Complex Dynamics of Jungck Ishikawa Iterates for Hyperbolic Cosine Function

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ABSTRACT
The dynamics of transcendental function is one of emerging and interesting field of research nowadays. We introduce in this paper the complex dynamics of hyperbolic cosine function of the type {cosh (z^n) + z + c = 0} and applied Jungck Ishikawa iteration to generate new Relative Superior Mandelbrot set and Relative Superior Julia set. In order to solve this function by Jungck –type iterative schemes, we write it in the form of Sz = Tz, where the function T, S are defined as Tz = cosh(z^n) +c and Sz = - z. Only mathematical explanations are derived by applying Jungck Ishikawa Iteration for transcendental function in the literature but in this paper we have generated relative Mandelbrot sets and Relative Julia sets.

Keywords
Complex dynamics, Relative Superior Mandelbrot set, Relative Julia set, Jungck Ishikawa Iteration

1. INTRODUCTION
The study of dynamical behavior of the transcendental functions was initiated by Fatou [12]. For transcendental function, points with unbounded orbits are not in Fatou sets but they must lie in Julia sets.

In complex analysis, the hyperbolic functions arise as the imaginary parts of sine and cosine .When considered defined by a complex variable, the hyperbolic functions are rational functions of exponentials, and are hence meromorphic.

In this past literature the cosine function was considered of the following forms:

(i) cos (z^n) + c = 0
(ii) $(\cos z + c)^n = 0$
(iii) acos(z^n)+c=0
(iv) $(\text{acos}(z) + c)^n = 0$

But now we have used hyperbolic cosine function of the type cosh(z^n) + z + c = 0 where $n \geq 2$ and applied Jungck Ishikawa iterates to develop fractal images of this transcendental function. Escape criteria of polynomials are used to generate Relative Superior Mandelbrot Sets and Relative Superior Julia Sets. Our results are different from existing results in literature.

2. PRELIMINARIES
The process of generating fractal images from $z \to \cosh(z^n) + z + c$ is similar to the one employed for the self-squared function [17]. Briefly, this process consists of iterating this function up to N times.

Starting from a value z_0 we obtain $z1_{,}$ $z_{2,}$ $z_{3,}$ z_4 ... by applying the transformation $z \to \cosh(z^n) + z + c$

2.1 Ishikawa Iteration [8]
Let X is a subset of real or complex numbers and T: X→ X for $x_0 \in X$, we have the sequences $\{x_n\}$ and $\{y_n\}$ in X in the following manner:

$$x_{n+1} = \alpha_n T y_n + (1- \alpha_n) x_n$$

$$y_n = \beta_n T x_n + (1- \beta_n) x_n$$

where $0 \leq \beta_n \geq 1$ and $0 \leq \alpha_n \geq 1$ and α_n & β_n both convergent to non zero number.

2.2 Definition [14]
The sequences $\{x_n\}$ and $\{y_n\}$ constructed above is called Ishikawa sequences of iteration or relative superior sequences of iterates. We denote it by (x_0, α_n , β_n ,t) .Notice that RSO (x_0, α_n , β_n ,t) with $\beta_n = 1$ is RSO(x_0, α_n ,t) i.e. Mann's orbit and if we place $\alpha_n = \beta_n = 1$ then RSO (x_0, α_n , β_n ,t) reduces to O ($x_{0,}$ t) .We remark that Ishikawa orbit RSO(x_0, α_n , β_n ,t) with $\beta_n = 1/2$ is Relative superior orbit. Now we define Julia set for function with respect to Ishikawa iterates. We call them as Relative Superior Julia sets.

2.3 Definition [14]
The set of points SK whose orbits are bounded under Relative superior iteration of function Q (z) is called Relative Superior Julia sets. Relative Superior Julia set of Q is a boundary of Julia set RSK.

2.4 Jungck Ishikawa Iteration [16]
Let (X, ‖.‖) be a Banach space and Y an arbitrary set. Let S, T: Y→X be two non self-mappings such that $T(Y) \subseteq S(Y)$, S(Y) is a complete subspace of X and S is injective. Then for $x_o \in Y$, define the sequence $\{S x_n\}$ iteratively by

$$S x_{n+1} = \alpha_n T y_n + (1- \alpha_n) S x_n$$

$$S y_n = \beta_n T x_n + (1- \beta_n) S x_n$$

where $0 \leq \beta_n \geq 1$ and $0 \leq \alpha_n \geq 1$ and α_n & β_n both convergent to non zero number.

3. GENERATING THE FRACTALS
Fractals have been generated from , using escape-time techniques-

3.1 Escape Criterion for Quadratics [14]

Suppose that |z | >max {|c |, 2 /s , 2 /s ' }, then $|z_n| > (1+\lambda)^n |z|$ and $|z_n| \to \infty$ as $n \to \infty$. So, |z | ≥ c |, and |z |>2/s as well as |z |>2/ s ' shows the escape criteria for quadratics.

3.2 Escape Criterion for Cubics[14]

Suppose that |z | >max {|b | , (a+ 2 /s) ½ , (a+ 2 /s ') ½ }, then $|z_n| \to \infty$ as $n \to \infty$ .This gives the escape criteria for cubic polynomials.

3.3 General Escape Criterion [14]

Suppose that |z | >max {|b | , (a+ 2 /s) ½ , (a+ 2 /s ') ½ }, then $|z_n| \to \infty$ as $n \to \infty$ is the general escape criteria.

4. FIXED POINTS

4.1 Fixed points of quadratic function

Table 1: Orbit of F (z) for (z_o= -0.8125-0.1125i) at α =0.5, β =0.5, c=0.1

No. of iterations	|Tz|	No. of iterations	|Tz|
1	1.30289	11	1.32199
2	1.31711	12	1.32223
3	1.32285	13	1.32217
4	1.32199	14	1.32219
5	1.32223	15	1.32219
6	1.32217	16	1.32219
7	1.32199	17	1.32219
8	1.32223	18	1.32199
9	1.32217	19	1.32223
10	1.32219	20	1.32217

Here we observe that the value converges to a fixed point **1.32219** after 6 iterations.

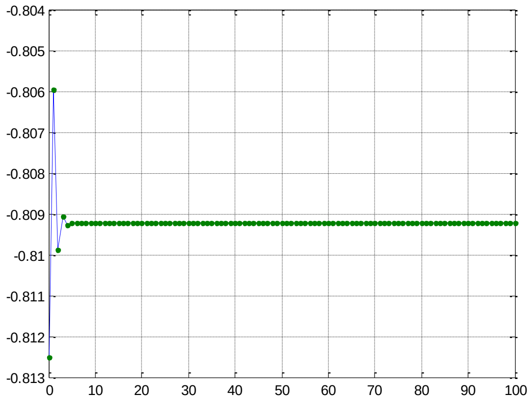

Figure 1: **Orbit of F (z) for (z_o= -0.8125-0.1125i) at α =0.5, β =0.5, c=0.1**

Table 2: Orbit of F (z) for (z_o= -2.6875-0.0625i) at α =0.8, β =0.1, c=0.1

No of Iterations	|Tx|	No of Iterations	|Tx|
71	74.025	86	1.1217
72	1.1217	87	74.0253
73	74.0256	88	1.1217
74	1.1217	89	74.0253
75	74.0251	90	1.1217
76	1.1217	91	74.0253
77	74.0255	92	1.1217
78	1.1217	93	74.0253
79	74.0252	94	1.1217
80	1.1217	95	74.0253
81	74.0254	96	1.1217
82	1.1217	97	74.0253
83	74.0253	98	1.1217
84	1.1217	99	74.0253
85	74.0254	100	1.1217

Here we 70 iterations and observed that the value converges to two fixed points **74.0253** and **1.1217** after 87 iterations.

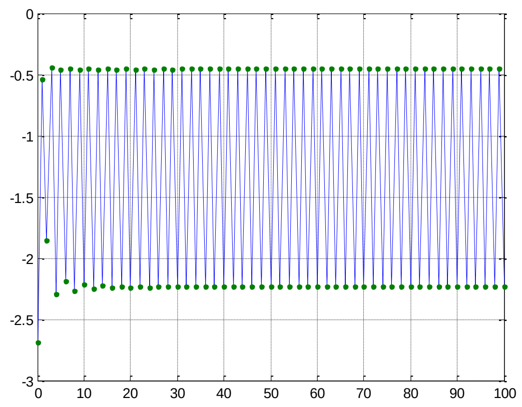

Figure 2. Orbit of F (z) for (z_o= -2.6875-0.0625i) at α =0.8, β =0.1, c=0.1

Table 3: Orbit of F (z) for (z_o=-0.35625-1.65i) at α =0.3, β =0.7, c=0.1

No. of iterations	|Tz|	No. of iterations	|Tz|
1	6.71552	11	1.26349
2	1.99943	12	1.26342
3	1.21717	13	1.26339
4	0.88466	14	1.26338
5	1.13628	15	1.26338
6	1.26549	16	1.26338
7	1.26856	17	1.26338
8	1.26571	18	1.26338
9	1.26426	19	1.26338
10	1.26370	20	1.26338

Here we observed that the value converges to a fixed point **1.26338** after 13 iterations.

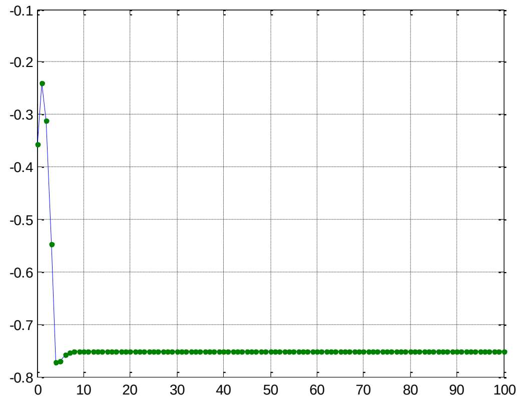

Figure 3: Orbit of F (z) for (z_o=-0.35625-1.65i) at α =0.3, β =0.7, c=0.1

4.2 Fixed points of cubic function

Table 1: Orbit of F (z) for (z_o=-0.9+1.04375) at α =0.5, β =0.7, c=0.1

No. of iterations	|Tz|	No. of iterations	|Tz|
51	2.889998	66	1.327397
52	1.327397	67	2.890001
53	2.890004	68	1.327397
54	1.327397	69	2.890001
55	2.89000	70	1.327397
56	1.327397	71	2.890001
57	2.89000	72	1.327397
58	1.327397	73	2.890001
59	2.890001	74	1.327397
60	1.327397	75	2.890001
61	2.890001	76	1.327397
62	1.327397	77	2.890001
63	2.890001	78	1.327397
64	1.327397	79	2.890001
65	2.890001	80	1.327397

Here we skipped 50 iterations and observed that the value converges to two fixed points **2.890001** and **1.327397** after 58 iterations.

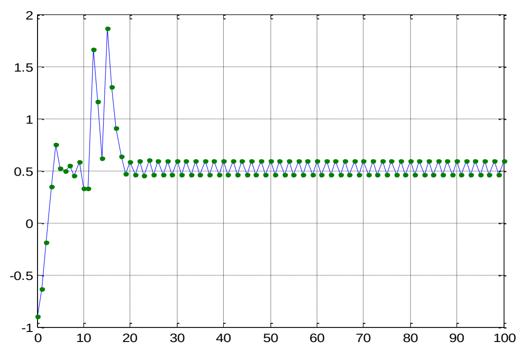

Figure 1: Orbit of F (z) for (z_o=-0.9+1.04375) at α =0.5, β =0.7, c=0.1

Table 2: Orbit of F (z) for (z_o=-1.31875+1.1225i) at α =0.3, β =0.5, c=0.1

No. of iterations	|Tz|	No. of iterations	|Tz|
11	1.3534	26	1.3392
12	1.3127	27	1.3384
13	1.3605	28	1.3388
14	1.3293	29	1.3387
15	1.3381	30	1.3386
16	1.3445	31	1.3388
17	1.3322	32	1.3386
18	1.3431	33	1.3387
19	1.3373	34	1.3387
20	1.3379	35	1.3387
21	1.3404	36	1.3387
22	1.3372	37	1.3387
23	1.3395	38	1.3387
24	1.3386	39	1.3387
25	1.3384	40	1.3387

Here we skipped 10 iterations and observed that the value converges to a fixed point **1.3387** after 32 iterations.

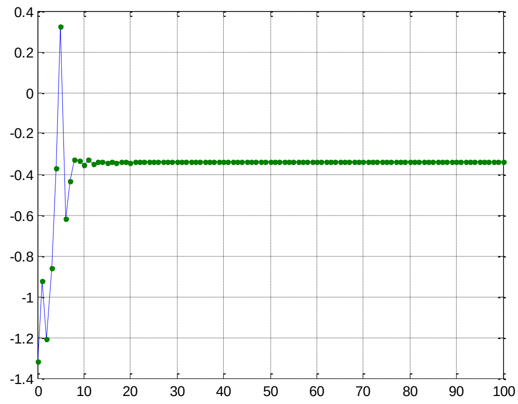

Figure 2: Orbit of F (z) for (z_o=-1.31875+1.1225i) at α =0.3, β =0.5, c=0.1

Table 3: Orbit of F (z) for (z_o = -1.925-1.7375i) at α =0.3, β =0.2, c=0.1

No of Iterations	|Tx|	No of Iterations	|Tx|
1	14879.0392	16	1.60569
2	17.11398	17	1.60569
3	1.50879	18	1.60566
4	4.78429	19	1.60567
5	1.53148	20	1.60567
6	1.47782	21	1.60567
7	1.66984	22	1.60567
8	1.60781	23	1.60567
9	1.59224	24	1.60567
10	1.61045	25	1.60567
11	1.60660	26	1.60567
12	1.60431	27	1.60567
13	1.60601	28	1.60567

14	1.60582	29	1.60567
15	1.60554	30	1.60567

Here we observe that the value converges to a fixed point **1.60567** after 18 iterations.

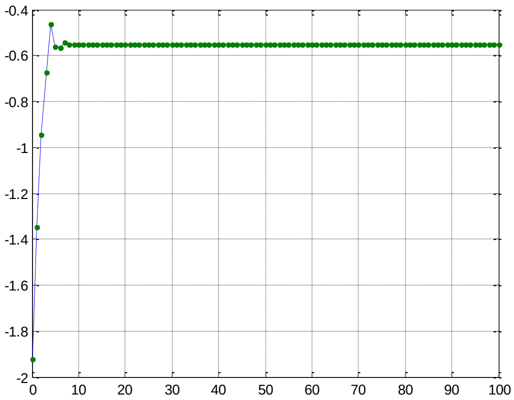

Figure 3: Orbit of F (z) for (z_o = -1.925-1.7375i) at α =0.3, β =0.2, c=0.1

4.3 Fixed points of biquadratic function

Table 1: Orbit of F (z) for (z_o= -2.66875+0.00625i) at α =0.5, β =0.5, c=0.1

No. of iterations	|Tz|	No. of iterations	|Tz|
21	1.1097	36	2.1494
22	2.1476	37	1.1097
23	1.1097	38	2.1494
24	2.1485	39	1.1097
25	1.1097	40	2.1494
26	2.1490	41	1.1097
27	1.1097	42	2.1494
28	2.1492	43	1.1097
29	1.1097	44	2.1494
30	2.1493	45	1.1097
31	1.1097	46	2.1494
32	2.1493	47	1.1097
33	1.1097	48	2.1494
34	2.1494	49	1.1097
35	1.1097	50	2.1494

Here we skipped 20 iterations and observed that the value converges to two fixed points **1.1097** and **2.1494** after 32 iterations.

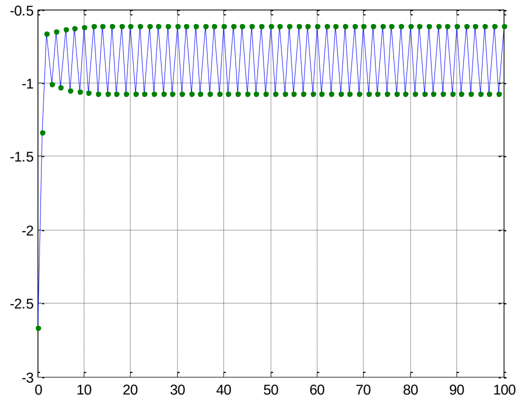

Figure 1: Orbit of F (z) for (z_o= -2.66875+0.00625i) at α =0.5, β =0.5, c=0.1

Table 2: Orbit of F (z) for (z_o= -1.6875+0.84375) at α =1, β =1, c=0.1

No. of iterations	|Tz|	No. of iterations	|Tz|
1	17.4696	11	1.11051
2	1.10000	12	1.11051
3	1.11399	13	1.11051
4	1.10956	14	1.11051
5	1.11078	15	1.11051
6	1.11043	16	1.11051
7	1.11053	17	1.11051
8	1.11050	18	1.11051
9	1.11051	19	1.11051
10	1.11051	20	1.11051

Here we observe that the value converges to a fixed point **1.11051** after 8 iterations.

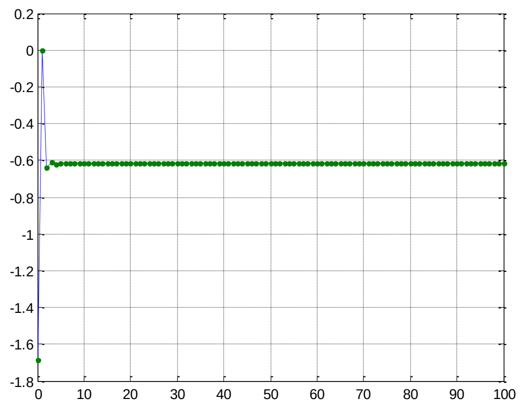

Figure 2: Orbit of F (z) for (z_o= -1.6875+0.84375) at α =1, β =1, c=0.1

Table 3: Orbit of F (z) for (z_0 = -2.24375+1.3375i) at α =0.3, β =0.2, c=0.1

No. of iterations	|Tz|	No. of iterations	|Tz|
1	58829601051.3444	16	1.3454
2	227.4253	17	1.3503
3	2.0861	18	1.3476
4	10.1152	19	1.3491
5	1.2891	20	1.3483
6	1.0992	21	1.3487
7	1.6548	22	1.3485
8	1.1541	23	1.3486
9	1.8486	24	1.3485
10	1.2591	25	1.3486
11	1.425	26	1.3485
12	1.3165	27	1.3485
13	1.3687	28	1.3485
14	1.3383	29	1.3485
15	1.3544	30	1.3485

Here we observe that the value converges to a fixed point **1.3485** after 25 iterations.

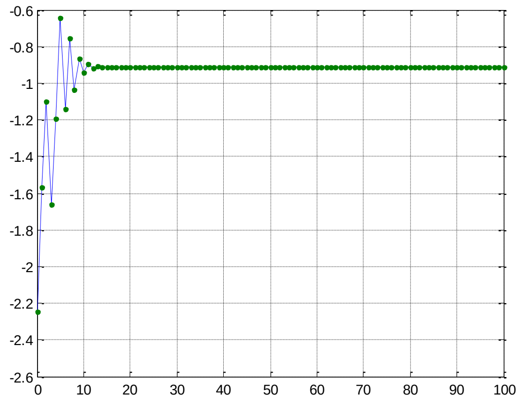

Figure 3: Orbit of F (z) for (z_0 = -2.24375+1.3375i) at α =0.3, β =0.2, c=0.1

5. GEOMETRY OF RELATIVE SUPERIOR MANDELBROT SETS AND RELATIVE SUPERIOR JULIA SETS

Relative Superior Mandelbrot Sets

- In case of quadratic function, the central body is divided into three parts or we can say it looks like a flower having 3 leaves. It is seen that the body is symmetric along the real axis only. Each part has one secondary lobe which is approximately equal in size.
- In case of cubic function, the central body is divided into 5 equal parts or we can say it looks like a flower having 5 leaves .Each part have one secondary lobe of different sizes. It is seen that the body is symmetric along the real axis only. For α =0.3, β =0.5, the size of the secondary lobes is larger as compared to other values.
- In case of biquadratic function, the central body is divided into seven equal parts or we can say it looks like a flower having 7 equal size leaves. Each part having one secondary lobe of different sizes. It is seen that the body is symmetric along the real axis only.

Relative Superior Julia Sets

- Relative Superior Julia Sets for the transcendental function cosh(z) appears to follow law of having 2n wings. These sets are symmetric along both the axes i.e. along real and imaginary axis.
- For quadratic function the Relative Superior Julia Set is divided into four wings having red central body. These sets are symmetric along both the axes.
- For cubic function the Relative Superior Julia Set is divided into six wings having reflectional and rotational symmetry, along with a larger red central region.
- For biquadratic function the Relative Superior Julia Set is divided into eight wings possessing the reflectional and rotational symmetry and it is having a larger escape region as compared to quadratic and cubic function.
- It is also observed from the graphical study of fixed points of Relative Superior Julia Sets that the convergence for α =0.5, β =0.5 and α =1, β =1 is quite fast for all polynomials in comparison to the convergence for other values.

6. GENERATION OF RELATIVE SUPERIOR MANDELBROT SETS

We generated the Relative Superior Mandelbrot sets. We present here some beautiful filled Relative Superior Mandelbrot sets for quadratic, cubic and biquadratic function.

6.1 Relative Superior Mandelbrot sets for Quadratic function

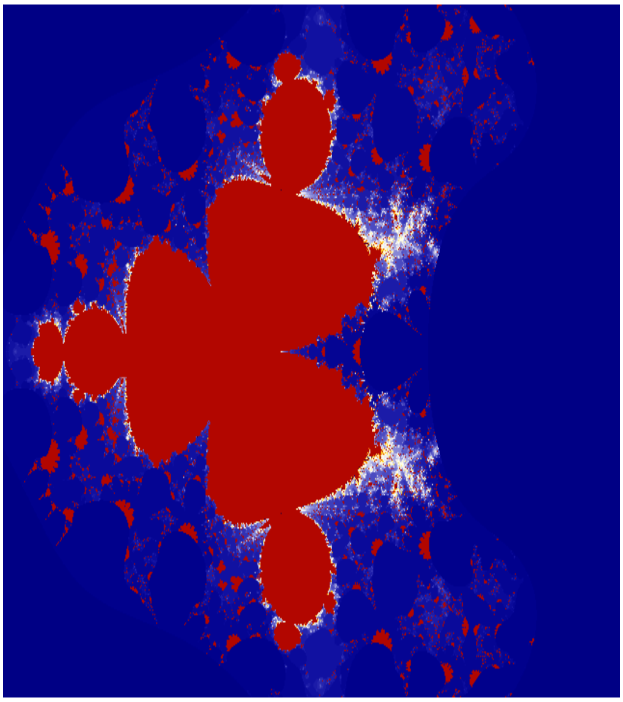

Figure 1: Relative Superior Mandelbrot Set for α = β =0.5 & c = -0.8125-0.1125i

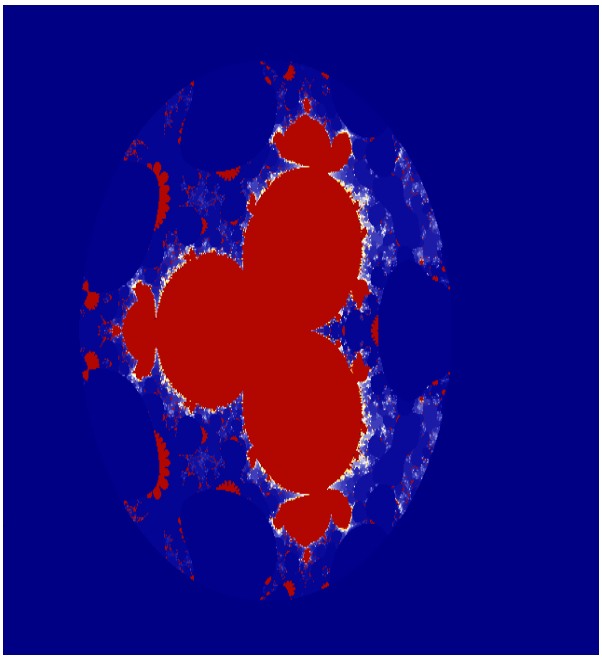

Figure 2: Relative Superior Mandelbrot Set for α =0.8, β =0.1, c=-2.6875-0.0625i

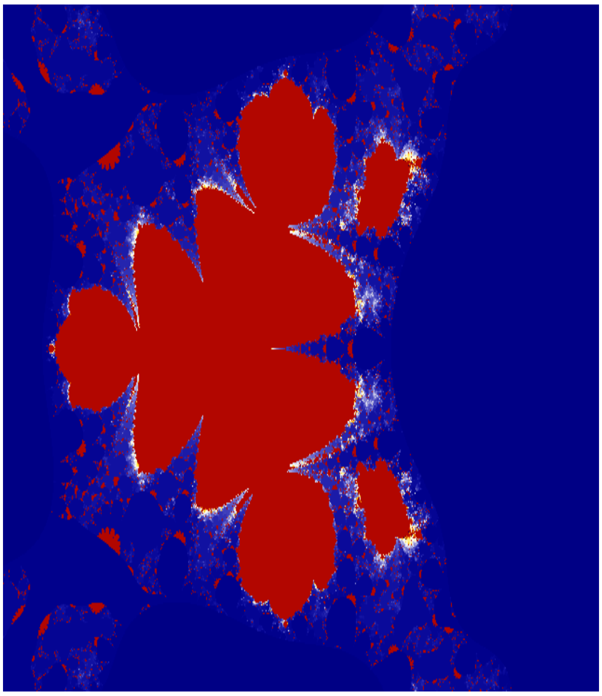

Figure 3: Relative Superior Mandelbrot Set for α =0.3, β =0.7, c=-0.35625-1.65i

6.2 Relative Superior Mandelbrot Sets for Cubic function

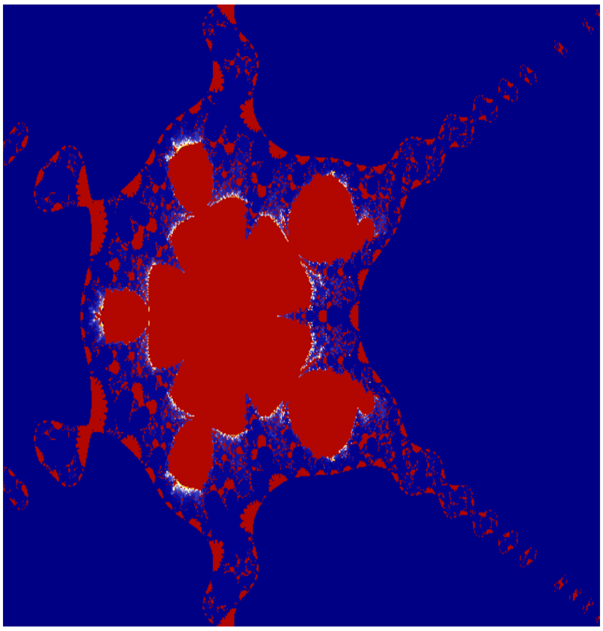

Figure 1: Relative Superior Mandelbrot Set for α =0.5 , β =0.7, c=-0.9+1.04375i

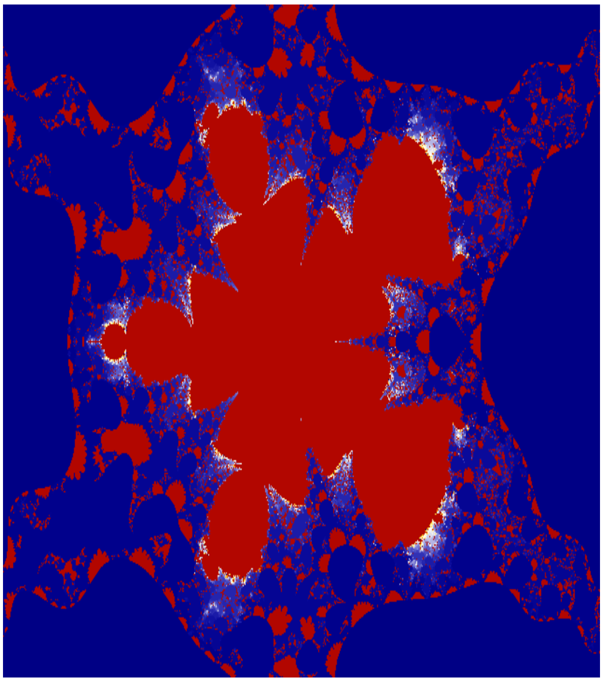

Figure 2: Relative Superior Mandelbrot Set for α =0.3, β =0.5, c = -1.31875+1.225i

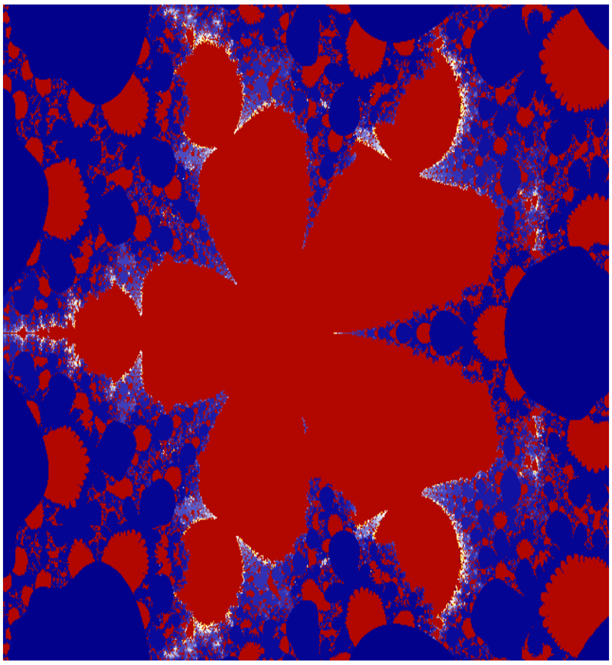

Figure 3: Relative Superior Mandelbrot Set for α =0.3, β =0.2, c = -1.925-1.7375i

6.3 Relative Superior Mandelbrot sets for biquadratic function

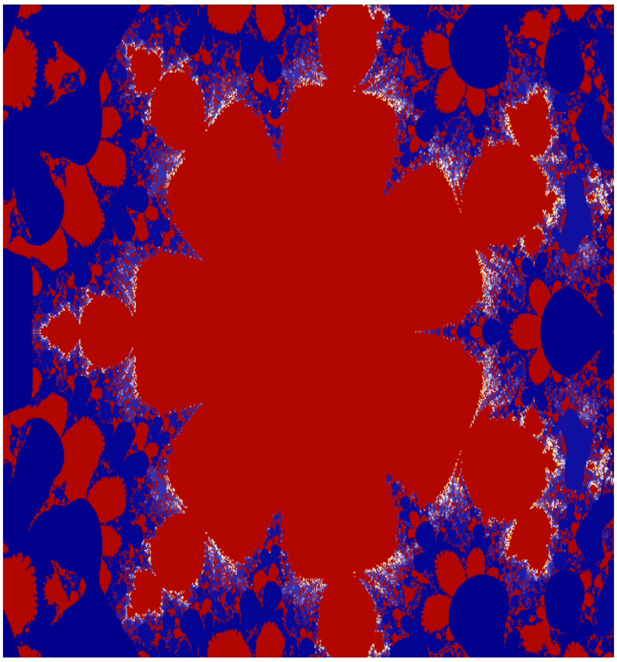

Figure 1: Relative Superior Mandelbrot Set for α =0.5, β =0.5, c = -2.66875+0.00625i

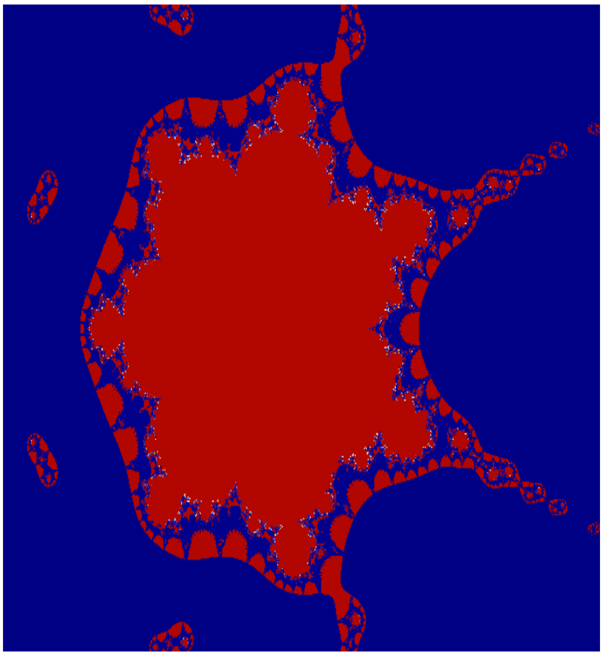

Figure 2: Relative Superior Mandelbrot Set for α =1, β =1, c = -1.6875+0.84375i

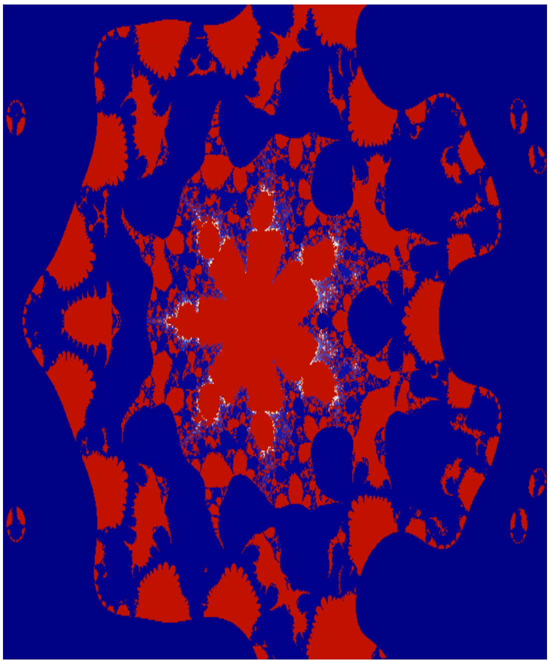

Figure 3: Relative Superior Mandelbrot Set for α =0.3, β =0.2, c = -2.24375+1.3375i

7. GENERATION OF RELATIVE SUPERIOR JULIA SETS

We generated the Relative Superior Julia sets. We have presented here some beautiful filled Relative Superior Julia sets for quadratic, cubic and biquadratic function.

7.1 Relative Superior Julia sets for Quadratic function

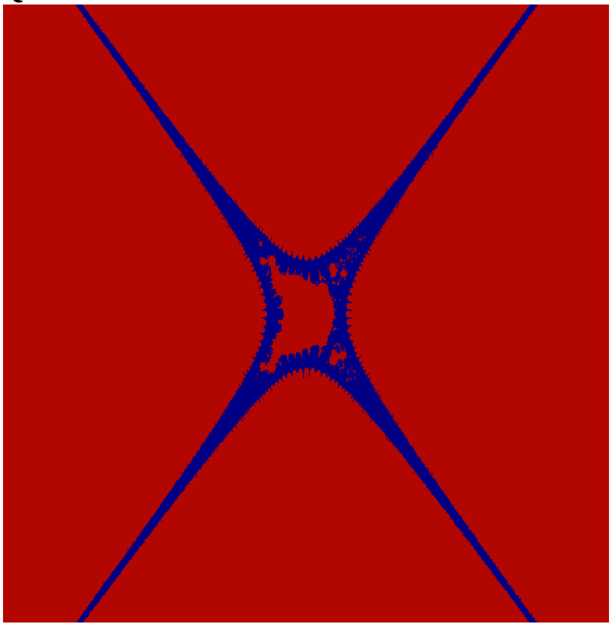

Figure 1: Relative Superior Julia Set for $\alpha = \beta$ =0.5 & c = -0.8125-0.1125i

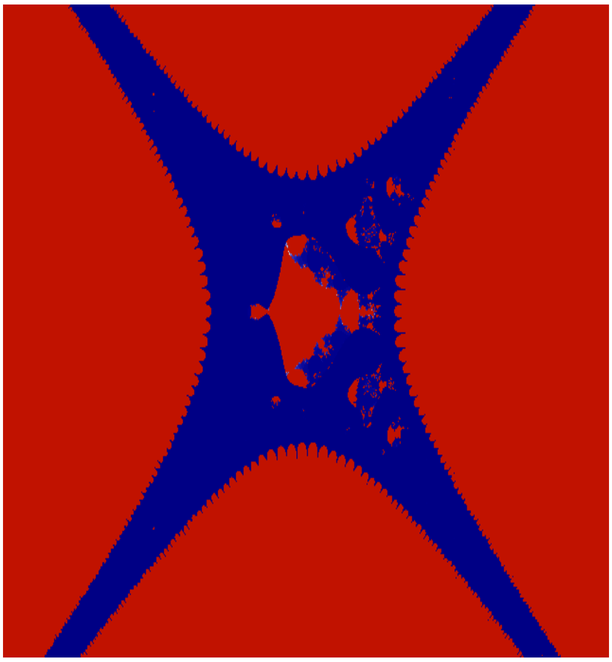

Figure 2: Relative Superior Julia Set for α =0.8, β =0.1, c=-2.6875-0.0625i

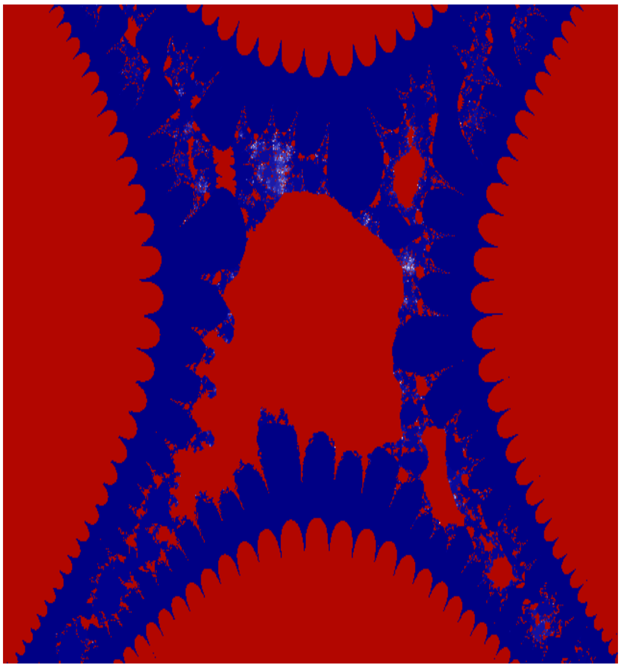

Figure 3: Relative Superior Julia Set for α =0.3, β =0.7, c=-0.35625-1.65i

7.2 Relative Superior Julia Sets for Cubic function

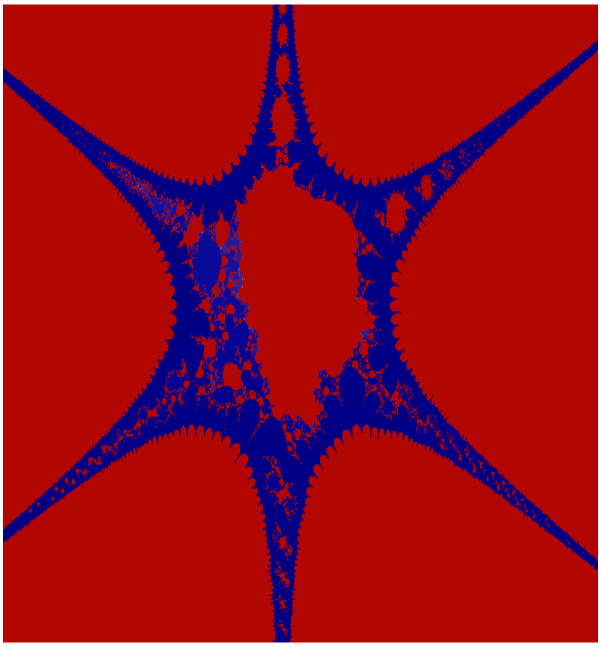

Figure 1: Relative Superior Julia Set for α =0.5 , β =0.7, c=-0.9+1.04375i

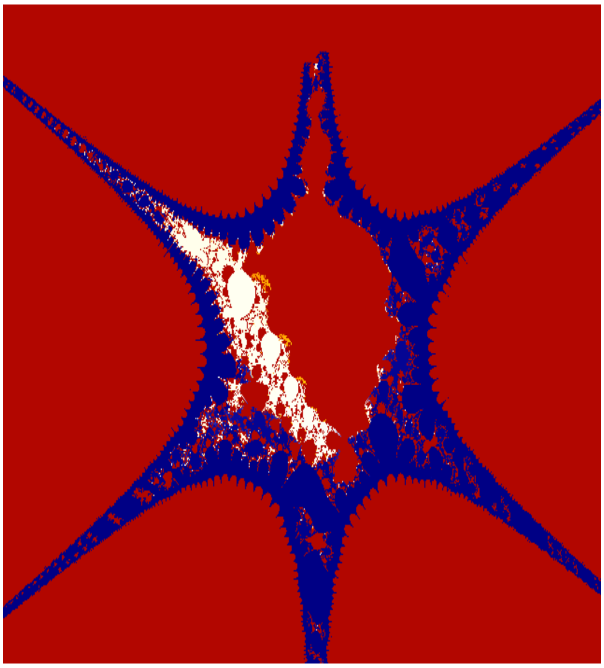

Figure 2: Relative Superior Julia Set for α =0.3, β =0.5, c = -1.31875+1.225i

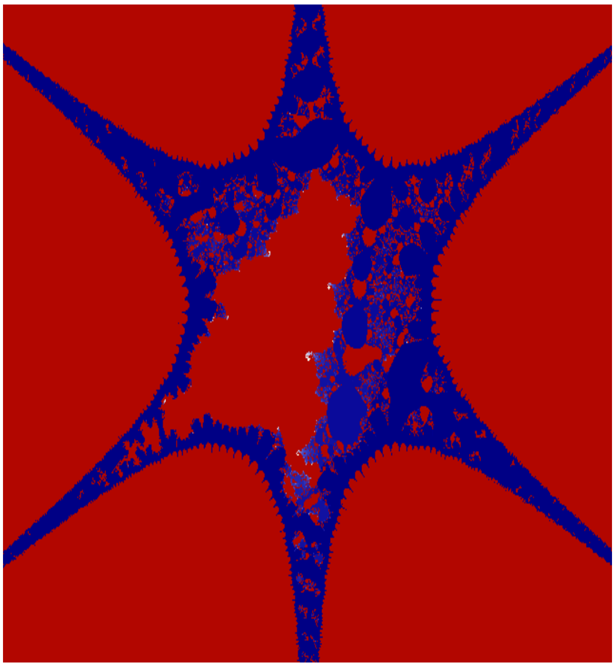

Figure 3: Relative Superior Julia Set for α =0.3, β =0.2, c = -1.925-1.7375i

7.3 Relative Superior Julia sets for biquadratic function

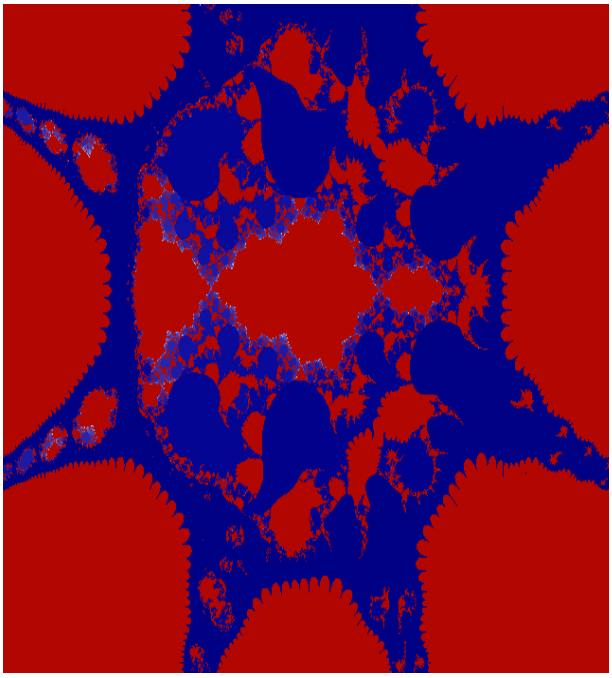

Figure 1: Relative Superior Julia Set for α =0.5, β =0.5, c = -2.66875+0.00625i

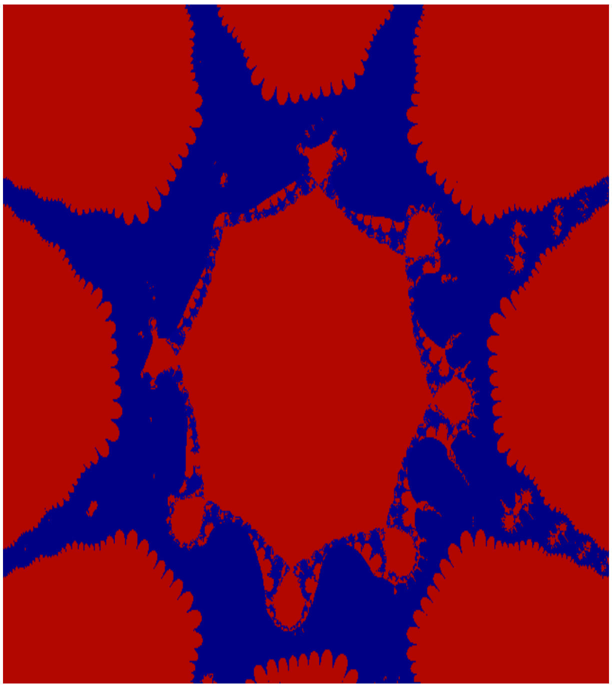

Figure 2: Relative Superior Julia Set for α =1, β =1, c = -1.6875+0.84375i

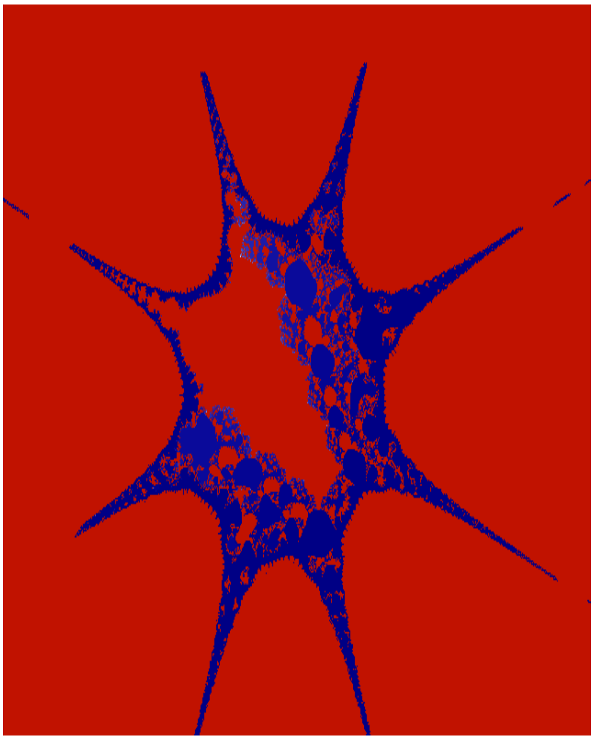

Figure 3: Relative Superior Julia Set for α =0.3, β =0.2,

c = -2.24375+1.3375i

8. CONCLUSION

In this paper we studied the hyperbolic cosine function which is one of the members of transcendental family. The fixed point 0 for S (z) = cosh (z^n) +z + c = 0 also satisfies $S'(0) = 1$. Relative Superior Mandelbrot sets for the hyperbolic transcendental function cosh(z) appear like beautiful flowers having the symmetry of 2n-1 petals/leaves while Relative Superior Julia Sets appears to follow law of having 2n wings. The surrounding region of the Mandelbrot set appears to be an invariant Cantor set in the form of curve or “hair” that extends to ∞. The orbit of any point on hair tends to infinity under iteration. Here the geometry of hairs is quite similar to that of exponential family and hence showed the property of transcendental function. The region filled up with large number of escaping points represents Julia set plane.

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