Line Graphs and Quasi-total Graphs

Bhavanari Satyanarayana
Department of Mathematics,
Acharya Nagarjuna University,
Nagarjuna Nagar – 522 510,
A. P., India.

Devanaboina Srinivasulu
Department of Mathematics,
Acharya Nagarjuna University,
Nagarjuna Nagar – 522 510,
A. P., India.

Kuncham Syam Prasad
Department of Mathematics,
Manipal Institute of Technology,
Manipal University,
Manipal – 576 104, India,

ABSTRACT
The line graph, 1-quasitotal graph and 2-quasitotal graph are well-known. It is proved that if G is a graph consist of exactly m connected components G_i, $1 \le i \le m$, then $L(G) = L(G_1) \oplus L(G_2) \oplus \ldots \oplus L(G_m)$ where L(G) denotes the line graph of G, and '$\oplus$' denotes the ring sum operation on graphs. The number of connected components in G is equal to the number of connected components in L(G) and also if G is a cycle of length n, then L(G) is also a cycle of length n. The concept of 1-quasitotal graph is introduced and obtained that $Q_1(G) = G \oplus L(G)$ where $Q_1(G)$ denotes 1-quasitotal graph of a given graph G. It is also proved that for a 2-quasitotal graph of G, the two conditions (i) $|E(G)|= 1$; and (ii) $Q_2(G)$ contains unique triangle are equivalent.

General Terms
Graph Theory, line graphs, ring sum operation on graphs.

Keywords
Line graph, quasi total graph, connected component.

1. LINE GRAPHS
All graphs considered are finite and simple. For standard literature on graph theory we refer Bondy and Murty [1], Harary [2], Satyanarayana and Syam Prasad [7, 8].

We start this section with the following remark.

1.1 ***Remark***: Let G be a graph with $E(G) \neq \phi$, and L(G) its line graph.

(i) $V(G) \cap V(L(G)) = \phi$.

(ii) $E(G) \cap E(L(G)) = \phi$.

1.2 ***Lemma***: Let G be a graph. Suppose G_1, G_2 are two connected components of a graph G. Write $G = G_1 \oplus G_2$. Suppose e_1, $e_2 \in G$ such that $s = \overline{e_1e_2} \in E(L(G))$. If $e_1 \in E(G_1)$, then $e_2 \in E(G_1)$ but not $e_2 \in E(G_2)$ (In other words, if $s_1 \in E(G_1)$ and $s_2 \in E(G_2)$, then s_1 and s_2 cannot be adjacent in L(G)).

Proof: Suppose $e_1 \in G_1$. Since $e_2 \in E(G) = E(G_1) \cup E(G_2)$, either $e_2 \in E(G_1)$ or $e_2 \in E(G_2)$. Now to show that $e_2 \notin E(G_2)$. If possible, suppose that $e_2 \in E(G_2)$.

Since $s = \overline{e_1e_2} \in E(L(G))$, we have that e_1 is adjacent to e_2. Then $e_1 = \overline{v_1v_2}$ and $e_2 = \overline{v_2v_3}$ for some $v_1, v_2, v_3 \in V(G)$. Since $e_1 \in E(G_1)$ and $e_2 \in E(G_2)$, we have that $v_1, v_2 \in V(G_1)$, $v_2, v_3 \in V(G_2)$. So $v_2 \in V(G_1) \cap V(G_2)$.

Take $x, y \in G_1 \cup G_2$. If $x, y \in G_1$ (or G_2), then since G_1 (or G_2) is a connected component, we have that there is a path from x to y. Suppose $x \in G_1$ and $y \in G_2$.

Since $x, v_2 \in G_1$, there is a path from x to v_2; and since $v_2, y \in G_2$ there is a path from v_2 to y in G_2. These paths combined together must provide a path from x to y. This shows that $G_1 \cup G_2$ is connected, a contradiction to the fact that G_1 and G_2 are two different connected components of a graph. Hence $e_2 \notin E(G_2)$ and $e_2 \in E(G_1)$.

1.3 ***Lemma***: If $G = G_1 \oplus G_2$ where G_1 and G_2 are two connected graphs with $V(G_1) \cap V(G_2) = \phi$, then $L(G) = L(G_1) \oplus L(G_2)$.

Proof: Since $G_1 \subseteq G$, $E(G_1) \subseteq E(G)$.

Also $E(G_2) \subseteq E(G)$. $V(L(G_1)) \cup V(L(G_2)) =$
$E(G_1) \cup E(G_2) = E(G) = V(L(G))$ … (i)

Let $s \in E(L(G_1))$. Then $s = \overline{e_1e_2}$ where $e_1, e_2 \in E(G_1)$, and e_1 and e_2 are adjacent in $G_1 \Rightarrow s = \overline{e_1e_2}$ and $e_1, e_2 \in E(G_1) \subseteq E(G)$, and e_1, e_2 are adjacent in $G_1 \subseteq G$.

$\Rightarrow s \in E(L(G))$. Therefore $E(L(G_1)) \subseteq E(L(G))$.

In a similar way we get $E(L(G_2)) \subseteq E(L(G))$.

So $L(G_1) \cup L(G_2) \subseteq L(G)$ …(ii)

Let $s \in E(L(G))$. Then $s = \overline{e_1e_2}$ for some $e_1, e_2 \in V(L(G)) = E(G)$, and e_1, e_2 are adjacent in G.

By Lemma 1.2, $e_1, e_2 \in E(G_1)$ or $e_1, e_2 \in E(G_2)$, but not both. If $e_1, e_2 \in E(G_1)$, then since e_1, e_2 are adjacent in G, e_1, e_2 are adjacent in G_1 and so $s = \overline{e_1e_2} \in E(L(G_1))$.

Similarly, if $e_1, e_2 \in E(G_2)$, then $s = \overline{e_1e_2} \in E(L(G_2))$.
Hence $E(L(G)) \subseteq E(L(G_1)) \cup E(L(G_2))$ … (iii)

From (ii) and (iii), $E(L(G)) = E(L(G_1)) \cup E(L(G_2))$ … (iv)

Since $G = G_1 \oplus G_2$, it follows that $E(G_1) \cap E(G_2) = \phi$.

Now from (i) and (iv), $L(G) = L(G_1) \oplus L(G_2)$, the proof is complete.

1.4 ***Example***: Consider the graphs G_1 and G_2 given in Fig. 1 and Fig. 2 respectively.

Fig 1

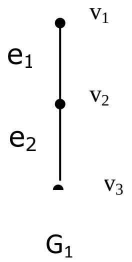
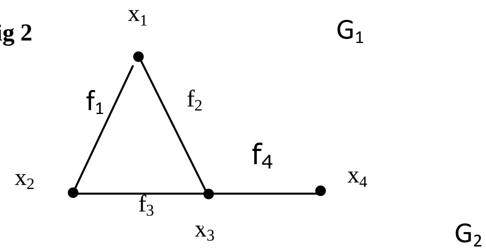


Fig 2



The ring sum of G1 and G_2 is given in Fig. 3

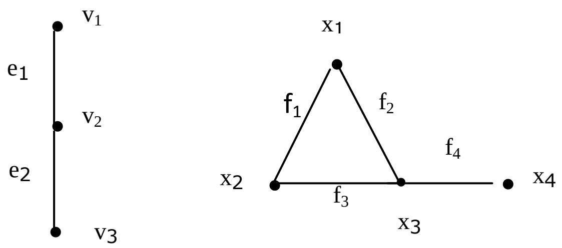
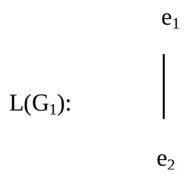


Fig 3 $G = G_1 \oplus G_2$

The line graph of G_1 and G_2 are given in Fig. 4 and Fig. 5 respectively.

Fig 4



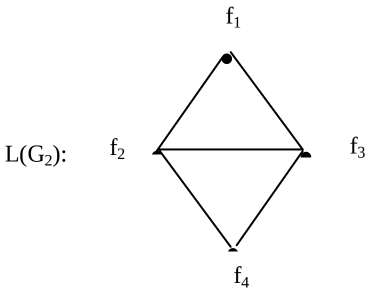


Fig 5

The line graph L(G) is given in Fig. 6

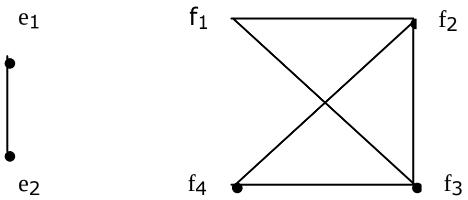


Fig 6

Now let us construct the ring sum of $L(G_1)$ and $L(G_2)$.

$V(L(G_1) \oplus L(G_2)) = \{e_1, e_2, f_1, f_2, f_3, f_4\}$, and

$E(L(G_1) \oplus L(G_2)) = \{\overline{e_1e_2}, \overline{f_1f_2}, \overline{f_1f_3}, \overline{f_2f_3}, \overline{f_3f_4}, \overline{f_2f_4}\}$.

The graph $L(G_1) \oplus L(G_2)$ is same as the graph given in Fig.6.

It is an easy observation that $L(G) = L(G_1) \oplus L(G_2)$.

1.5 ***Theorem***: If G is a graph consists of exactly m connected components G1, $G_2, \ldots, G_m$, then $L(G) = L(G_1) \oplus L(G_2) \oplus \ldots \oplus L(G_m)$.

Proof: The proof is by induction on m.

If m = 2, then it follows through the above Lemma 1.3. Suppose that the result is true for m = k.

Now take a graph G with m = k + 1 connected components $G_1, G_2, \ldots, G_{k+1}$.

Now $G = G_1 \oplus G_2 \oplus \ldots \oplus G_k \oplus G_{k+1} = (G_1 \oplus G_2 \oplus \ldots \oplus G_k) \oplus G_{k+1}$.

Now $L(G) = L((G_1 \oplus G_2 \oplus \ldots \oplus G_k) \oplus (G_{k+1}))$

$= L(G_1 \oplus G_2 \oplus \ldots \oplus G_k) + L(G_{k+1})$ (by Lemma 1.3)

$= L(G_1) \oplus L(G_2) \oplus \ldots \oplus L(G_k) \oplus L(G_{k+1})$ (by induction hypothesis), the proof is complete.

1.6 ***Lemma***: If G is a connected graph, then L(G) is also a connected graph.

Proof: Let G be a connected graph. To show that L(G) is connected, let $e_1, e_2 \in V(L(G)) = E(G)$.

Suppose $e_1 = \overline{uv}$ and $e_2 = \overline{xy}$ for some $u, v, x, y \in V(G)$. Since G is connected, there exists a path from v to x. Suppose this path is $vf_1v_1f_2v_2 \ldots f_kv_k$ with $v_k = x$.

Since f_1 is adjacent to f_2, f_2 is adjacent to f_3, … f_{k-1} is adjacent to f_k, it follows that $\overline{f_1f_2}, \overline{f_2f_3}, \ldots \overline{f_{k-1}f_k}$ is a path from f_1 to f_k in L(G).

If $e_1 = f_1$ and $f_k = e_2$, then $\overline{f_1f_2}, \overline{f_2f_3}, \ldots, \overline{f_{k-1}f_k}$ is a path from e_1 to e_2 in L(G).

If $e_1 \neq f_1$ and $f_k = e_2$, then $\overline{e_1f_1}, \overline{f_1f_2}, \overline{f_2f_3}, \ldots, \overline{f_{k-1}f_k}$ is a path from e_1 to e_2 in L(G).

If $e_1 \neq f_1$ and $f_k \neq e_2$, then $\overline{e_1f_1}, \overline{f_1f_2}, \ldots, \overline{f_{k-1}f_k}, \overline{f_ke_2}$ is a path from e_1 to e_2 in L(G).

If $e_1 = f_1$ and $f_k \neq e_2$, then $\overline{f_1f_2}, \overline{f_2f_3}, \ldots, \overline{f_{k-1}f_k}, \overline{f_ke_2}$ is a path from e_1 to e_2 in L(G).

Hence for any $e_1, e_2 \in V(L(G))$, there is a path between e_1 & e_2 in L(G). This shows that L(G) is connected, the proof is complete.

1.7 ***Theorem***: The number of connected components in G is equal to the number of connected components in L(G).

Proof: Suppose the connected components of G are $G_1, G_2, \ldots, G_k$.

Then $G = G_1 \oplus G_2 \oplus \ldots \oplus G_k$.

By Theorem 1.6, $L(G) = L(G_1) \oplus L(G_2) \oplus \ldots \oplus L(G_k)$.

Since G_i is connected by Lemma1.6, the graph $L(G_i)$ is connected and so $L(G_i)$ is a connected component of L(G).

Hence $L(G) = L(G_1) \oplus L(G_2) \oplus \ldots \oplus L(G_k)$ and each $L(G_i)$ is connected.

Thus the number of components of G = k = the number of components of L(G), the proof is complete.

1.8 ***Theorem***: If G is a cycle of length n, then L(G) is also a cycle of length n.

Proof: Suppose G is a cycle of length n.

Then $V(G) = \{v_1, v_2, \ldots, v_n\}$, and $E(G) = \{e_1, e_2, \ldots, e_n\}$ with the cycle $v_1e_1v_2e_2\ldots v_ne_nv_1$.

Now $V(L(G)) = E(G) = \{e_1, e_2, \ldots, e_n\}$.

Since e_{i-1} and e_i for $2 \leq i \leq n$ are adjacent in G, we get that $\overline{e_{i-1}e_i} \in L(G)$ for $2 \leq i \leq n$. Since e_1 and e_n have common vertex v_1 in G, we have that $\overline{e_1e_n} \in L(G)$.

Thus $\overline{e_1e_2}$, $\overline{e_2e_3}$, …, $\overline{e_{n-1}e_n}$, $\overline{e_ne_1} \in E(L(G))$.

Since these are only edges in L(G) we get that E(L(G))

$= \{\overline{e_1e_2}, \overline{e_2e_3}, \ldots, \overline{e_{n-1}e_n}, \overline{e_ne_1}\}$.

Thus L(G) is a cycle of length n.

2. 1-QUASITOTAL GRAPHS

We start this section by introducing a new concept "1-quasitotal graph".

2.1 ***Definition***: Let G be a graph with vertex set V(G) and edge set E(G). The **1–quasitotal graph**, (denoted by $Q_1(G)$) of G is defined as follows:

The vertex set of $Q_1(G)$, that is $V(Q_1(G)) = V(G) \cup E(G)$.

Two vertices x, y in $V(Q_1(G))$ are adjacent if they satisfy one of the following conditions:

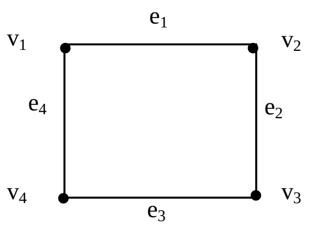
(i). x, y are in V(G) and $\overline{xy} \in E(G)$.

(ii). x, y are in E(G) and x, y are incident in G.

2.2 ***Note***: (i) G is a subgraph of $Q_1(G)$; and

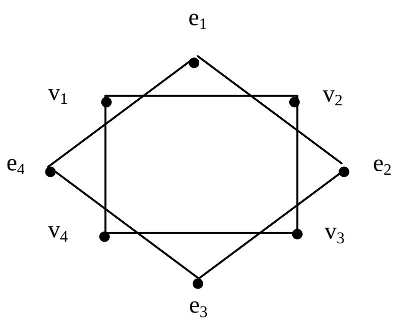
(ii) $Q_1(G)$ is a subgraph of T(G).

2.3 ***Example***: Consider the graph G given in Fig. 7. Let us construct the 1-quasitotal graph $Q_1(G)$ of the graph G.



G

Fig 7



$Q_1(G)$

Fig 8

$V(Q_1(G)) = \{V(G) \cup E(G)\} = \{v_1, v_2, v_3, v_4, e_1, e_2, e_3, e_4\}$

It is clear that $E(G) \subseteq E(Q_1(G))$.

So $\overline{v_1v_4}$, $\overline{v_4v_3}$, $\overline{v_3v_2}$, $\overline{v_2v_1} \in E(Q_1(G))$.

Since e_1 and e_2 are incident in G, there is an edge $\overline{e_1e_2} \in E(Q_1(G))$. Since e_1 and e_4 are incident in G, there is an edge $\overline{e_1e_4} \in E(Q_1(G))$. Since e_2 and e_3 are incident in G, there is an edge $\overline{e_2e_3} \in E(Q_1(G))$. Since e_3 and e_4 are incident in G, there is an edge $\overline{e_3e_4} \in E(Q_1(G))$. Therefore $E(Q_1(G)) = \{\overline{v_1v_4}, \overline{v_4v_3}, \overline{v_3v_2}, \overline{v_2v_1}, \overline{e_1e_2}, \overline{e_1e_4}, \overline{e_2e_3}, \overline{e_3e_4}\}$.

The 1-quasitotal graph $Q_1(G)$ is given by the Fig. 8.

2.4 ***Theorem***: $Q_1(G) = G \oplus L(G)$.

Proof: By the definition of $Q_1(G)$,

$V(Q_1(G)) = V(G) \cup E(G) = V(G) \cup V(L(G))$ (since $V(L(G)) = E(G)$). Let $s \in E(G)$. If s is an edge in G, then $s \in E(G)$.

If $s \notin E(G)$, then (by the definition of $Q_1(G)$) $s = \overline{e_1e_2}$ where $e_1, e_2 \in E(G)$ and e_1, e_2 are adjacent edges in G. By the definition of L(G) it follows that $s \in E(L(G))$.

Therefore $E(Q_1(G)) \subseteq E(G) \cup E(L(G))$. By Note 2.2, $E(G) \cup E(L(G)) \subseteq E(Q_1(G))$. Hence $Q_1(G) = G \cup L(G)$, the union of the two graphs G and L(G). Since $V(G) \cap V(L(G)) = V(G) \cap E(G) = \phi$, there exists no common edge in G and L(G). This means that $E(G) \cap E(L(G)) = \phi$. This implies that $G \cup L(G) = G \oplus L(G)$.

Hence $Q_1(G) = G \cup L(G) = G \oplus L(G)$, the proof is complete.

2.5 ***Corollary***: If G is a cycle of length n, then $Q_1(G)$ is the ring sum of exactly two disjoint cycles of length n.

Proof: Suppose G is a cycle of length n.

By Theorem1.8, L(G) is a cycle of length n.

Since $Q_1(G) = G \oplus L(G)$ (by Theorem 2.4) $Q_1(G)$ is equal to the ring sum of two disjoint cycles of length n, the proof is complete.

3. 2-QUASITOTAL GRAPHS

We start this section by introducing a new concept "2-quasitotal graph".

3.1 ***Definition***: Let G be a graph with vertex set V(G) and edge set E(G).

The **2-quasitotal graph** of G, denoted by $Q_2(G)$ is defined as follows: The vertex set of $Q_2(G)$, that is, $V(Q_2(G)) = V(G) \cup$

E(G). Two vertices x, y in $V(Q_2(G))$ are adjacent in $Q_2(G)$ in case one of the following holds:

(i) x, y are in V(G) and $\overline{xy} \in E(G)$.

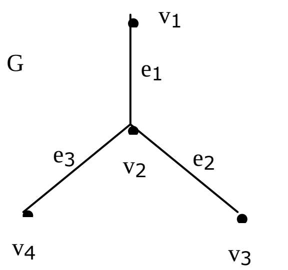
(ii) x is in V(G); y is in E(G); and x, y are incident in G.

3.2 ***Note***: (i) G is a subgraph of $Q_2(G)$; and

(ii) $Q_2(G)$ is a subgraph of T(G).

3.3 ***Example***: Consider the graph given in fig 9.

Fig 9



Let us construct $Q_2(G)$ of G.

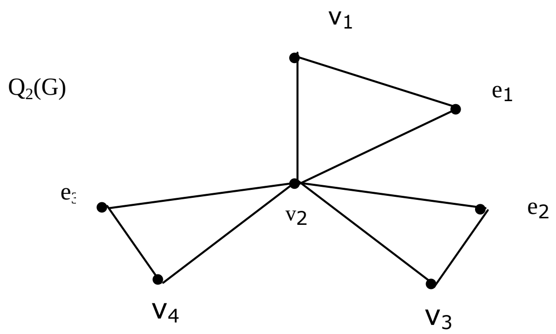
$V(Q_2(G)) = V(G) \cup E(G)$.

$= \{v_1, v_2, v_3, v_4\} \cup \{e_1, e_2, e_3\}$

$= \{v_1, v_2, v_3, v_4, e_1, e_2, e_3\}$.

$E(Q_2(G)) = \{\overline{v_1v_2}, \overline{v_2v_3}, \overline{v_2v_4}, \overline{v_2e_3}, \overline{v_1e_1}, \overline{e_1v_2}, \overline{v_2e_2}, \overline{e_2v_3}, \overline{e_3v_4}\}$.

Fig 10



The $Q_2(G)$ is given in Fig 10

3.4 ***Lemma***: If $e = \overline{uv} \in E(G)$, then there exist a triangle in $E(Q_2(G))$ containing e as one of the edges.

Proof: *Suppose G is a graph with |E(G)| = 1. Let $e \in E(G)$ and $e = \overline{vu}$ for some v, $u \in V(G)$. Now e, v, $u \in V(G) \cup E(G) = V(Q_2(G))$.*

Now $\overline{vu} \in E(G) \subseteq E(Q_2(G))$. Since e and u are incident in G, we have that $\overline{ue} \in E(Q_2(G))$. Since e and v are incident in G, we have that $\overline{ev} \in E(Q_2(G))$.

So $\overline{vu}$, $\overline{ue}$, $\overline{ev} \in E(Q_2(G))$ and these edges put together form a triangle, the proof is complete.

3.5 ***Lemma***: If e is not in a triangle of G and $e = \overline{uv} \in E(G)$ is only the edge between the vertices u and v in G, then there is only one triangle in $E(Q_2(G))$ containing e as one of the edges.

Proof: By Lemma 3.4, we know that $\overline{uv}$, $\overline{ve}$, $\overline{eu}$ is a triangle (in $Q_2(G)$) containing the edge $e = \overline{uv}$.

Let $\overline{ab}$, $\overline{bc}$, $\overline{ca}$ be a triangle in $Q_2(G)$ containing $e = \overline{uv}$. Without loss of generality, we assume that $\overline{ab}$ = e and so $\overline{ab} = \overline{uv}$. Further, we we assume that a = u and b = v.

If $\overline{bc}$ and $\overline{ca}$ are edges in G, then $\overline{ab}$, $\overline{bc}$, $\overline{ca}$ is a triangle in G containing e, a contradiction to our hypothesis. So $\overline{bc}$ or $\overline{ca}$ is not in E(G).

Suppose $\overline{bc}$ is not in E(G). Since $b \in V(G)$ it follows that $c \in E(G)$.

Since $\overline{bc}$ is in $E(Q_2(G))$, by the definitions of $Q_2(G)$, it follows that the edge c of G is incident on the vertex b = v of G.

Now $\overline{ca} \in E(Q_2(G))$ implies that the edge c is incident on the vertex a = u.

Thus c is an edge between its end points u and v.

Since there is only one edge (in G) between the vertices u and v, we get c = e.

Thus the triangle $\overline{ab}$, $\overline{bc}$, $\overline{ca}$ taken in $E(Q_2(G))$ is nothing but $\overline{uv}$, $\overline{ve}$, $\overline{eu}$. Hence there is only one triangle in $Q_2(G)$ containing (or corresponding to) e.

3.6 ***Note***: If $\overline{xy} \in E(Q_2(G)) \setminus E(G)$, then by the definition of $Q_2(G)$ it follows that one of the x, y is edge (say x) in G and the edge x is incident on the vertex y (of G) in G.

3.7 ***Lemma***: Every triangle in $Q_2(G)$ contains an edge of G.

Proof: Let $\overline{ab}$, $\overline{bc}$, $\overline{ca}$ be a triangle in $Q_2(G)$.

If possible suppose that neither $\overline{ab}$ nor $\overline{bc}$ nor $\overline{ca}$ is an edge in G.

Since $\overline{ab}$ is an edge in $E(Q_2(G)) \setminus E(G)$, one of the a, b is an edge and the other is a vertex in G

Without loss of generality, assume that $a \in E(G)$ and $b \in V(G)$ and a is incident on b.

Since $b \in V(G)$, $\overline{bc} \in E(Q_2(G)) \setminus E(G)$, we have that $c \in E(G)$ and c is incident on b.

Since $\overline{ca} \in E(Q_2(G)) \setminus E(G)$ and $c \in E(G)$, it follows that $a \in V(G)$ and c is incident on a. This fact $a \in V(G)$ is a contradiction to the fact $a \in E(G)$.

Thus one of the $\overline{ab}$, $\overline{bc}$, $\overline{ca}$ is an edge in G.

3.8 ***Theorem***: If G is a graph containing only one edge (that is |E(G)| = 1) then the graph $Q_2(G)$ contains unique triangle.

Proof: (**Existence**): Let E(G) = {e} and u, $v \in V(G)$ with $e = \overline{uv}$. By Lemma 3.4,

$\overline{uv}$, $\overline{ve}$, $\overline{eu}$ is a triangle in $Q_2(G)$ containing the edge e.

(**Uniqueness**): Since e is only the edge in G, the graph G contains no triangles. So the statement "e is not in any triangle of G" is true.

Thus by using Lemma 3.5, we can conclude that $Q_2(G)$ contains only one triangle containing "e"… (i)

Now we verify that any triangle in $Q_2(G)$ contains e.

Let $\overline{xy}$, $\overline{yz}$, $\overline{zx}$ be a triangle in $Q_2(G)$.

By Lemma 3.7, this triangle contains an edge of G. Since G contains only one edge e, it follows that the triangle ($\overline{xy}$, $\overline{yz}$, $\overline{zx}$) contains e. From the above steps, every triangle in $Q_2(G)$ contains the edge "e" … (ii)

From (i) & (ii), we get that $Q_2(G)$ contains unique triangle.

3.9 ***Lemma***: Suppose G contains two distinct edges $e_1 = \overline{uv}$ and $e_2 = \overline{xy}$ (i) If $\{u, v\} \cap \{x, y\} = \varnothing$, then $Q_2(G)$ contains two distinct triangles one containing e_1 and other containing e_2. Moreover there is no common vertex between two triangles.

(ii) If $\{u, v\} \cap \{x, y\} \neq \varnothing$, then $Q_2(G)$ contains two distinct triangles one containing e_1 and other containing e_2. Moreover if $\{a\} = \{u, v\} \cap \{x, y\}$, then a is a common vertex to these two triangles.

Proof: Given that $e_1 = \overline{uv}$ and $e_2 = \overline{xy}$ are two edges in G.

By Lemma 3.4, $\overline{ue_1}$, $\overline{e_1v}$, $\overline{vu}$ is a triangle in $Q_2(G)$ containing $e_1 = \overline{uv}$; and

$\overline{xe_2}$, $\overline{e_2y}$, $\overline{yx}$ is a triangle in $Q_2(G)$ containing $e_2 = \overline{xy}$.

Clearly these are two distinct triangles.

Suppose that $\{u, v\} \cap \{x, y\} = \phi$.

If possible suppose that the triangles $\{\overline{ue_1}, \overline{e_1v}, \overline{vu}\}$ and $\{\overline{xe_2}, \overline{e_2y}, \overline{yx}\}$ have a common vertex.

The vertex sets of these triangles are $\{u, v, e_1\}$ and $\{x, y, e_2\}$.

Since $e_1 \neq e_2$ (as edges in G), $e_1 \neq e_2$ (as vertices in $Q_2(G)$).

The remaining part is clear because $\{u, v\} \cap \{x, y\} = \phi$.

Hence there is no common vertex between the two triangles.

(ii)Suppose $\{u, v\} \cap \{x, y\} \neq \phi$. If $\{u, v\} = \{x, y\}$, then $e_1 = \overline{uv} = \overline{xy} = e_2$, a contradiction. So $\{u, v\} \neq \{x, y\}$, and $\{u, v\} \cap \{x, y\} \neq \phi$.

Without loss of generality, assume that u = x and v ≠ y.

In this case, the vertex sets of the triangles are $\{u, v, e_1\}$ and $\{x, y, e_2\} = \{u, y, e_2\}$.

This shows that the two triangles are having a common vertex u, the proof is complete.

3.10 ***Theorem***: Let G be a graph. Then the following conditions are equivalent:

(i) $|E(G)| = 1$;

(ii) $Q_2(G)$ contains unique triangle.

Proof: (i) ⇒ (ii): Theorem 3.8

(ii) ⇒ (i): Suppose $Q_2(G)$ contains unique triangle. By Lemma 3.7, every triangle of $Q_2(G)$ contains at least one edge of G. Since $Q_2(G)$ contains a triangle, $|E(G)| \geq 1$. If $|E(G)| > 1$, then E(G) contains two distinct edges. By Lemma 3.9, it follows that $Q_2(G)$ contains two distinct triangles, a contradiction to (ii).

Thus $|E(G)| = 1$, the proof is complete.

4. CONCLUSIONS

There is a scope for concepts of total graphs, quasi-total graphs can be extended to finite directed graph with suitable assumptions.

5. ACKNOWLEDGEMENT

The first and the second authors are grateful to Acharya Nagarjuna University for the kind encouragement. The third author expresses thanks to Manipal Institute of Technology, Manipal University for the kind encouragement.

6. REFERENCES

[1] Bondy J. A. and Murty U. S. R. "Graph Theory with Applications", *The Macmillan Press Ltd*, (1976).

[2] Harary F. "Graph Theory", Addison-Wesley Publishing Company, USA (1972).

[3] Narsing Deo "Graph Theory with Applications to Engineering and Computer Science", *Prentice Hall of India Pvt. Ltd*, New Delhi (1997).

[4] Armugam S. & Ramachandran S. "Invitation to Graph Theory", *Scitech Publications (India) Pvt. Ltd*, Chennai, (2001).

[5] Kulli V. R. "Minimally Non outerplanar Graphs: A survey", (in the Book: "Recent Studies in Graph theory" ed: V. R. Kulli), *Vishwa International Publication*, (1989) 177-189.

[6] Satyanarayana Bh. and Syam Prasad K. "An Isomorphism Theorem on Directed Hypercubes of Dimension n", *Indian J. Pure & Appl. Math* 34 (10) (2003) 1453-1457.

[7] Satyanarayana Bh. and Syam Prasad K. "Discrete Mathematics and Graph Theory", Prentice Hall India Pvt. Ltd., ISBN 978-81-203-3842-5 (2009).

[8] Satyanarayana Bh. and Syam Prasad K. "Nearrings, Fuzzy Ideals and Graph Theory" CRC Press (Taylor & Francis Group, London, New York), 2013 ISBN 13: 9781439873106.

[9] Satyanarayana Bh., Srinivasulu D., and Syam Prasad K. "Some Results on Degree of Vertices in Semitotal Block Graph and Total – Block Graph", International Journal of Computer Applications (0975 - 8887) Vol. 50, No.9, (July 2012) 19-22.