q ∈ Q, it follows that $\mu(x + y, q) = 1- \mu^c(x + y,q)$ $\geq 1 - \max \{ \mu^c(x ,q), \mu^c(y,q)\}$ $= \min \{\mu(x ,q), \mu(y,q)\}$ $\mu(xy,q) = 1- \mu^c(xy, q) \geq 1- \mu^c(y, q) = \mu(y, q)$. Let x,z,a,b ∈ R be such that x+a+z = b + z, Then $\mu(x ,q) = 1- \mu^c(x ,q)$ $\leq 1 - \max \{ \mu^c(a ,q), \mu^c(b,q)\}$ = min $\{\mu(a,q),\mu(b,q)\}$=min$\{\mu(a ,q), (b,q)\}$

Hence μ is a Q- fuzzy left h – ideal of R .

Proposition 3.2 let 'μ' be Q – fuzzy left h – ideal in a hemi ring R such that L(μ:α) is a left h–ideal of R for each $\alpha \in I_m(\mu)$,α∈[0,1].Then μ is Q-fuzzy left h-ideal in R.

Proof: Let x, y ∈ R, q ∈ Q be such that $\mu(x ,q) = \alpha_1$, $\mu(y ,q) = \alpha_2$.Then x+y∈L(μ: α). Without loss of generality, we may assume that $\alpha_1 > \alpha_2$. It follows that L(μ;α_2) ⊆ L(μ;α_1) o that x ∈ L(μ;α_1) and y ∈ L(μ;α_2). Since L(μ;α_1) is a left is a left h–ideal of R. It implies that x + y ∈ L(μ;α_1). Thus $\mu(x + y,q) \geq \alpha_1 = \min \{ \mu(x ,q), \mu (y,q)\}$. $\mu (xy,q) \geq \alpha_1 = \mu(y,q)$. Let x, z, a, b ∈ R be such that x+ a + z = b + z. Then $\mu(x ,q) \geq \alpha_1 = \max \{ \mu(a ,q), \mu(b,q)\}$. This shows that μ is an Q-fuzzy left h-ideal in R.

Corollary 3.3: Let μ be Q- fuzzy left h-ideal in R then μ is an Q- fuzzy left h-ideal in R if and only if L(μ;α) is a left h-ideal in R for every α∈[μ(0,q),1] with α∈[0,1].

Proposition 3.4 : let 'μ' be Q – fuzzy set in a hemi ring R then two lower level subsets L(μ;t_1) and L(μ;t_2), ($t_1 < t_2$) are equal if and only if there is no x ∈ R such that $t_1 \leq \mu(x,q) \leq t_2$.

Proof: From the definition of L(μ;α), it follows that L(μ ; t) = μ^{-1}([μ(0, q); t]) for t ∈ [0. 1]. Let t_1 , t_2 ∈ [0, 1] be such that $t_1 < t_2$ then L(μ ; t_1) = L(μ ; t_2)

⇔ $\mu^{-1}([\mu(0,q);t_1]) = \mu^{-1}([\mu(0,q);t_2])$

⇔ $\mu^{-1}(t_1, t_2) = \varphi$

⇔There is no x ∈R with $t_1 \leq \mu(x,q) \leq t_2$.

Definition 3.5: A left h – ideal A of hemi ring R is said to be characteristic if and only if f(A) = A , for all f ∈ Aut (R), where Aut (R) is the set of all automorphisms of R. Q–fuzzy left h – ideal μ of hemi ring R is said to be Q – fuzzy characteristic if μ^f (x ,q) = μ(x ,q) for all and f ∈ Aut (R).

Lemma 3.6: Let μ be an Q-fuzzy left h–ideal of a hemi ring R and x ∈ R. Then μ(x ,q) = s if and only if x ∈ L(μ ; s) and x ∉ L(μ ; t) for all s > t.

Proof: Straight forward.

Proposition 3.7: let 'μ' be an Q–fuzzy left h – ideal of a hemi ring R then each level left h–ideal of μ is characteristic if and only if μ is a Q-fuzzy Characteristic.

Proof: Suppose that μ is an Q-fuzzy characteristic and S ∈ $I_m(\mu)$, f ∈ Aut (R) and x ∈ L(μ ; s). Then μ^f (x ,q) = μ(x ,q) ≥ s ⇒ μ(f(x, q)) ≥ s ⇒ f(x, q) L(μ ; s). Thus f (L(μ ; s)) and y ∈R such that f(y ,q) = (x ,q) ⇒ μ(y ,q) = μ^f (y ,q) = μ(f(y ,q)) = μ(x, q) ≥ s ⇒ y ∈ L(μ : s) so that (x ,q) = f(y ,q) ∈ L(μ ; s). Consequently, L(μ; s) ⊆ f (L(μ ; s)). Hence f (L(μ; s)) = L(μ; s) ⇒ L(μ; s) is characteristic.

Conversely, suppose that each level h–ideal of μ is characteristic and let x ∈ R, f ∈ Aut (R) and μ(x ,q) = s. Then x ∈ L(μ; s) by (3.3) and x ∉ L(μ ; t) for all s > t. It follows from the assumption that f(x ,q) ∈ f (L(μ ; s))= L(μ) so that μ^f (x ,q) = μ(f(x ,q)) ≥ s .

Let t = μ^f (x ,q) and assume that s > t. then f(x ,q) ∈ L(μ ; t)= f(L(μ ; t)) which implies from the injectivity of f that x ∈ L(μ : t), a contradiction. Hence μ^f (x, q) = μ(f(x ,q)) = s = μ(x ,q) showing that μ is an Q – anti fuzzy characteristic.

Proposition 3.8: Let f: $R_1 \to R_2$ be an epimorphism of hemi-rings. If V is an Q-fuzzy left h-ideal of R_2, and μ is the pre-image of V under f, then μ is an Q-fuzzy left h-ideal of R_1.

Proof: For any x, y ∈ R_1 and q ∈ Q, it implies that

$\mu(x + y,q) = V(f(x + y,q))$

$= V(f(x ,q) + f(y,q))$

$\geq \min \{V(f(x ,q), V(f(y,q))\}$

$= \min \{ \mu(x ,q), \mu(y,q)\}$

Also $\mu (xy,q) = V(f(xy,q)) = V(f(x ,q). f(y,q)) \geq V(f(x ,q)) = \mu(y,q)$

Let x,z,a,b ∈R be such that x+a+z = b + z.

Then it becomes that

$\mu(x, q) = V (f(x, q)) \geq \min \{V(f(a ,q)), V(f(b,q))\} = \min \{ \mu(a ,q), \mu(b,q)\}$

Hence μ is an Q – fuzzy left h- ideal of $R_{1.}$

Definition 3.9: Let R_1 and R_2 be two hemi rings and f be a function of R_1 into R_2. If μ is a Q–fuzzy in R_2 then the Pre– image of μ under f then μ is the Q – fuzzy in R_1 defined by $f^{-1}(\mu) (x, q) = \mu (f (x, q))$, for all x ∈ R_1 , and q ∈ Q.

Proposition 3.10: Let f: $R_1 \rightarrow R_2$ be an onto homomorphism of hemi rings. If μ is a Q-fuzzy left h-ideal of R2, then $f^{-1} (\mu)$ is a Q-anti fuzzy left h-ideal of R_1.

Proof: Let $x_1, x_2 \in R_1$, then it implies that

$f^{-1}(\mu)(x_1+x_2,q) = \mu(f(x_1,q) + f(x_2,q)) \geq \min \{\mu (f(x_1 ,q)) + \mu (f(x_2 ,q))\}$

$= \min\{f^{-1}(\mu)(x_1,q) ,f^{-1}(\mu)(x_2,q)\}$

$f^{-1}(\mu)(x_1x_2,q) = \mu(f(x_1,q) f(x_2,q)) \geq \mu(f(x_2,q)) = f^{-1}(\mu)(x_2,q)$

Let x,z,a,b ∈ R_1 be such that x+a +z = b+z. Then it gives that $f^{-1}(\mu) (x, q) = \mu(f(x, q)) \geq \min \{\mu(f(a ,q)) , \mu(f(b ,q))\} = \min \{f^{-1} (\mu)(a, q) ,f^{-1}(\mu)(b, q)\}$. Hence $f^{-1}(\mu)$ is an Q – fuzzy left h-ideal of $R_{1.}$

Definition 3.11: Let R_1 and R_2 be any sets and let f: $R_1 \rightarrow R_2$ be any function. A be Q – fuzzy subset μ of R_1 is called f- invariant if f(x) = f(y) implies μ (x, q) = μ (y, q)) for all x, y ∈ R , and q ∈ Q.

Proposition 3.12: Let f: $R_1 \rightarrow R_2$ be an epimorphism of hemi rings. Let μ be an f – invariant Q – fuzzy left h-ideal of $R_{1,}$ then f (μ) is an Q – fuzzy left h- ideal of $R_{2.}$

Proof: Let x, y ∈ R_2**. T**hen there exists a, b ∈ $R_{1,}$ such that f(a) = x and f(b) = y ⇒ x + y = f (a + b) and xy = f(ab). Since μ is invariant, it follows that

$f(\mu) (x + y), q) = \mu (x + y, q)$

$\geq \min \{ \mu(a, q), \mu(b,q)\}$

$= \min \{ f(\mu)(x, q), f(\mu)(y, q)\}$

$f(\mu)(xy, q) = \mu (ab, q)$

$\geq \mu(x, q)$

$\geq f(\mu)(y, q).$

Let x, z, a, b ∈ R_2 be such that x + a + z = b + z. There exists x, z, a, b such that f(x) = x, f(y) = y; f(a) = a; f(b) = b. Since μ is f-invariant, it gives that

$f(\mu)(x, q) = \mu(x, q)$

$\geq \min \{ \mu(a, q), \mu(b, q)$

$= \min \{f(\mu)(a, q), f(\mu)(b, q)\}$

Thus f(μ) is a Q-fuzzy left h-ideal of R_2.

Definition 3.13: A Q- fuzzy left h-ideal μ of a hemi ring R is said to be normal if there exist x ∈ R such that μ (x, q) = 1. Note that if μ is a normal Q – anti fuzzy left h – ideal of R_1 then μ(0,q) = 1 and hence μ is normal if and only if μ(0,q) = 1

Proposition 3.14: Let μ be an Q – fuzzy left h – ideal of a hemi ring R. Let μ^+ be a Q–fuzzy set in R defined by $\mu^+ (x,q) = \mu (x,q) = 1- \mu(0,q)$ for all x ∈ R. Then μ^+ is a normal Q– fuzzy left h – ideal of R which contains μ.

Proof: For any x, y ∈ R, it finds that $\mu^+ (x,q) = \mu(0,q) + 1- \mu(0,q) = 1$, and $\mu^+ (x + y, q) = \mu (x + y, q) + 1- \mu(0,q) \geq \min \{ (\mu (x ,q), \mu (y, q)) + 1- \mu(0,q) \}$

$= \min \{ \mu^+ (x, q), \mu^+ (y, q)\}$

$\mu^+ (xy, q) = \mu (xy, q) + 1- \mu(0,q)$

$\geq \mu (y, q) + 1- \mu(0,q) = \mu^+ (y, q)$

This shows that μ^+ is a Q – fuzzy left h – ideal of R. Let a, b, x, z ∈ R be such that x+ a + z = b + z.

Then $\mu^+ (x, q) = \mu (x ,q) + 1- \mu(0,q) \geq \min \{ (\mu (a ,q)), \mu (b,q)) + 1- \mu(0,q) = \min (\mu (a ,q) + 1- \mu(0,q)), (\mu(b ,q) + 1- \mu(0,q)) \} = \min \{\mu^+ (a,q), \mu (b,q) \}$

So μ^+ is a normal Q–anti fuzzy left h–ideal of hemi ring of R. Clearly $\mu \leq \mu^+$.

Definition 3.15 : Let N(R) denote the set of all normal Q –fuzzy left h – ideals of R. Note that N(R) is a poset under the set inclusion. A Q – fuzzy set μ in a hemi ring

R is called a maximal Q –fuzzy left h – ideal of R if it is non – constant and μ^+ is a maximal element of (N(R),⊆).

Proposition 3.16: Let μ ∈N(R) be non – constant such that it is a maximal element of (N(R) , ⊆) then it takes only two values {0,1}.

Proof: Since μ is normal, μ(0,q)=1. We claim that μ (x, q) = 0. If not, then there exists x_0 ∈ R such that 0 □ μ(x_0,q) □ 1. Define on R a Q – fuzzy set V by putting V(x, q) = ½ { μ (x ,q) + μ(x_0,q)} for each x ∈R then clearly V is well defined and for all x, y ∈ R. it gives that

$V(x + y, q) = ½\ \mu(x + y ,q) + ½\ \mu(x_0,q)$

$\geq ½ \{\min\{\mu (x ,q), \mu (y ,q)\}+ \mu(x_0,q) \}$

$= \min \{V(x,q) ,V(y,q)\}$

$V(xy,q) = ½\ \mu (xy ,q) + ½\ \mu(x_0,q)$

$\geq ½ (\mu (y ,q)+ \mu(x_0,q)) = V(y,q)$

Thus V is a Q –fuzzy left h- ideal of R.

Let a, b, x, z,∈ R be such that x+ a + z = b + z. Then it follows that $V(x, q) = ½\ \mu (x ,q) + ½\ \mu(x_0,q) \geq ½ \{\min\{\mu (a ,q), \mu (b ,q)\}+ \mu(x_0,q) \}$

$= \min\{½(\mu (a ,q) + \mu(x_0,q), ½(\mu (b ,q) + \mu(x_0,q)\} = \min \{V(a, q) ,V(b, q)\}$

Hence V is a Q – anti fuzzy left h – ideal of R. By (3.15), V^+ is a maximal Q- anti fuzzy left h – ideal of R.

Note that $V^+ (x,q) = V (x ,q) + 1- V(0,q)$

$= ½(\mu (x ,q) + \mu(x_0,q) + 1- ½(1+ \mu(x_0,q)) = ½(\mu(x_0,q) +1) = V(x_0,q)$

and $V^+ (x_0,q) \geq 1 = V^+ (0,q)$.

Hence V^+ is a non – constant and μ is not a maximal non – constant and μ is not a maximal element of N(R). This is a contradiction.

3. CONCLUSION

Y. B. Jun [2004] introduced the concept on fuzzy h-ideals in hemi rings and investigated the idea of anti fuzzy left h- ideals in hemi rings. In this paper, some structure properties of Q- fuzzy left h- ideals is established in a hemi rings.

4. REFERENCES

[1] R. Biswas, Fuzzy subgroups and anti fuzzy subgroups, Fuzzy sets and systems, 44 (1990), 121-124

[2] .M. Henriksen, Ideals in Semi rings with commutative addition, Am. Math. Soc. Notices, Volume 6, (1958), 03-21

[3] K.Izuka, On the Jacobson radical of a semi ring, Tohoku, Math. J., Volume 11(2), (1959), 409-421.

[4] Y.B.Jun, M.A.Ozturk and S.Z.Song, On fuzzy h-ideals in hemi rings, info. Scien.162(2004), 211-226.

[5] R.LaTorre, On h-ideals and k-ideals in hemi rings, Publ. Math. Debrecen, Vol. 12 (1965), 219-226.

[6] A.Rosenfeld, Fuzzy groups, J. Math. Anal. Appl. 35 (1971), 512 - 517.

[7] A.Solairaju and R.Nagarajan, Q- fuzzy left R-subgroups of near rings with respect to T-norms, Antarctica Journal of mathematics, 5 (2), (2008), 59-63.

[8] A.Solairaju and R.Nagarajan, A New Structure and Constructions of Q- Fuzzy groups, Advances in Fuzzy mathematics, 4(1), (2009), 23-29.

[9] A.Solairaju and R.Nagarajan, Lattice valued Q-fuzzy sub modules of near rings with respect to T-norms, Advances in Fuzzy mathematics, 4(2), (2009),137-145.

[10] A.Solairaju and R.Nagarajan, Q- fuzzy subgroups of Beta fuzzy congruence relations on a group, International Journal of Computer Science, Network and Security(IJCSNS),2010.

[11] A.Solairaju and R.Nagarajan, characterization of interval valued anti fuzzy left h- ideals over hemi rings , Advances in Fuzzy Mathematics, Volume 4(2), (2009) , 129-136.

[12] L.A.Zadeh, Fuzzy sets, Information control, Volume 8, (1965), 338-353.