Quartic Spline Interpolation

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ABSTRACT
In this paper, we have obtained existence, uniqueness, and error bound of deficient quartic spline interpolation.

Keywords and Phrases
Deficient, Quartic Spline, Interpolation Error Bounds

1. INTRODUCTION
Piecewise linear and higher degree interpolation are widely used schemes for Piecewise Polynomial approximation. But at joint of two linear pieces, piecewise linear functions have corners and therefore to achieve a prescribed accuracy usually more data are required then higher order method therefore higher order method are beneficial for best approximation. In the direction of higher order method, Kopotun [4] has obtained univariate splines equivalence of moduli of smoothness and application. Marker and Remier [5] have investigate an unconditionally convergent method for computing zero's of splines and polynomials.(also we refer to Howell and Varma [6], Dikshit and Rana [2] ,Rana [7.8], Agrwal and Wong [9] and Gmeling –Mayling [10])

2. EXISTENCE AND UNIQUENESS
Let a mesh on [0, 1] be given by $P: 0 = x_0 {}_j < x_1 < .... < x_n = 1$ which such that

$h_i = x_{1+i} - x_i$ for i = 0, 1,....n-1. Let π_4 denotes the set of all algebraic polynomials of degree not greater than 4. For a function s defined over p, we denote the restriction of s over $\lfloor x_i, x_{i+1} \rfloor$ by s_i. The class $S(4,P)$ of deficient quartic splines defined over p is given by

$S(4,P) = \{s : s \in C^1[0,1], s_i \in \pi_4$ for i = 0, 1,....n-1}

Where in $S^*(4,P)$ denotes the class of all deficient quartic spline $S(4,P)$ which satisfies the boundary condition.

$$s(x_0) = f(x_0)$$

$$s(x_n) = f(x_n) \qquad (2.1)$$

For a given function *f*, are introduced the following interpolatory condition.

$$s(\alpha_i) = f(\alpha_i) \qquad (2.2)$$

$$s(\beta_i) = f(\beta_i) \qquad (2.3)$$

$$s'(\gamma_i) = f'(\gamma_i) \qquad (2.4)$$

Where $\alpha_i = x_{i-1} + \frac{1}{3} h_i$

$\beta_i = x_{i-1} + \frac{1}{2} h_{i-1} = \gamma_i$ for i = 1,2,......n

Problem 1.1 : For given functional values and derivative $f(\alpha_i), f(\beta_i), f'(\gamma_i)$ along with $f(x_0)$ and $f(x_n)$. There exist a unique $s \in S(4,P)$ which satisfy (2.2) - (2.4) condition.

Let *Q(Z)* be a Polynomial of degree 4 on [0,1], then it is easy to verify that

$$Q(z) = Q\left(\frac{1}{3}\right)P_1(t) + Q\left(\frac{1}{2}\right)P_2(t) + Q'\left(\frac{1}{2}\right)P_3(t) + Q(0)P_4(t) + Q(1)P_5(t) \qquad (2.5)$$

Where $P_1(t) = t\left[\frac{81}{2} - \frac{405}{2}t + 324t^2 - 162t^3\right]$

$$P_2(t) = t[32 - 176t + 288t^2 - 144t^3]$$

$$P_3(t) = t\left[4 - 24t + 44t^2 - 24t^3\right]$$

$$P_4(t) = \left[1 - 8t + 23t^2 - 28t^3 + 12t^4\right]$$

$$P_5(t) = \left[-\frac{t}{2} + \frac{7}{2}t^2 - 8t^3 + 6t^4\right] \qquad (2.6)$$

We are now set to answer Problem 1.1 in theorem 2.1.

Theorem 2.1: There exist a unique deficient quartic spline in $S^*(4,P)$ which satisfies the interpolatory condition (2.2) - (2.4).

Proof of Theorem 2.1 : Let $t = \frac{(x - x_i)}{h_i}$ $0 \le t \le 1$ then in view of condition (2.1) - (2.4), we now express equation (2.5) in terms of restriction S_i of s in $\lfloor x_i, x_{i+1} \rfloor$ as follows :-

$$s_i(x) = f(\alpha_i)P_1(t) + f(\beta_i)P_2(t) + h_i f'(\gamma_i)P_3(t) + s(x_i)$$
$$P_4(t) + s(x_{i+1})P_5(t) \qquad (2.7)$$

Since $s \in C^1[a,b]$, we have form (2.7)

$$2h_i\, s_{i-1} + s_i(x)\left(8h_{i-1} - \frac{13}{2}h_i\right) + \frac{1}{2}h_{i-1}\, s_{i+1}$$

$$= \frac{81}{2}\left[h_{i-1}\, f(\alpha_i) - f(\alpha_{i-1})h_i\right] + 32\left[h_i f(\beta_i) - h_{i-1} f(\beta_{i-1})\right]$$

$$+ \left[8h_i\, f'(\gamma_{i-1}) + 4h_{i-1}\, f'(\gamma_i)\right] \qquad (2.8)$$

We can easily see that excess of the absolute value of the coefficient of $s(x_i) = m_i$ (say) dominant for the sum of the absolute values of the coefficient of m_{i-1} and m_{i+1} in (2.8) under the condition of Theorem 2.1. Therefore the coefficient matrix of the system of equation (2.8) is diagonally dominant and hence invertible. Thus, the system of equation has unique solution, this complete the proof of theorem (2.1).

3. ERROR BOUNDS

In this section of the paper error bounds i.e. $e^r(x)=f^{(r)}-s^r(x)$ r = 0,1 are obtained for the spline interpolant of Theorem 2.1 by following approach used by Hall and Meyer [3]. We shall denote by $L_i[f,x]$ the unique quartic agreeing with

$f(\alpha_i), f(\beta_i), f'(\gamma_i), f(x_i) \& f(x_{i+1})$

and let $f \in C^5[0,1]$. Now consider a first continuously differentiable quartic spline s of theorem 2.1. We have for $x_i \le x \le x_{i+1}$

$$|f(x)-s(x)|=|f(x)-s_i(x)|$$

$$\le |f(x)-L_i[f,x]|+|L_i[f,x]-s_i(x)| \qquad (3.1)$$

Thus it is clear from (3.1) that in order to get the bounds of e(x), we have to estimate pointwise bounds of both the terms on the right hand side of (3.1). By a well known remainder theorem of Cauchy (See Davis [1]), we see that

$$|f(x)-Li[f,x]\le F\frac{h_i^5}{5!}\left|t\left(t-\frac{1}{3}\right)\left(t-\frac{1}{2}\right)^2(1-t)\right| \qquad (3.2)$$

Where $F=\max_{0\le x\le 1}\left|f^{(5)}(x)\right|$

We next proceed to obtain bound for $|L_i[f,x]-s_i(x)|$. It follows from (2.4) that

$$|L_i[f,x]-s_i(x)|\le|e(x_i)P_4(t)+e(x_{i+1})P_5(t)| \qquad (3.3)$$

Thus

$$|L_i[f,x]-s_i(x)|\le|e(x_i)||P_4(t)|+|e(x_{i+1})||P_5(t)| \qquad (3.4)$$

Now since $P_4(t)=\left[1-8t+23t^2-28t^3+12t^4\right]$

and $P_5(t)=\left[-\frac{t}{2}-\frac{7}{2}t^2-8t^3+6t^4\right]$ therefore

$$|P_4(t)|+|P_5(t)|=\left|(1-2t)^2(1-3t)\left(1-\frac{t}{2}\right)\right| \quad 0\le t\le 1$$

$$=K(t) \qquad \text{(Say)} \qquad (3.5)$$

Now, using (3.5) in (3.4), we have

$$|L_i[f,x]-s(x)|\le\max\{|e(x_i)|,|e(x_{i+1})|\}k(t) \quad (3.6)$$

Setting $\left|e(x_j)\right|=\max_{i=1,2.....n-1}|e(x_i)|$

and $h=\max_{i=1,2....n-1} h_i \qquad (3.7)$

We see that (3.6) may be written as

$$\left|L_i[f,x]-s(x)\right|\le\left|e(x_j)k(t)\right| \qquad (3.8)$$

It is clear from (3.8) that in order to estimate the bounds of $|e(x)|$ first we have to obtain the upper bounds of $\left|e(x_j)\right|$.

Replacing $s(x_j)$ by $e(x_j)$ in (2.8), we get

$$2h_j e_{j-1}+e_j(x)\left(8h_{j-1}-\frac{13}{2}h_j\right)+\frac{1}{2}h_{j-1}e_{j+1}$$

$$=\frac{81}{2}\left[h_{j-1}f(\alpha_j)-h_j f(\alpha_{i-1})\right]+32\left[h_{j-1}f(\beta_j)-h_j f(\beta_{j-1})\right]$$

$$+\left[8h_j f'(\gamma_{j-1})+4h_{j-1}f'(\gamma_j)\right]-2h_j f(x_{j-1})-\left(8h_{n-1}-\frac{13}{2}h_j\right)f(x_j)$$

$$-\frac{1}{2}h_{j-1}f(x_{j+1})=E(f) \qquad (3.9)$$

Where $E\left[(x-y)^4\right]=\frac{81}{2}\left[h_{j-1}(\alpha_j-y)_+^4-h_j(\alpha_{j-1}-y)_+^4\right]$

$$+32\left[h_{j-1}(\beta_j-y)_+^4-h_j(\beta_{j-1}-y)_+^4\right]+32h_jh_{j-1}(\gamma_{j-1}-y)_+^3+$$

$$+16h_jh_{j-1}(\gamma_j-y)_+^3-\left(8h_{j-1}-\frac{13}{2}h_j\right)$$

$$(x_j-y)_+^4-\frac{1}{2}h_{j-1}\left(x_{j+1}-y\right)_+^4 \qquad (3.10)$$

Observing that $E(f)$ is a linear functional which is zero for polynomials of degree 4 or less and applying the Peano theorem (See Davis [1]). We have

$$E(f)=\int_{x_{j-1}}^{x_{j+1}}\frac{f^5(y)}{4!}E\left[(x-y)_+^4\right]dy \qquad (3.11)$$

Now from (3.11), it follows that

$$|E(f)|=\frac{1}{4!}F\int_{x_{j-1}}^{x_{j+1}}\left|E(x-y)_+^4\right|dy \qquad (3.12)$$

In order to evaluate the integral of the right hand side of (3.12) we rewrite the expression (3.10) in the following symmetric form of x_j, thus

$$E\left[(x-y)_+^4\right]$$

$$=-\frac{1}{2}h_{j-1}\left[x_j+h_j-y\right]^4$$

$$\beta_j\le y\le x_{j+1}$$

$$=h_{j-1}\left[\frac{63}{2}(x_j-y)^4+78h_j(x_j-y)^3+69h_j^2(x_j-y)^2+ \atop 26(x_j-y)h_j^3+\frac{7}{2}h_j^4\right]$$

$$\alpha_j\le y\le\beta_j=\gamma_j$$

$$= h_{j-1}\begin{bmatrix} 72(x_j - y)^4 + 132(x_j - y)^3 h_j + 96(x_j - y)^2 h_j^2 + \\ 32(x_j - y)h_j^3 + 4h_j^4 \end{bmatrix}$$
$$x_j \le y \le \alpha_j$$

$$= -h_i\left[\frac{63}{2}(x_j - y)^4 - 78h_{j-1}(x_j - y)^3 + 69h_{j-1}^2(x_j - y)^2 - 26h_{j-1}^3(x_j - y) + \frac{7}{2}h_{j-1}^4\right] \gamma_{i-1} = \beta_{i-1} \le y \le x_j$$

$$= -h_i\left[72(x_j - y)^4 - 132(x_i - y)^3 h_{j-1} + 96(x_i - y)^2 h_{j-1}^2 - 32(x_j - y)h_{j-1}^3 + 4h_{j-1}^4\right]$$
$$\alpha_{j-1} \le y \le \beta_{j-1} \le \gamma_{j-1}$$

$$\Rightarrow \frac{1}{2}h_j\left[x_y - y - h_{j-1}\right]^4 \qquad x_{j-1} \le y \le \alpha_{j-1} \quad (3.13)$$

From the above expression it is follows that $E\left[(x-y)^4\right]$ is non-negative for $x_{j-1} \le y \le x_{j+1}$.

Thus, we see that

$$\int_{x_{j-1}}^{x_{j+1}} \left|E(x-y)_+^4\right| dy = 4\left[h_j h_{j-1}^5 + h_{j-1} h_j^5\right] \quad (3.14)$$

Thus, we have following from (3.12) when, we appeal to (3.14).

$$|E(f)| \le \frac{F\, h_{j-1} h_j\left[h_{j-1}^4 + h_j^4\right]}{3!} \quad (3.15)$$

Combining (3.7), (3.9), (3.12) with (3.15) we have

$$\max_{i=1,2,.....n-1}\left|e(x_j)\right| = \left|e_j\right| \le \frac{F\, h_j h_{j-1}\left[h_{j-1}^4 + h_j^4\right]}{3!\left[\frac{9}{2}h_j + \frac{17}{2}h_{j-1}\right]} \quad (3.16)$$

Now making the use of equation (3.2) and (3.7) in (3.1) and then using (3.16) along with (3.8) we see that

$$|e(x)| \le F\frac{h^5}{5!}\left|t\left(t-\frac{1}{3}\right)\left(t-\frac{1}{2}\right)^2(1-t)\right| + \left|e(x_j)\right|k(t) \quad (3.17)$$

$$= \frac{h^5}{5!}F\left|t\left(t-\frac{1}{3}\right)\left(t-\frac{1}{2}\right)^2(1-t)\right| + \frac{Fh^5}{39}K(t)$$

$$= h^5 F\,|C(t)| \quad (3.18)$$

Where $\left|C(t) = \left[\frac{1}{5!}\left|t\left(1-\frac{1}{3}\right)\left(t-\frac{1}{2}\right)^2(1-t)\right|\right] + \frac{K(t)}{39}\right|$

Thus, we prove the following.

Theorem 3.1 : Suppose *s(x)* is the quartic spline interpolant of Theorem 2.1 and $f \in C^5[0,1]$ then

$$|e(x)| \le K\frac{h^5}{5!}\left|f^5(x)\right| \quad (3.19)$$

Where $K = \max_{0 \le t \le 1}|C(t)|$ given by (3.18) Also we have

$$\left|e(x_j)\right| \le \frac{h^5}{39}\max_{0 \le x \le 1} f^5(x) \quad (3.20)$$

Equations (3.18) and (3.16) respectively prove the inequality (3.20) and (3.19) of Theorem 3.1.

Now, we shall show that inequality (3.19) is best possible in the limit. Consider $f(x) = \frac{x^5}{5!}$, we can easily see that by Cauchy formula given in [1] that

$$\frac{x^5}{5!} - L_i\left[\frac{t^5}{5!}, x\right] = \frac{h^5}{5!}\left[t(1-t)\left(t-\frac{1}{3}\right)\left(1-\frac{1}{2}\right)^2\right] \quad (3.21)$$

Moreover, for equally spaced knots we have from (3.9) that

$$E\left(\frac{x^5}{5!}\right) = 2e_{j-1} + \frac{3}{2}e_i + \frac{1}{2}e_{j+1} = \frac{h^5}{3} \quad (3.22)$$

Consider for a moment

$$e(x_j) = \frac{h^5}{12} = e(x_{j-1}) = e(x_{j+1}) \quad (3.23)$$

We have from (3.8)

$$L_i[f,x] - s(x) = \frac{h^5}{12}\left(Q_4(t) + Q_5(t)\right) = \frac{h^5}{12}K(t) \quad (3.24)$$

Combine (3.21) and (3.24) we have

$$f(x) - s(x) = \frac{h^5}{12}\left[\frac{t(1-t)\left(t-\frac{1}{2}\right)^2\left(t-\frac{1}{3}\right)}{10} + K(t)\right] \quad (3.25)$$

From (3.25), it is clearly observed that (3.19) is best possible proved that we could prove that

$$e(x_{j-1}) = e(x_j) = e(x_{j+1}) = \frac{h^5}{12} \quad (3.26)$$

In fact (3.26) is attained only in the limit, the difficulty will take place in the boundary condition $e(x_0) = e(x_n) = 0$. However it can be shown that as we move many subinterval away from the boundaries $e(x_j) \to \frac{h^5}{12}$. For that we shall apply (3.22) indutively to move away from the end condition $e(x_0) = e(x_n) = 0$.

First step in this direction is to show that $e(x_j) \ge 0$ for j=0,......n which can be prove by contradiction assumption.

Let $e(x_j)<0$ for some j=1,....n-1.

Now, a making use of (3.20)

$$\frac{h^5}{39} \geq \left|e(x_j)\right| > 2e_{j-1} + \frac{3}{2}e_j + \frac{1}{2}e_{j+1} = \frac{h^5}{3} \text{ i.e. } 3 \geq 39$$

This is a contradiction. Hence $e(x_j) \geq 0$ for j = 0,.....n.

Now from equation (3.22)

$$\frac{3}{2}e_j = \frac{h^5}{3} - 2e_{j-1} - \frac{1}{2}e_{j+1}$$

Since $e_j \geq 0$

$$\Rightarrow e_j \leq \frac{2}{9}h^5 \text{ for j = 1,2,.....n-1} \qquad (3.27)$$

Now again using (3.27) in (3.22) we have

$$e(x_i) \leq \frac{2}{3}\frac{h^5}{3}\left[1 - \frac{5}{3}\right]$$

Repeated use of (3.22) follows that

$$e(x_j) \leq \frac{2h^5}{9}\left[1 - \frac{5}{3} + \left(\frac{5}{3}\right)^2 .....\right] \qquad (3.28)$$

Now it can be easily see that r.h.s. of (3.28) $\rightarrow \frac{h^5}{12}$ and hence in the limiting case

$$e(x_j) \rightarrow \frac{h^5}{12} \qquad (3.29)$$

which verifies (3.19) inequality. Thus corresponding to the function $f(x) = \frac{x^5}{5!}$, (3.28) imply $\rightarrow \frac{h^5}{12}$ in the limit for equally spaced knots. This completes the proof of theorem 3.1.

4. CONCLUSION

In this paper, we have obtained existence, uniqueness, and error bound of deficient quartic spline interpolation.

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