Numerical Solution of Fourth Order Integro-differential Boundary Value Problems by Optimal Homotopy Asymptotic Method

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ABSTRACT
In the course of this paper, the Optimal Homotopy Asymptotic Method (OHAM) introduced by Marica is applied to solve linear and nonlinear boundary value problems both for fourth-order integro-differential equations. The following analysis is accompanied by numerical examples whose results show that the Optimal Homotopy Asymptotic Method is highly accurate, convenient and relatively efficient for solving fourth order integro-differential equations.

Keywords
Fourth Order Integro-differential Equations, Boundary Value problem, Optimal Homotopy Asymptotic Method.

1. INTRODUCTION
The Integro-differential equation (IDE) is one that takes into account both integrals and derivatives of an unknown function. Mathematical modeling of real-life problems usually results in functional equations, like ordinary or partial differential equations, integral and integro-differential equations, and stochastic equations. Many mathematical formulations of physical phenomena contain Integro-differential equations; these equations pop up in many fields, namely physics, astronomy, potential theory, fluid dynamics, biological models, and chemical kinetics. Integro-differential equations; are usually difficult to solve analytically; it is, therefore, required to obtain an efficient approximate solution [1-3]. Recently, several numerical methods to solve IDEs have been proposed such as the Wavelet-Galerkin method [4], Lagrange interpolation method[5], Variational Iteration Method[6,7], Homotopy Perturbation Method [8,9], Tau method [10], Adomian's decomposition method [11,12], Taylor polynomials [13],Spline Function Expansion[14,15] and Collocation Method [16-19].

The work in question is motivated by the desire to obtain analytical and numerical solutions to boundary value problems for fourth-order integro-differential equations. In recent years, Optimal Homotopy Asymptotic Method (OHAM), which was introduced by Marica et al [20], has been used in obtaining approximate solutions of a wide class of differential, integral and the elusive Integro-differential equations. The method provides the solution in a rapidly convergent series with components that are elegantly computed. The main advantage of the method is that it can be used directly without using assumptions or transformations.

In this work, we aim to implement this reliable technique to Integro-differential equations with two-point boundary value problems (BVPs) for the fourth-order Integro-differential equations

$$y^{(iv)}(t) = f(t) + \gamma y(t) + \int_0^t \begin{pmatrix} g(t)y(t)+ \\ h(t)F(y(t)) \end{pmatrix} dt \qquad (1)$$

$$y(t_o) = \alpha_o\,,\ y''(t_o) = \alpha_1\,,\ y(t_f) = \beta_o\,,\ y''(t_f) = \beta_1$$

where $t \in (t_o, t_f)$ and F is a real non-linear continuous function $\gamma, \alpha_o, \alpha_1, \beta_o$ and β_1 are real constants, and f, g and h are given and can be approximated by Taylor polynomials.The conditions for existence and uniqueness of solutions of (1) are given in [1]. The boundary conditions will be imposed on various approximants of the obtained series solution to complete the determination of the remaining constants. Several numerical methods to solve the fourth-order Integro-differential equations have been given such as the Variational Iteration Method [7], Chebyshev Cardinal Functions [21], Pseudospectral Method [22], Homotopy Perturbation Method [23], Spectral Method [24], Adomian decomposition method [11] , Reproducing kernel theory [25] and a number of others that work but are somehow inferior to the new method under our belts.

The rest of this paper is organized as follows. In Section 2, we review the Optimal Homotopy Asymptotic Method (OHAM). In Section 3, illustrative examples are provided for the confirmation of the effectiveness of the presented method. Section 4 contains conclusive notes and notations about future work.

2. REVIEW OF OHAM
We review the fundamental ideas of Optimal Homotopy Asymptotic Method, as discussed in [20], for solving nonlinear differential equation. Consider the following differential equation

$$A(v(\tau)) + h(\tau) = 0\,;\ \tau \in \alpha \qquad (2)$$

where α is problem domain. The operator A usually consists of two parts

$$A(v) = L(v) + N(v) \qquad (3)$$

where $L(v)$ is the linear part of the operator and $N(v)$ is nonlinear part of the operator. $v(\tau)$ is unknown function and $h(\tau)$ is known function. Now, we construct an optimal homotopy equation as follows:

$$(1-q)[L(\phi(\tau,q)) + h(\tau)] - H(q)[A(\phi(\tau,q)) + h(\tau)] = 0 \quad (4)$$

where q is an embedding parameter ranges from zero to one. $H(q)$ is an auxiliary function on which convergence of Eq.(4) depends. It can be given as

$$H(q) = \sum_{t=1}^{s} q^t c_t \tag{5}$$

This function also adjusts the convergences domain as well as convergence region. The following approximate solution has been obtained if we expand $\phi(\tau; q, c_j)$ in a Taylor's series about q

$$\phi(\tau; q, c_j) = v_o(\tau) + \sum_{K=1}^{\infty} v_k(\tau, c_j) q^k, \quad j = 1,2,3 \tag{6}$$

The convergence of Equation (6) depends upon c_j . If it is convergent, then we get

$$\tilde{v} = v_o(\tau) + \sum_{k=1}^{s} v_k(q, c_j) \tag{7}$$

By substituting Eq.(7) in Eq.(24), the following residual has been achieved

$$R(\tau, c_j) = L(v(\tau; c_j)) + h(\tau) + N(v(\tau; c_j)) \tag{8}$$

By minimizing the residual, we will get the approximate solution. If $R(\tau, c_j) = 0$, then $\tilde{v}$ will be the exact solution. In general, such case will not arise for nonlinear problem. For determining the value of c_j, different methods such as least square method or Galarkin's method can be used. After substituting the values of c_j in Eq.(7), one can obtained the corresponding approximate solution.

3. NUMERICAL EXAMPLES

Example 1:

We first consider the linear boundary value problem for the integro-differential equation

$$y^{(iv)}(x) = x(1 + e^x) + 3e^x + y(x) - \int_0^x y(t)dt \quad ; 0 < x < 1 \tag{9}$$

subject to the boundary conditions

$$y(0) = 1 \ ; \ \ y'(0) = 1 \ , \ \ y(1) = 1 + e \ , \ \ y'(1) = 2e$$

The exact Solution of this problem is $y(x) = 1 + xe^x$ (10)

The Optimal homotopy asymptotic method formulation of equation (9) is

$$L\{y(x,q)\} = y^{(iv)}(x) \tag{11}$$

$$N\{y(x,q)\} = y(x) - \int_0^x y(t)dt \tag{12}$$

$$g(x) = x(1 + e^x) + 3e^x \tag{13}$$

which satisfies

$$(1-q)\begin{bmatrix} \begin{pmatrix} u_o(x)+ \\ u_1(x)+ \\ u_2(x)+ \\ \cdot\cdot \end{pmatrix} + \\ x(1+e^x)+ \\ 3e^x \end{bmatrix} = \begin{pmatrix} c_1 q + \\ c_2 q^2 + \\ \cdots\cdot \end{pmatrix} \begin{bmatrix} \begin{pmatrix} u_o(x)+ \\ u_1(x)+ \\ u_2(x)+ \\ \cdots \end{pmatrix} + \\ x(1+e^x)+ \\ 3e^x + \\ y(x) - \\ \int_0^x y(t)dt \end{bmatrix} \tag{14}$$

By equating the coefficients of the same power of q , one obtain

$$q^0 : y_o^{(iv)}(x) = x(1+e^x) + 3e^x \tag{15}$$

$$q^1 : y_1^{(iv)}(x, c_1) = c_1\left[y_o(x) + \int_0^x y_o(t)dt \right] \tag{16}$$

$$q^2 : y_2^{(iv)}(x, c_1, c_2) - y_1^{(iv)}(x, c_1) = c_1\left[y_1(x) + \int_0^x y_1(t)dt \right] + c_2\left[y_o(x) + \int_0^x y_o(t)dt \right] \tag{17}$$

By solving the above equations, we can easily obtain $y_o(x)$, $y_1(x, c_1)$ and $y_2(x, c_1, c_2)$ which are as follows:-

$$y_o(x) = e^x(x-1) - x^2(x-2)e + \frac{1}{120}\begin{pmatrix} x^5 + 357x^3 - \\ 598x^2 + 120x + \\ 240 \end{pmatrix} \tag{18}$$

$$y_1(x, c_1) = \frac{1}{7257600} c_1 \begin{pmatrix} -7257600 + 7257600e^x - \\ 7257600x + 35979673x^2 - \\ 14446440ex^2 - 21330310x^3 + \\ 7161840ex^3 - 60480x^5 - \\ 110544x^6 + 40320ex^6 + \\ 40056x^7 - 114400ex^7 - \\ 3213x^8 + 1080ex^8 + \\ 20x^9 - 2x^{10} \end{pmatrix} \tag{19}$$

$$y_2(x,c_1,c_2) = A\left\{1441440c_1\left(\begin{array}{l} 7257600e^x + \\ 360ex^2\begin{pmatrix}-40129+19894x+ \\ 112x^4-40x^5+3x^6\end{pmatrix}- \\ (x-1)^2\begin{pmatrix}7257600+21772800x+ \\ 308327x^2+174164x^3+ \\ 40001x^4-33682x^5+ \\ 3179x^6-16x^7+2x^8\end{pmatrix}\end{array}\right)+\right.$$

$$1441440c_2\left(\begin{array}{l} 7257600e^x + \\ 360ex^2\begin{pmatrix}-40129+19894x+ \\ 112x^4-40x^5+3x^6\end{pmatrix}- \\ (x-1)^2\begin{pmatrix}7257600+21772800x+ \\ 308327x^2+174164x^3+ \\ 40001x^4-33682x^5+ \\ 3179x^6-16x^7+2x^8\end{pmatrix}\end{array}\right)+$$

$$\left. c_1^2x^2\left(\begin{array}{l} 26961954643-37555567572x+ \\ 15859232592x^4-57183184961x^5+ \\ 4575351495x^6-28828800x^7- \\ 28732704x^8+10164336x^9- \\ 997360x^{10}+31668x^{11}-240x^{12}+8x^{13}- \\ 120e\begin{pmatrix}824541453-1148028766x+ \\ 482029548x^4-171275676x^5+ \\ 12801789x^6-96096x^8+ \\ 30576x^9-2912x^{10}+84x^{11}\end{pmatrix}\end{array}\right)\right\} \quad (20)$$

where $A = \dfrac{1}{1046139494000}$

The solution of equation (9) can be obtained approximately in the form

$$y^2(x,c_1,c_2) = y_o(x) + y_1(x,c_1) + y_2(x,c_1,c_2) \quad (21)$$

where $y^2(x,c_1,c_2)$ is the second order approximate solution. Substituting equation (21) into (9), the residual has been obtained

$$R(x,c_1,c_2) = y^2(x,c_1,c_2) + x(1+e^x) + 3e^x + y(x) - \int_0^x y(t)dt \quad (22)$$

By using least square method, the values of constant is obtained as

$c_1 = 1.00067388853492054,$

$c_2 = -1.0013478258763215$

Comparison of results with exact solution is presented in Table 1, which contains percentage error of the approximate solution obtained by Optimal Homotopy Asymptotic Method.

Table 1: Percentage error of Example 1

x	Numerical Solution	Exact Solution	% Error
0.0	1.00000000000	1.00000000000	0.00000000000
0.1	1.11051709180	1.11051709180	0.00000000000
0.2	1.24428055161	1.24428055163	0.00000000001
0.3	1.40495764226	1.40495764227	0.00000000000
0.4	1.59672987908	1.59672987905	0.00000000001
0.5	1.82436063541	1.82436063535	0.00000000003
0.6	2.09327128033	2.09327128023	0.00000000004
0.7	2.40962689533	2.40962689522	0.00000000004
0.8	2.78043274286	2.78043274279	0.00000000002
0.9	3.21364280006	3.21364280004	0.00000000000
1.0	3.718281828459	3.71828182845	0.00000000000
% Mean Error			0.00000000001

Table 1 clearly shows accuracy of the Optimal Homotopy Asymptotic Method with mean percentage error 0.000000000018. Fig 1 shows the accuracy of the approximate solution. It's obvious that good accuracy is achieved with a minimum amount of computation.

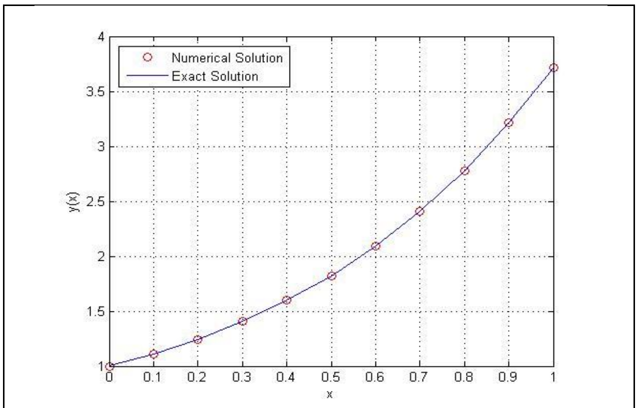

Figure 1: Comparison of Numerical and Exact solution of Example 1

Example 2:

Consider the fourth order BVP

$$y^{(iv)}(x) = 1 + \int_0^x e^{-t} y^2(t)\,dt \quad ; \quad 0 < x < 1 \tag{23}$$

With boundary condition

$$y(0)=1 \ ; \quad y'(0)=1 \ , \quad y(1)=e \quad , \quad y'(1)=e$$

By Optimal Homotopy Asymptotic technique, one can construct

$$L\{\phi(x,q)\} = y^{(iv)}(x) ;\ N\{\phi(x,q)\} = \int_0^x e^{-t} y^2(t)\,dt \ ;$$

$$g(x)=1 \tag{24}$$

which satisfies

$$(1-q)\left[\begin{pmatrix} u_o(x)+ \\ u_1(x)+ \\ u_2(x)+ \\ \ldots\ldots \end{pmatrix} + 1\right] = \begin{pmatrix} c_1 q + \\ c_2 q^2 + \\ \ldots\ldots \end{pmatrix} \left[\begin{matrix} \begin{pmatrix} u_o(x)+ \\ u_1(x)+ \\ u_2(x)+ \\ \ldots\ldots \end{pmatrix} + \\ 1 + \\ \int_0^x e^{-t} y^2(t)\,dt \end{matrix}\right] \tag{25}$$

Equating the coefficients of the same power of q gives:

$$q^0 : y_o^{(iv)}(x) = 1 \tag{26}$$

$$q^1 : y_1^{(iv)}(x,c_1) = c_1\left[\int_0^x e^{-t} y_o^2(t)\,dt\right] \tag{27}$$

$$\begin{aligned} q^2 : y_2^{(iv)}(x,c_1,c_2) - \\ y_1^{(iv)}(x,c_1) &= c_1\left[\int_0^x e^{-t}\, 2 y_o(t) y_1(t)\,dt\right] + \\ & c_2\left[\int_0^x e^{-t} y_o^2(t)\,dt\right] \end{aligned} \tag{28}$$

By solving equations (26-28), we can easily obtain $y_o(x)$, $y_1(x,c_1)$ and $y_2(x,c_1,c_2)$ which are as follows:-

$$y_o(x) = \frac{1}{24}\begin{pmatrix} x^4 - 24ex^3 + 70x^3 + \\ 48ex^2 - 119x^2 + 24x + 24 \end{pmatrix} \tag{29}$$

$$y_1(x) = \frac{c_1 e^{-(x+1)}}{576}\left\{\begin{matrix} \left(\begin{matrix} \left(-840e^x x^2(3876229x - 5490855)\right) + \\ 8064e^{x+3}(x-1)^2\left(x^2 + 11280x + 6960\right) - \\ 48e^{x+2}\begin{pmatrix} 1207x^4 + 18893878x^3 - \\ 26311097x^2 - \\ 3193200x + 8462400 \end{pmatrix} + \\ 3e^{x+1}\begin{pmatrix} 34835x^4 + 996951906x^3 - \\ 1399259085x^2 - \\ 92009368x + 245133872 \end{pmatrix} \end{matrix}\right) - \\ \left(\begin{matrix} 576e^3\begin{pmatrix} x^6 + 26x^5 + 354x^4 3080x^3 + \\ 17520x^2 + 60480x + 97440 \end{pmatrix} + \\ 48e^2\begin{pmatrix} x^7 + 103x^6 + 2411x^5 + \\ 31737x^4 + 271916x^3 + \\ 1533972x^2 + 5269200x + \\ 8462400 \end{pmatrix} - \\ e\begin{pmatrix} x^8 + 180x^7 + 10402x^6 + \\ 223208x^5 + 2846769x^4 + \\ 24024548x^3 + 13440560x^2 + \\ 459373512x + 735401616 \end{pmatrix} \end{matrix}\right) \end{matrix}\right\} \tag{30}$$

$$y_2(x) = B\left\{ D\left(\begin{matrix} -840e^x x^2\begin{pmatrix} 3876229x - \\ 5490855 \end{pmatrix} + \\ 8064e^{x+3}(x-1)^2\begin{pmatrix} x^2 + \\ 11280x + \\ 6960 \end{pmatrix} - \\ 48e^{x+2}\begin{pmatrix} 1207x^4 + 18893878x^3 - \\ 26311097x^2 - 3193200x + \\ 8462400 \end{pmatrix} + \\ 3e^{x+1}\begin{pmatrix} 34835x^4 + 996951906x^3 - \\ 1399259085x^2 - 92009368x + \\ 245133872 \end{pmatrix} - \\ 576e^3\begin{pmatrix} x^6 + 26x^5 + 354x^4 + 3080x^3 + \\ 17520x^2 + 60480x + 97440 \end{pmatrix} + \\ 48e^2\begin{pmatrix} x^7 + 103x^6 + 2411x^5 + 31737x^4 + \\ 271916x^3 + 1533972x^2 + 5269200x + \\ 8462400 \end{pmatrix} - \\ e\begin{pmatrix} x^8 + 180x^7 + 10402x^6 + 223208x^5 + \\ 2846769x^4 + 24024548x^3 + 134405604x^2 + \\ 459373512x + 735401616 \end{pmatrix} \end{matrix}\right) + \right.$$

$$
C\left\{\left(\left(\begin{array}{l}
-840e^{x}x^{2}(3876229x-5490855)+\\
8064e^{x+3}(x-1)^{2}\left(x^{2}+11280x+6960\right)-\\
48e^{x+2}\begin{pmatrix}1207x^{4}+18893978x^{3}-\\ 26311097x^{2}-3193200x+\\ 8462400\end{pmatrix}+\\
3e^{x+1}\begin{pmatrix}34835x^{4}+996951906x^{3}-\\ 1399259085x^{2}-\\ 92009368x+245133872\end{pmatrix}-\\
576e^{3}\begin{pmatrix}x^{6}+26x^{5}+354x^{4}+\\ 3080x^{3}+17520x^{2}+\\ 60480x+97440\end{pmatrix}
\end{array}\right)+\right.\right.
$$

$$
\left.\left.\left(\begin{array}{l}
48e^{2}\begin{pmatrix}x^{7}+103x^{6}+2411x^{5}+\\ 31737x^{4}+271916x^{3}+\\ 1533972x^{2}+5269200x+\\ 8462400\end{pmatrix}-\\
e\begin{pmatrix}x^{8}+180x^{7}+10402x^{6}+\\ 223208x^{5}+2846769x^{4}+\\ 24024548x^{3}+134405604x^{2}+\\ 459373512x+735401616\end{pmatrix}
\end{array}\right)\right)\right\}+
$$

$$
B=\frac{e^{-2(x+1)}}{442368},
$$

$$
C=768c_{2}e^{x+1},
$$

$$
D=768c_{1}e^{x+1}
$$

$$
c_{1}^{2}\left(\left(\begin{array}{l}
105e^{2x}x^{2}\begin{pmatrix}79318153591579423x-\\ 11227583492230458\end{pmatrix}-\\
6912e^{2x+5}(x-1)^{2}\begin{pmatrix}319262696x^{2}+\\ 341956261730x+\\ 211416784105\end{pmatrix}+\\
864e^{2x+4}\begin{pmatrix}34663718962x^{4}+446139192032118x^{3}-\\ 61906962809449x^{2}-\\ 87753618746840x+22969842805725\end{pmatrix}-\\
30e^{2x+1}\begin{pmatrix}947426746120x^{4}+43378899036780394x^{3}-\\ 61065336887690631x^{2}-\\ 23940660116402880x+62896910612336640\end{pmatrix}-\\
6e^{2x+3}+\begin{pmatrix}25295678399352x^{4}+417281268056946134x^{3}-\\ 58160441380624349x^{2}-\\ 63996641466428840x+167729890158595335\end{pmatrix}\\
3e^{2x+2}\begin{pmatrix}113383947602937x^{4}+26974997501616137009x^{3}-\\ 37778091984056181443x^{2}-\\ 28667842608597389x+75228762237370806\end{pmatrix}+\\
12386304e^{x+5}\begin{pmatrix}x^{7}+11311x^{6}+300755x^{5}+4156233x^{4}+\\ 36554900x^{3}+\\ 209657340x^{2}+728466240x+1179821040\end{pmatrix}+\\
53760e^{x+1}\begin{pmatrix}3876229x^{7}+401513190x^{6}+9571748419x^{5}+\\ 127726275166x^{4}+1105672748396x^{3}+\\ 62878997074780x^{2}+217389790646280x+\\ 3509872249920\end{pmatrix}-\\
73728e^{x+4}\begin{pmatrix}7x^{8}+80923x^{7}+27136043x^{6}+698961901x^{5}+\\ 9566187078x^{4}+83774515792x^{3}+\\ 4793856900408x^{2}+1663432388700x+\\ 2691853233600\end{pmatrix}+\\
1536e^{x+3}\begin{pmatrix}2414x^{8}+38157801x^{7}+6917051363x^{6}+\\ 17312726181x^{5}+2348035642947x^{4}+\\ 20478351865132x^{3}+116927921298372x^{2}+\\ 405210829711440x+6552082342020560\end{pmatrix}-\\
192e^{x+2}\begin{pmatrix}34835x^{8}+1000783756x^{7}+1294394117070x^{6}+\\ 315729856928x^{5}+42460035091011x^{4}+\\ 368883677336156x^{3}+210189616391300x^{2}+\\ 7275170322848776x+11754626533070208\end{pmatrix}
\end{array}\right)+\right)+
$$

$$\left.\left.\left(\begin{array}{l} -6912e^{5}\left(\begin{array}{l} 4x^{9}+186x^{8}+4208x^{7}+ \\ 59608x^{6}+572060x^{5}+3790750x^{4}+ \\ 17086080x^{3}+50173560x^{2}+ \\ 88188030x+71462475 \end{array}\right)+ \\ 288e^{4}\left(\begin{array}{l} 12x^{10}+1492x^{9}+56730x^{8}+ \\ 1191320x^{7}+16261064x^{6}+ \\ 152894400x^{5}+1002167772x^{4}+ \\ 4499173440x^{3}+13236002550x^{2}+ \\ 23404938507x+19142584425 \end{array}\right)- \\ 6e^{3}\left(\begin{array}{l} 24x^{11}+4852x^{10}+347068x^{9}+ \\ 11378646x^{8}+223956392x^{7}+ \\ 2953554620x^{6}+27232128021x^{5}+ \\ 176621077926x^{4}+789910563438x^{3}+ \\ 2328201749942x^{2}+4141505711070x+ \\ 3417796748025 \end{array}\right)+ \\ e^{2}\left(\begin{array}{l} 2x^{12}+560x^{11}+60506x^{10}+3221854x^{9}+ \\ 94039110x^{8}+1747905008x^{7}+ \\ 22324233660x^{6}+202007479996x^{5}+ \\ 1296837219491x^{4}+ \\ 5778786330690x^{3}+17065989071685x^{2}+ \\ 30537630812259x+25421577376518 \end{array}\right) \end{array}\right)\right)\right) \quad (31)$$

Calculate second order approximate solution by putting the values of $y_o(x)$ $y_1(x,c_1)$ and $y_2(x,c_1,c_2)$ in (23). The Residual has been achieved as

$$R(x,c_1,c_2)=y^2(x,c_1,c_2)+1+\int_0^x e^{-t}y^2(t)dt \qquad (32)$$

Using method of least square, one may easily get values of c_1 & c_2 as

$$c_1=0.3187036408266732,\quad c_2=0.3638105496011505$$

Comparison of the numerical results with exact solution $y(x)$ and the percentage error of Example 2 is given in Table 2. The algorithm produces results which are of excellent accuracy

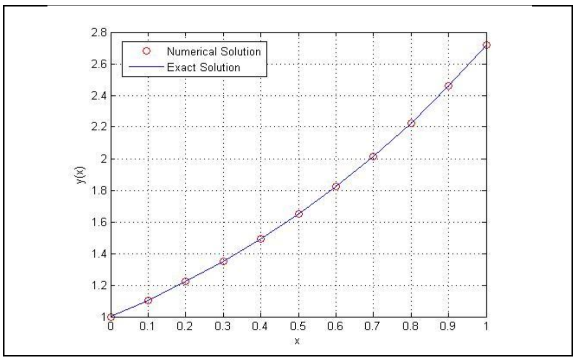

Figure 2: Comparison of Numerical and Exact solution of Example 2

Table 2: Percentage error of Example 2

x	Numerical Solution	Exact Solution	% Error
0.00	1.000000000	1.000000000	0.000000000
0.10	1.105170945	1.105170918	0.000000024
0.20	1.221402835	1.221402758	0.000000063
0.30	1.349858920	1.349858808	0.000000083
0.40	1.491824815	1.491824698	0.000000079
0.50	1.648721366	1.648721271	0.000000058
0.60	1.822118859	1.822118800	0.000000032
0.70	2.013752734	2.013752707	0.000000013
0.80	2.225540937	2.225540928	0.000000004
0.90	2.459603113	2.459603111	0.000000001
1.00	2.718281828	2.718281828	0.000000000
% Mean Error			0.000000032

Fig 2 shows the numerical result of exact solution and OHAM solution, it is clear that the results are in excellent agreement.

4. DISCUSSION & CONCLUSION

Integro-differential equations are typically cumbersome and hard to solve analytically; so, it is required to obtain the approximate solution. In this paper, we proposed the optimal Homotopy Asymptotic Method (OHAM) for solving fourth order Integro-differential equations. From our obtained results, we have been able to conclude that the proposed method gives solutions in excellent agreement with the exact solution and better than the other methods. Optimal Homotopy Asymptotic Method (OHAM) provides a simple and easy way to control and adjust the convergence region for strong nonlinearity and is also applicable to higher order Integro-differential equations. All computation has been conducted using MATHEMATICA 9.0.

5. CONFLICT OF INTERESTS

The authors declare that there is no conflict of interest regarding the publication of this paper.

6. ACKNOWLEDGMENTS

The authors would like to thank the reviewers for the comments which led to an improvement in the paper. Special Thanks to Ms Wishaal Khalid for proof reading.

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