# A Study on Double, Triple and N - Tuple Domination of Fuzzy Graphs

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## ABSTRACT
In a graph G, a vertex dominates itself and its neighbors. A subset S of V is called a dominating set in G if every vertex in *V-S* is adjacent to at least one vertex in S. The minimum cardinality taken over all, the minimal double dominating set which is called Fuzzy Double Domination Number and which is denoted as $\gamma fdd(G)$. A set $S \subseteq V$ is called a Triple dominating set of a graph G if every vertex in V dominated by at least three vertices in S. The minimum number of colours required to colour all the vertices such that adjacent vertices do not receive the same colour is the chromatic number $\gamma(G)$. The minimum cardinality of a triple dominating set is called Triple domination number of G and is denoted by $T\gamma(G)$. The connectivity $\kappa(G)$ of a connected graph G is the minimum number of vertices whose removal results in a disconnected or trivial graph. For any graph, G a complete sub graph of G is called a clique of G. For a fixed positive integer k, the n-tuple domination problem is to find a minimum vertex subset such that every vertex in the graph dominated by at least k vertices in this set. In this paper we find an upper bound for the sum of the Fuzzy Double Domination, Triple domination, Chromatic Number in fuzzy graphs and characterize the corresponding extremal fuzzy graphs.

## Keywords
Domination Number, Double Domination Number, Triple Domination Number, n-tuple Domination Number, Chromatic Number, Clique, Fuzzy graphs and Connectivity.

AMS Subject Classification: 05C72

## 1. INTRODUCTION
Let $G(\mu,\sigma)$ be a simple undirected fuzzy graph. The degree of any vertex u in G is the number of edges incident with u and is denoted by $d(u)$. The minimum and maximum degree of a vertex is denoted by $\delta(G)$ and $\Delta(G)$ respectively, $P_n$ denotes the path on n vertices. The vertex connectivity $\kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected graph. The Chromatic Number $\chi$ is defined to be the minimum number of colours required to colour all the vertices such that adjacent vertices do not receive the same colour. For any graph G a complete sub graph of G is called a clique of G. The number of vertices in a largest clique of G is called the clique number of G. A subset S of V is called a dominating set in G if every vertex in V-S is adjacent to at least one vertex in S. The minimum cardinality taken over all dominating sets in G is called the domination number of G and is denoted by $\gamma$. A Dominating set is said to be Fuzzy Double Dominating set if every vertex in V-S is adjacent to at least two vertices in S. The minimum cardinality taken over all, the minimal double dominating set is called Fuzzy Double Domination Number and is denoted by $\gamma fdd(G)$. If X is a collection of objects denoted generically by x, then a Fuzzy set A in X is a set of ordered pairs: $A = \{(x, \mu(A(x) / x \in x\}$, A μ(x) is called the membership function of x in A that maps X to the membership space M when M contains only the two points 0 and 1. Let E be the crisp set of nodes. A Fuzzy graph is then defined by, $G(x_i, x_j) = \{((x_i, x_j), G\mu(x_i, x_j)), \forall(x_i, x_j) \in ExE\}$. H(xi, xj) is a Fuzzy Sub graph of $G(x_i, x_j)$, if $H\mu(x_i, x_j) \le G\mu(x_i, x_j), \forall(x_i, x_j) \in ExE$, $H(x_i, x_j)$ spans graph $G(x_i, x_j)$ if the node set of H(xi, xj) and $G(x_i, x_j)$ are equal, that is if they differ only in their arc weights. $\mu(x_1)= 0.1$, $\mu(x_2)= 0.5$, $\mu(x_3)= 0.4$, $\mu(x_4)= 0.2$ by [6].

## 2. FUZZY GRAPH
The first definition of Fuzzy graphs proposed [4] from the fuzzy relations introduced [20] and [9] introduced another elaborated definition, including fuzzy vertex and fuzzy edges. Several fuzzy analogs of graph theoretic concepts such as paths, cycle's connectedness etc., the concept of domination in fuzzy graphs was investigate and presents the concepts of independent domination, total domination, connected domination and domination in Cartesian product and composition of fuzzy graphs [19]. Several authors have studied the problem of obtaining an upper bound for the sum of a domination parameter and a graph theoretic parameter and characterized the corresponding extremal graphs. In [8] proved that, $\gamma + k \le p$, $\gamma_c + \chi \le p+1$ also characterized the class of graphs for which the upper bound is attained also proved similar results for $\gamma$ and $\gamma$t. The concept of complementary perfect domination number $\gamma$cp and proved that, $\gamma_{cp} + \chi \le 2n-2$, and characterized the corresponding extremal graphs. In [5], they proved the result $\gamma dd + \chi \le 2n$. They also characterized the class of graphs for which the upper bound is attained. Let v∈V. The open neighborhood and closed neighborhood of v are denoted by N(v) and N[v] = N(v) ∪ {v} respectively. If $S \subseteq V$ then N(S) = $(v)$ $v \in S$, $\forall v \in V$ and N[S] = N(S) ∪ S. If $S \subseteq V$ and u∈S then the private neighbor set of u with respect to S is defined by $p_n$ [u, S] = {v: N[v] ∩S = {u}}. We denote a cycle on n vertices by Cn, a path on n vertices by $P_n$ and a complete graph on n vertices by Kn. A bipartite graph is a graph whose vertex set can be divided into two disjoint sets $V_1$ and $V_2$ such that every edge has one end in $V_1$ and another in $V_2$. A complete bipartite graph is a bipartite graph where every vertex of $V_1$ is adjacent to every vertex in $V_2$. The complete bipartite graph with partitions of order $v_1 = m$, and $v_2 = n$ is

denoted by Km, n. A wheel graph, denoted by Wn is a graph with n vertices formed by connecting a single vertex to all vertices of $C_{n-1}$. H{m1, m2,........,mn) denotes the graph obtained from the graph H by pasting mi edges to the vertex $v_i \epsilon V(H)$, $1 \leq i \leq n$ , $H(P_{m1}, P_{m2}\ldots, P_{mn})$ is the graph obtained from the graph H by attaching the end vertex of $P_{mi}$ to the vertex vi in H, $1 \leq I \leq n$. Bistar B(r,s) is a graph obtained from $K_{1,r}$ and $K_{1,s}$ by joining its Centre vertices by an edge.

In a graph G, a vertex dominates itself and its neighbors. A subset S of V is called a dominating set in G if every vertex in V is dominated by at least one vertex in S. The domination number γ(G) is the minimum cardinality of a dominating set. In [2] introduced the concept of double domination in graphs. A set $S \subseteq V$ is called a double dominating set of a graph G if every vertex in V is dominated by at least two vertices in S The minimum cardinality of double dominating set is called double domination number of G and is denoted by dd(G). A vertex cut, or separating set of a connected graph G is a set of vertices whose removal results in a disconnected graph. Let κ(G) denoted by connectivity or vertex connectivity of a graph G. The Connectivity κ(G) of a connected graph G is the minimum number of vertices whose removal results in a disconnected or trivial graph. A matching M is a subset of edges so that every vertex has degree at most one in M. Several authors have studied the problem of obtaining an upper bound for the sum of a domination parameter and a graph theoretic parameter and characterized the corresponding extremal graphs. In [8] proved that $\gamma(G) + \kappa(G) \leq n$ and characterized the corresponding extremal graphs .

We use the following theorems.

**Theorem 1.1:** For any connected graph G,
$$\gamma dd(G) \leq n.$$

**Theorem 1.2: For any connected graph G,** $\chi(G) \leq \Delta(G) + 1$.

**Theorem 1.3: For any graph G,** $dd(G) \leq n$ **by [3]**.

**Theorem 1.4: For any graph G,** $\kappa(G) \leq \delta(G)$ **by [3].**

The main aim of domination in graph theory is a good model for many location problems in operations research and other application-oriented problems. In a graph G, a vertex is to be dominating itself and all of its neighbors. A dominating set of G = (V, E) is a subset D of V such that every vertex in V is dominated by some vertex in D. The domination number n(G) is the minimum size of a dominating set of G. Domination and its variations have been extensively studied in the literature then among the variations of domination, the n-tuple domination was introduced in [2, 3 and 8]. For a fixed positive integer n, an n-tuple dominating set of G = (V, E) is a subset D of V such that every vertex in V is dominated by at least k vertices of D. The n-tuple domination number γxn(G) is the minimum cardinality of an n-tuple dominating set of G. The special case when n = 1 is the usual domination. The case when n = 2 was called double domination in [2], where exact values of the double domination numbers for some special graphs are obtained. The same paper also gives various bounds of double and n-tuple domination number terms of other parameters. Nordhaus - Gaddum type inequality for double domination was given in [1].

This paper contains sharp upper bound for the sum of the Fuzzy Double and Triple Domination Number, chromatic number, connectivity of a characterized the corresponding extremal Fuzzy graphs. Finally, the n-tuple domination problems from an algorithmic point of view and in particular a linear-time algorithm for the 2-tuple domination problem in trees were studied. Note that not every graph has an n-tuple dominating set. In fact, a graph G has an n-tuple dominating set if and only if $\delta(G) + 1 \geq n$, where δ(G) is the minimum degree of a vertex in G. As any nontrivial tree has at least two leaves, only consider 2 - tuple domination for trees. To establish our algorithm, we employ a labeling method similar to those for variations of domination in tree-type graphs. Suppose G = (V, E) is a graph in which every vertex v is associated with a label, $M(v) = (t(v), k(v))$, where, $t(v) \in \{B, R\}$ and k(v) is a nonnegative integer. The interpretation of the label is that we want to find a dominating set D containing all vertices u with t(u) = R is called required vertices, such that each vertex v is dominated by at least n(v) vertices in D. More precisely, an M-dominating set of $G = (V, E)$, is a subset D of V satisfying the following conditions [17]:

**Theorem 1.5: If t (v) = R, then v ϵ D.**

**Theorem 1.6: If** $|NG[v] \cap D| \geq n(v)$, $\forall v \in V$ **, where,** $NG[v] = \{v\} \cup \{u \in v, uv \in E\}$ **is the closed neighborhood of the vertex v.**

The M-domination number γM(G) is the minimum cardinality of an M-dominating set in G. Notice that 2-tuple domination is M-domination with M(v) = (B, 2) for all vertices v in V. Also, G has an M-dominating set, i.e., $\gamma_M(G)$ is finite, if and only if |NG[v]| > n(v) for all vertices $v \in V$ . If G contains exactly one vertex x, then $\gamma_M(G) = 0$. When M(x) = (B,0), $\gamma_M(G) = 1$,when M(x)ϵ{(B,1); (R,0); (R,1)}, and $\gamma_M M(G) = \infty$ , otherwise.

# 3. MAIN RESULTS

Theorem 3.1: For any connected fuzzy graph G, $\gamma fdd(G) + \gamma(G) \leq 2n$ and the equality holds if and only if $G \cong K2$.

Proof:
$$\gamma fdd(G) + \chi(G) \leq n + \Delta + 1 = n + (n-1) \leq 2n.$$
If $\gamma fdd(G) + \chi(G) = 2n$. Then the only possible case is γfdd = n and χ = n. Since χ = n, G = Kn. But for Kn, γfdd = 2, so that $G \cong K2$. Converse is obvious.

Theorem 3.2: For any connected fuzzy graph G, γfdd (G) + χ(G) = 2n –1 if and only if $G \cong K3$.

Proof: Assume that γfdd (G) + χ(G) = 2n – 1.

This is possible only if γfdd = n and χ = n - 1 (or)

γfdd = n – 1, χ = n.

Case (i): Let γfdd = n and χ = n – 1.

Since χ = n – 1, G contains a clique K on n – 1 vertices. Let x be a vertex other than the vertices of Kn – 1. Since G is connected, x is adjacent to ui for some i in Kn – 1. Then {x,ui,uj} is γfdd– set, so that γfdd = 3. Since γfdd = n, we have n = 3. Hence K = K2. Let u, v be the vertices of K2. Let x be adjacent to u. Then γfdd = 2, which is a contradiction. Hence, no fuzzy graph exists.

Case (ii): If γfdd = n – 1 and χ = n.

Since χ = n, G = Kn. But for Kn, γfdd (G) = 2, so that n = 3. Hence, $G \cong K3$. Converse is obvious.

Theorem 3.3: For any connected fuzzy graph G [6], γfdd(G) + χ(G) = 2n–2 if and only if K4 or G1 given in Fig. 1.

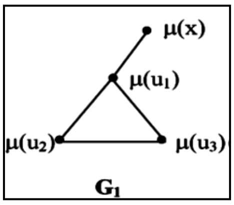


Fig. 1

Proof: If G is K4 or G1, then clearly, γfdd(G) +χ(G =2n–2.

Conversely, assume that

γfdd(G) + χ(G) = 2n – 2.

This is possible only if γfdd = n

and χ = n – 2(or) γfdd = n – 1

and χ = n – 1 (or) γfdd = n – 2

and χ = n.

Case (i): Let γfdd = n and χ = n – 2.

Since χ = n–2, G contains a clique K on n –2 vertices. Let S={x, y} ϵ V - S. Then <S> = K2 or K2.

Subcase (a): Let <S>=K2. Since G is connected, x is adjacent to some ui of Kn – 2. Then {y, ui, uj} for i ≠ j in Kn – 2 is an γfdd - set, so that γfdd= 3 and hence n = 3. But χ = n – 2= 1, which is a contradiction. Hence no fuzzy graph exists.

Subcase (b): Let <S> = K2. Since G is connected, x is adjacent to some ui of Kn – 2. Then y is adjacent to the same ui of Kn – 2 or adjacent to uj of Kn – 2 for i ≠j. In both the cases {x, y, ui, and uj} is an γfdd set. Since γfdd = n, we have n=4. Hence K=K2. Let u, v be the vertices of K2. Without loss of generality, let x and y both be adjacent to u. Then γfdd = 3, which is a contradiction. Hence no fuzzy graph exists. Now without loss of generality, let x be adjacent to u and y be adjacent to v. In this case also no fuzzy graph exists.

Similarly, we prove the following cases.

Case (ii): Let γf dd = n – 1 and χ = n – 1.

Case (iii): Let γfdd = n – 2 and χ = n.

Theorem 3.4: For any connected graph G [6], γfdd(G)+χ(G) = 2n – 3 if and only if $G \cong P4$ or any one of the following fuzzy graphs in the Fig. 2.

Proof: If G is any one of the graph given in the figure, then clearly γfdd (G) + χ(G) = 2n – 3.

Conversely assume that γfdd(G) + χ(G) = 2n–3. This is possible only if γfdd = n and χ = n – 3

(or) γfdd = n – 1 and χ = n – 2

(or) γfdd = n – 2 and χ = n – 1

(or) γfdd = n – 3 and χ = n.

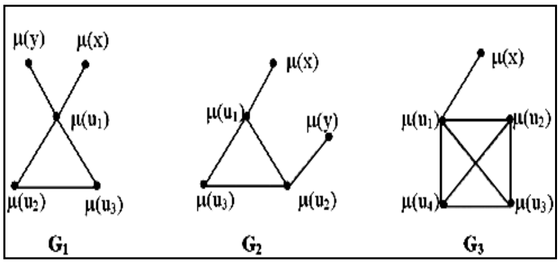


Fig. 2

Similarly we prove the following cases:

Case (i): Let γfdd = n and χ = n – 3.

Case (ii): Let γfdd = n–1 and χ = n – 2.

Case (iii): Let γfdd = n – 2 and χ = n – 1.

Case (iv): Let γfdd = n – 3 and χ = n.

Theorem 3.5: For any connected graph G [6], γfdd(G)+χ(G) = 2n – 4 if and only if G ≅ K6 or any one of the following graphs given in the Fig. 3.

Proof: If G is any one of the graph given in the figure, then clearly γfdd (G) + χ(G) = 2n – 4.

Conversely, assume that γfdd (G) + χ(G) = 2n – 4. This is possible only if, γfdd = n and χ = n – 4

(or) γfdd = n – 1 and χ = n – 3

(or) γfdd = n – 2 and χ = n – 2

(or) γfdd = n – 3 and χ = n – 1

(or) γfdd = n – 4 and χ = n.

We can prove the similar result based on the following cases:

Case (i): If γfdd = n and χ = n – 4.

Case (ii): Let γfdd = n –1 and χ = n – 3.

Let u1, u2, u3 be the vertices of K3. Without loss of generality, let u1 be adjacent to all the vertices of S and if d(x) = d(y) = d(z) = 1, then G ≅ G1.

In all other cases, no new graph exits.

Case (iii) Let γfdd= n – 2 and χ = n – 2.

Case (iv) Let γfdd = n – 3 and χ = n –1.

Case (v) Let γfdd = n – 4 and χ = n.

Since χ = n then G = Kn. But for Kn, γfdd = 2,

so that n = 6. Hence $G \cong K6$.

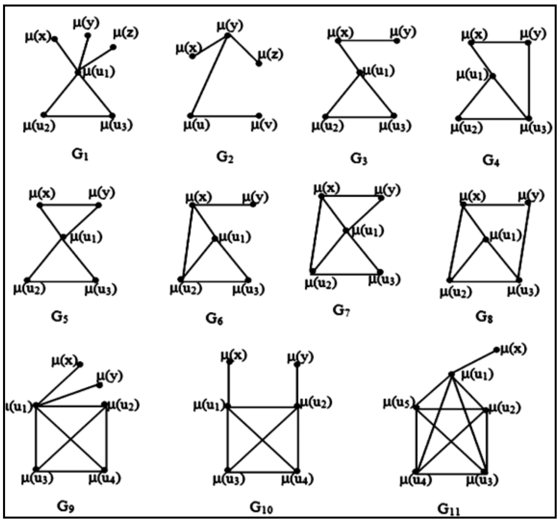


**Fig. 3**

The authors can obtain large classes of graphs with very lengthy proof for which γfdd (G) + χ(G) = 2n – 5, γfdd(G) + χ(G) = 2n –6 and γfdd(G) + χ(G) = 2n – 7.

**Definition 3.1:** If every vertex in V is dominated by at least three vertices in S. Then a set S ⊆ $V$ is called Triple dominating set of a graph G. The minimum cardinality of Triple dominating set is called Triple domination number of G and is denoted by Tγ(G). The connectivity κ(G) of a connected graph G is the minimum number of vertices whose removal results in a disconnected or trivial graph. Choose S={v1, v2, v3} ϵ V(G). If N[S] = V(G), a dominating set obtained in the way given above is called a Triple dominating set [7].

**Example 3.1:** For the graph K3, Tγ(G) = 3, and κ(G) = 2.

We have Tγ(G) + κ(G) = 5

**Example 3.2:** For the graph K4, Tγ(G) = 3, and κ(G) = 3.

We have Tγ(G) + κ(G) = 6

**Example 3.3.** For the graph K5, Tγ(G) = 3, and κ(G) = 4.

We have Tγ(G) + κ(G) = 7

**Example 3.4.** For the graph K6, Tγ(G) = 3, and κ(G) = 5.

We have Tγ(G) + κ(G) = 8

**Theorem 3.6:** For the complete graph Kn, we have

(i) Tγ(G) + κ(G) = 2n - 1, when n = 3

(ii) Tγ(G) + κ(G) ≤ 2n - 1, when n = 4

(iii) Tγ(G) + κ(G) ≤ 2n - 2, when n = 5

(iv) Tγ(G) + κ(G) ≤ 2n - 3, when n = 6.

**Theorem 3.7:** [1] For any graph Kn, we have γ(G)≤γN(G) ≤ Tγ(G) .Where γN(G) is the degree equitable domination number of G.

**Theorem 3.8:** For any graph G, Tγ(G) ≤ n.

**Theorem 3.9:** For any connected graph G, κ(G) = n - 1 if G is isomorphic to Kn.

**Theorem 3.10:** For any connected graph G, Tγ(G) = 3 if G is isomorphic to Kn , n ≥ 3

**Theorem 3.11:** For the graph Km,n where m = n. There exists a Triple dominating set with matching M

**Theorem 3.12:** Let G1 and G2 be any two graphs of Triple dominating sets then G1 + G2 is a graph of Triple dominating set of G1 or G2.

**Proof :** Let G1 and G2 be any two graphs having triple dominating sets .By taking sum of G1 and G2, we have every vertex in G1 is adjacent to every vertex in G2. Therefore by the definition of triple dominating set, we have By choosing S is the Triple dominating set of G1 or G2 and N[S]=V(G1+G2). Hence S is the Triple dominating set of G1 or G2.

**Theorem 3.13:** Every complete graph Kn has a Triple dominating set if n ≥ 3.

**Proof :** Given the graph G is complete when n ≥ 3, Choose S={v1,v2,v3}∈V(G), If N[S] = V(G), A dominating set obtained is a Tγ(G) = 3.

**Theorem 3.14:** For any connected graph G, Tγ(G)+κ(G)=2n - 1 if and only if G is isomorphic to K3. **Proof: Case 1:** Tγ(G) + κ(G) ≤ n + δ ≤ n + n - 1= 2n - 1. Let Tγ(G) + κ(G) = 2n - 1 then Tγ(G) = n and κ(G) = n - 1. Then G is a complete graph on n vertices. Since Tγ(Kn) = 3 we have n = 3. Hence G is isomorphic to K3. The converse is obvious.

**Case 2:** Suppose Tγ(G) = n - 1 and κ(G) = n then n ≤ δ(G) is impossible which is a contradiction to κ(G) = n - 1. Hence Tγ(G) = n - 1 and κ(G) = n is not possible.

**Theorem 3.15:** For any connected graph G, Tγ(G) + κ(G) = 2n - 2 if and only if G is isomorphic to K4 or C4.

**Proof:** Tγ(G) + κ(G) = 2n - 2, then there are two cases to be considered.

Tγ(G) = n - 1 and κ(G) = n - 1 (i)

Tγ(G) = n and κ(G) = n – 2 (ii)

**Case 1:** Tγ(G) = n - 1 and κ(G) = n - 1 , Then G is a complete graph on n vertices, Since Tγ(G) = 3, We have n = 4 . Hence, G is isomorphic to K4.

**Case 2:** Tγ(G) = n and κ(G) = n – 2, Then n – 2 ≤ δ(G). If δ = n - 1, Then G is a complete graph, which is a contradiction. Hence δ(G) = n – 2. Then G is isomorphic to Kn – M where M is a matching in Kn. Then Tγ(G) = 3 or 4, If Tγ(G) = 3 then n =3. Which is a contradiction to κ(G) =1≠ n-2 and N[S] ≠ V(G), Thus Tγ(G) = 4. Then n = 4 and hence G is isomorphic toK4 - e or C4 with $M$ =1 or 2 respectively.

**Theorem 3.16:** For any connected graph G, Tγ(G)+κ(G)=2n – 3, if and only if G is isomorphic to K4 (or) K4 – e (or) K1,3 (or) K5 – M, Where M is a matching on K5 with $M$ =1.

**Proof:** Let Tγ(G) + κ(G) = 2n – 3, Then there are three cases to be considered

(i) Tγ(G) = n - 2 and κ(G) = n – 1

(ii) Tγ(G) = n - 1 and κ(G) = n – 2

(iii) Tγ(G) = n and κ(G) = n – 3.

**Case 1:** Tγ(G) = n - 2 and κ(G) = n - 1, Then G is a complete graph on n vertices, Since Tγ(G) = 3 and $\delta(G)$ = n is not possible, We have n = 5. Hence G is isomorphic to K5.

**Case 2:** Let Tγ(G) = n - 1 and κ(G) = n– 2, then n - 2 ≤ δ(G). If δ = n - 1 then G is a complete graph, Which gives a contradiction to κ(G) = n – 2. If δ(G) = n - 2, then G is isomorphic to Kn – M, where M is a matching in Kn, then Tγ(G) = 3 or 4. If Tγ(G) = 3 then n = 4, then G is either C4 or K4 – e. But Tγ(G) = 4 ≠ n – 1. Hence G is isomorphic to K4 – e where 'e' is a matching in K4. If Tγ(G) = 4, then n = 5 and hence G is isomorphic to K5 - M where M is a matching on K5 with $M$ = 1.

**Case 3:** Let Tγ(G) = n and κ(G) = n – 3 then n - 3 ≤ δ(G). If δ = n - 1 then G is a complete graph, which gives a contradiction to κ(G) = n-3. If δ(G) = n - 2, Then G is isomorphic to Kn – M, where M is a matching in Kn, then Tγ(G) = 3 or 4. Then n = 3 or 4, Since n = 3 is impossible, We have n=4, Then G is either K4 – e or C4. For these two graphs κ(G) ≠ n - 3 which is a contradiction. Hence δ = n - 3. Let X be the vertex cut of G with $X$ = n - 3 and let V– X ={$x_1$, $x_2$, $x_3$}, X = {$v_1$, $v_2$, ............, $v_{n-3}$}.

**Sub Case 3.1:** $V - X = K3$ Then every vertex of V - X is adjacent to all the vertices of X. Then {(V-X)∪{$v_1$}} – M is a Triple dominating set of G, where M is a matching in Kn and hence Tγ(G) ≤ 4, this gives n ≤ 5, since n ≤ 3 is impossible, we have n = 4 or 5. If n = 4 then G is isomorphic to $K_{1,3}$, which is a contradiction to the definition of matching. If n = 5 then the graph G has Tγ(G) = 3 or 4, which is a contradiction to κ(G) = n – 3.

**Sub case 3.2:** $V - X = K1 \cup K2$. Let $x_1$, $x_2$ ϵ E(G), then $x_3$ is adjacent to all the vertices in X and $x_1$, $x_2$ are not adjacent to at most one vertex in X, if $v_1$ ∉ N($x_1$) ∪ N($x_2$) then {(V-X) ∪ {$v_1$}} is a triple dominating set of G and hence Tγ(G) ≤ 4, this gives n = 4. For this graph κ(G) = 1 which is a contradiction to κ(G) = n - 3. So all vi ϵ N(x1) or vi ϵ N($x_2$) or both, then {( V- X) ∪{$v_i$} } is a triple dominating set of G. Hence Tγ(G) ≤ 4, then n = 4. Hence G is isomorphic to C4 or k3(1, 0', 0). But Tγ[k3(1 ,0`, 0)] = 3 ≠ n. Which are contradiction then the converse is obvious.

**Theorem 3.17:** For any connected graph G, Tγ(G) + κ(G) = 2n – 4 if and only if G is isomorphic to K6 or K4-e or K1,4 or K3(1,0,0) or B(1,2) or K5 - M where M is a matching on K5 with $M$ = 1.

**Proof:** Let Tγ(G) + κ(G) = 2n – 4. Here there are four cases to be discussed

(i) Tγ(G) = n – 3 and κ(G) = n – 1

(ii) Tγ(G) = n – 2 and κ(G) = n – 2

(iii)Tγ(G) = n – 1 and κ(G) = n – 3

(iv)Tγ(G) = n and κ(G) = n – 4

**Case 1:** Tγ(G) = n – 3 and κ(G) = n – 1, Then G is a complete graph on n vertices, since Tγ(G) = 3 we have n = 6 . Hence, G is isomorphic to K6.

**Case 2:** Tγ(G) = n – 2 and κ(G) = n – 2, then n – 2 ≤ $\delta$. If $\delta = n - 1$, then G is a complete graph which is a contradiction. If $\delta = n - 2$, then G is isomorphic to Kn-M is a matching in Kn, then Tγ(G) = 3 or 4. If Tγ(G) = 3 then n = 5 then G is either C5 or K5 - e. If Tγ(G) = 4 then n = 6 and hence G is isomorphic to K6 - e.

**Case 3:** Tγ(G) = n – 1 and κ(G) = n – 3, Then n – 3 ≤ $\delta$. If $\delta = n - 1$ then G is isomorphic to Kn-M where M is a matching in Kn. Then Tγ(G)=3 or 4, then n = 3 or 4. Since n=3 is impossible, we have n = 4, then G is either K4 - e or C4 for these to graphs Tγ(G) = 2 ≠ $n - 3$, Which is a contradiction. Hence $\delta = n - 3$. Let X is the vertex cut of G with $X = n - 3$ and V – X = {$x_1$,$x_2$,$x_3$}, X = {$v_1$,$v_2$, ,,., $v_{n-3}$}

**Subcase 3.1:** $V - X = K3$. Then every vertex of V - X is adjacent to all the vertices of X then {(V - X) ∪ {$v1$}} $- M$ is a Triple dominating set of G where M is a matching in Kn and hence Tγ(G) ≤ 4, this gives n ≤ 5, since n ≤ 3 is impossible, we have n = 4 or 5. If n = 4 then G is isomorphic to $K_{1,3}$. This is a contradiction. If n = 5 then the graph G has Tγ(G) = 3 or 4, which is a contradiction to κ(G) = n – 3.

**Subcase 3.2:** $V - X = K1 \cup K2$. Let $x_1x_2 \in E(G)$, then $x_3$ is adjacent to all the vertices in X and $x_1$, $x_2$ are not adjacent to at most one vertex in X. If $v_1$ ∉ N($x_1$) ∪ N($x_2$) then {(V-X)∪{$v_1$}} is a Triple dominating set of G and hence Tγ(G) ≤ 4, this gives n = 5 , for this graph κ(G) = 2, which is a contradiction. So all $v_i$ ϵ N($x_1$) or $v_i$ ϵ N($x_2$) or both, then {(V-X) ∪ {$v_1$}} is a Triple dominating set of G. Hence Tγ(G) ≤ 4 and then n = 4 or 5. If n = 4 is impossible, we have n = 5. Then G is isomorphic to C5 or C3 (P3, 0, 0). But κ[C3(P3,0,0)] = 1 ≠ n - 3. Hence G is isomorphic to C5.

**Case 4:** Tγ(G) = n and κ(G) = n – 4. Then, n – 4 ≤ $\delta$. If $\delta$ = n - 1 then G is a complete graph which is a contradiction. If $\delta$ = n - 2 then G is isomorphic to Kn-M, where M is a matching in Kn. Then Tγ(G) = 3 or 4 then n = 3 or 4, which is a contradiction to κ(G) = n – 4. Suppose $\delta$ = n - 3. Let X be the vertex cut of G with $X = n - 4$ and let X = {$v_1$,$v_2$, . ., $v_{n-4}$}, V - X = {$x_1$, $x_2$, $x_3$, $x_4$}. If $V - X$ contains an isolated vertex then $\delta \leq n - 4$, Which is a contradiction. Hence $V - X$ is isomorphic to K2 ∪ K2. Also every vertex of V - X is adjacent to all the vertices of X. Let $x_1x_2$, $x_3x_4$ ∈ E(G). Then {$x_1$, $x_2$, $x_3$, $v_1$} is a triple dominating set of G. Then Tγ(G) ≤ 4. Hence n ≤ 4, Which is a contradiction to κ(G) = n – 4. Thus $\delta(G)$ = n - 4.

**Subcase 4.1:** $V - X = K4$. Then every vertex of V - X is adjacent to all the vertices in X. Suppose E($X$) = ∅. Then $X$ ≤ 4 and hence G is isomorphic to $K_{s,4}$ where S = 1, 2, 3, 4. If S ≠ 1, 2 then Tγ(G) = 3 or 4, which is a contradiction to Tγ(G) = n. Hence G is isomorphic to $K_{3,4}$ or $K_{4,4}$. Suppose E($X$) = ∅. If any one of the vertex in X say vi is adjacent to all the vertices in X and hence Tγ(G) ≤ 3 which gives n ≤ 3, which is a contradiction. Hence every vertex in X is not adjacent to at least one vertices in X then {$v_1$, $v_2$, $v_3$, $v_4$} is a triple dominating set of G and hence Tγ(G) ≤ 4 then n ≤ 4. Which is a contradiction to κ(G) = n - 4.

**Subcase 4.2:** $V - X = P3 \cup K1$. Let x1 be the isolated vertex in $V - X$ and ($x_1$, $x_2$, $x_3$) be a path then x1 is adjacent to all the vertices in X and $x_2$, $x_4$ are not adjacent to at most two vertices in X and hence {$x_1$. $x_5$, $x_2$. $v_1$, $v_2$. $x_5$, $v_1$. $v_2$, $x_3$. $v_1$}, where $v_1$ ∈ N(x1) ∩ X, $v_2$ ∈ N($x_2$) ∩ X and $v_3$ ∈ N($x_3$) ∩ X, is a triple dominating set of G and hence Tγ(G) ≤ 5, thus n = 5, then G is isomorphic to P5 or C4(1, 0, 0) or K3(1,1,0) or (K4-e)(1, 0, 0, 0). All these graph Tγ(G) ≠ n. This is a contradiction.

**Subcase 4.3:** V – X = K3 ∪ K1. Let $x_1$ be the isolated vertex in V − X and {$x_2$, $x_3$, $x_4$} be a complete graph, Then $x_1$ is adjacent to all the vertex in X and $x_2$, $x_3$, $x_4$ are not adjacent to at most two vertices in X and hence {$x_2$, $x_3$, $x_4$, $x_5$, $v_1$.$x_1$} where $v_1$, $v_2$ ∈ X - N($x_2$ ∪ $x_3$) is a Triple dominating set of G and hence n = 5. All these graph Tγ(G) ≠ n.

**Subcase 4.4:** V – X = K2 ∪ K2. Let x1x2, x3x4 ∈ E(G). Since $\delta$(G) = n – 4, each xi is non-adjacent to at most one vertex in X then at most one vertex say v1 ∈ X, such that N($v_1$)∩(V−X) = 1. If all $v_i$ ∈ X such that N($v_i$) ∩ (V−X) ≥ 3, then {$x_1$. $v_1$, $x_2$, $x_3$, $x_4$} is a triple dominating set of G and hence n = 4. This is a contradiction. Then each $x_i$ is non-adjacent to at most one vertex in X then at most one vertex say v1 or v2 ∈ X such that N(v1) ∩ (V − X) = 2 and N(v2)∩(V−X) = 2 and N(vi) ∩ (V−X) ≥ 3, i ≠1 then {v1. v2, x1, x2, x3, x4} is a triple do$_m$inating set of G and hence n = 5. Which is a contradiction to (v1) ∩ (V − X) = 2 and N(v2)∩(V−X) = 2 and N(vi) ∩ (V − X) ≥ 3. The converse is obvious.

## 4. DOMINATION IN TREES

To give an algorithm for the 2-tuple domination problem in trees, in fact establish one for the M-domination problem in trees. Believe that the approach has potential for other classes of graphs. First give the following theorem which is the base of the algorithm notice that it works for general graphs.

**Theorem 4.1:** [17] Suppose G = (V, E) is a nontrivial graph in which every vertex v has a label M(v) = (t(v), n(v)). Let x be a leaf adjacent to y.

(1) If k(x) > 2 or n(y) > $|N_G[y]|$, then G has no M- dominating set.

(2) If k(x) = 2 or n(y) = $|N_G[y]|$, then γM(G) = γM'(G') + 1, where G' is obtained from G by deleting x and M' is obtained from M by relabeling y with t'(y) = R and n'(y) = max {n(y) –1, 0}.

(3) If t(x) = R and k(x) < 2 and k(y) < |NG[y]|, then γM (G) = γM'(G') + 1; where G' is obtained from G by deleting x and M' is obtained from M by re labeling y with k'(y) = max {n(y) – 1, 0}.

(4) If M(x) = (B, 1) and n(y) < $|N_G[y]|$, then γM(G) = γM'(G'); where G' is obtained from G by deleting x and M is obtained from M by re labeling y with t'(y) = R.

(5) If M(x) = (B, 0) and n(y) < |NG[y]|, then γM(G) =$\gamma_M$(G-x).

### Proof:

(1). This follows from the definition of M-domination.

(2). Suppose D' is a minimum M'-dominating set of G'.

Then y ϵ D', since t'(y) = R. Hence, D = D' U {x} is an M-dominating set of G, since $|N_G[x] \cap D| \geq 2 \geq$ n(x).

Thus, γM'(G') + 1 = |D'| + 1 = |D| ≥ γM (G).

On the other hand, suppose D is a minimum M- dominating set of G. Then x,y∈D, since n(x) = 2 or k(y)= $|N_G[y]|$. Hence, D' = D\{x} is an M'-dominating set of G', since y € D' and $|N_{G'}[y] \cap D'| = |N_G[y] \cap D|$ - 1 > max{n(y)–1, 0} = n'(y).

So, γM (G) = |D| =|D'| + 1 ≥ γM'(G') + 1. These complete the proof of $\gamma_M$ (G) = γM'(G') + 1.

(3). Suppose D' and M'- dominating set of G, since $|N_G[x] \cap D| \geq 1 >$ k(x). Thus, $\gamma_M$>(G')+1=|D'|+1 = |D| > γM(G).

On the other hand, suppose D is a minimum M - dominating set of G. Then x∈D, since t(x) = R. Hence, D' = D\{x} is an M' - dominating set of G,

since $|N_{G'}[y] \cap D| = |N_G[y] \cap D|$ - 1 > max{n(y) –1, 0}= n'(y). So, $\gamma_M$ (G) = |D| = |D'| + 1 ≥ γM'(G') +1. These complete the proof of γM(G) = γM'(G') +1.

(4). Suppose D' is a minimum M'- dominating set of G. Then y ϵ D', since t(y) = R. Consequently, D is an M-dominating set of G as M(x) = (B, 1). Thus, γM'(G') = |D'| > γM (G).

On the other hand, suppose that D is a minimum M-dominating set of G. If x ∉ D, then y ϵ D, since n(x) = 1. And D is an M' - dominating set of G'. Therefore, $\gamma_M$(G) = |D| > γM'(G'). We may now assume that x ϵ D. Let D = D\{x}. If y ϵ D' and $|N_G'[y] \cap D'|$ > k(y), then D' is an M-dominating set of G' and so $\gamma_M$(G) = |D| > |D'| > γM > (G). So now y ∉ D$^1$ or |NG'[y] ∩ D| < k(y) $|N_G[y]|$ - 1 = |NG'[y]|. For the case when y∉ D, let z = y, for the case when y ϵ D', choose a vertex z ϵ $N_{G-X}$ [y] \ D'. Then, in any case, y ϵ DU{z} and so D' U{z} is an M' - dominating set of G'.

Hence, γM (G) = |D| = |D' U {z}| > γM'(G').

These complete the proof of γM(G) = γM'(G').

(5). Suppose D' is a minimum M-dominating set of G - x. Then D' is also an M-dominating set of G, since t(x) = B and n(x) = 0. Therefore, $\gamma_M$ (G - x) =|D| > γM(G).

On the other hand, suppose that D is a minimum M-dominating set of G. If x ∉ D, then D is also an M-dominating set of G-x. Thus, $\gamma_M$ (G) = |D| > $\gamma_M$ (G-x). We may now assume that x ϵ D. Let D' = D\{x}. If | $N_G$-x[y]∩ D' | > k(y), then D' is an M - dominating set of G-x and $\gamma_M$(G) = |D| > |D'| > γM (G - x). So, now $|N_{G-x}[y] \cap D'|$<n(y) $|N_G[y]|$-1=|NG-x[y]|.

Choose a vertex Z ϵ $N_{G-x}$[y]\D. Then D' U {z} is an M-dominating set of G-x. Hence, γ(G)=|D|= |D'U{z}|> γM(G-x).

These complete the proof of $\gamma_M$ (G) = $\gamma_M$(G - x). The following linear-time algorithm for the M-domination problem in trees can studied based on the above theorem.

## 5. ALGORITHM

The Algorithm for an M-dominating set of a tree [17].

**INPUT:** A tree T= (V,E) in which each vertex v is labeled by M(v) = (t(v), n(v)).
**OUTPUT:** A minimum M-dominating set D of T.
**METHOD:**
D ← 0;
T' ← T;
**While** (T' has at least two vertices) **do**
choose a leaf x adjacent to y in T';
**if** (k(x) > 2 or n(y) >$|N_{T'}[y]$\) **then**
**stop** since there is no M-dominating set;
**else if** (n(x) = 2 or n(y) = $|N_{T'}[y]|$) **then**
t(y)=R and n(y)=max{n(y)-1,0}and D←DU{x};
**else if** {* now n(x) < 2 and n(y) < $|N_{T'}[y]|$ *}
(t(x) = R) **then**
k(y) = max {n(y) – 1, 0} and D ← D U {x};
**else if** {* now t(x) = B, n(x) < 2, n(y) $|N_{T'}[y]$ *}
(n(x) = 1) **then**
t(y) = R; T' ← T' - x {* delete x from T' *};
**end while**;
suppose the only vertex of T' is x;
**if** (n(x) > 1) **then**
**STOP** as there is no M-dominating set;
**else if** (t(x) = R or n(x) = 1) **then**
D ← D U {x}.

## 6. CONCLUSION

In this paper found an upper bound for the sum of Triple domination number and connectivity of graphs and characterized the corresponding extremal graphs. Similarly Triple domination number with other graph theoretical parameters can be considered. And finally the N - tuple domination problem from an algorithmic aspect studied. In particular, it gives a linear-time algorithm for the 2 - tuple domination problem in trees. Note that not every graph has an N - tuple dominating set. In fact, a graph G has an N - tuple dominating set if and only if $\delta(G) + 1 \geq n$, where $\delta(G)$ is the minimum degree of a vertex in G. As any nontrivial tree has at least two leaves only consider 2 - tuple domination for trees.

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