The Geodetic Parameters of Strong Product Graphs

Ashalatha K.S
Research Scholar,Sri Gauthama Research Center,
(Affiliated to Kuvempu university),Government First Grade College,
Gubbi, Tumkur,Karnataka,India

Venkanagouda M Goudar
Sri Gauthama Research Center, Department of Mathematics,
Sri Siddhartha Institute of Technology, Tumkur,
Karnataka, India

Venkatesha
Department of Mathematics, Kuvempu University
Shankarghatta, Shimoga, Karnataka, India.

ABSTRACT

A set $S \subseteq V(G)$ is a split geodetic set of G, if S is a geodetic set and $\langle V - S\rangle$ is disconnected. The split geodetic number of a graph G, is denoted by $g_s(G)$, is the minimum cardinality of a split geodetic set of G. A set $S \subseteq V(G)$ is a strong split geodetic set of G, if S is a geodetic set and $\langle V - S\rangle$ is totally disconnected. The strong split geodetic number of a graph G, is denoted by $g_{ss}(G)$, is the minimum cardinality of a strong split geodetic set of G. In this paper we obtain the geodetic number, split geodetic number, strong split geodetic number and non split geodetic number of strong product graphs, composition of graphs and join of graphs.

Keywords:

Cartesian product, Distance, Edge covering number, Split geodetic number, Vertex covering number

1. INTRODUCTION

In this paper we follow the notations of [4]. As usual $n = |V|$ and $m = |E|$ denote the number of vertices and edges of a graph G respectively.
The graphs considered here have at least one component which is not complete or at least two non trivial components.
The distance $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u - v$ path in G. It is well known that this distance is a metric on the vertex set $V(G)$. For a vertex v of G, the eccentricity $e(v)$ is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is radius, rad G, and the maximum eccentricity is the diameter, diam G. A $u - v$ path of length d(u, v) is called a $u - v$ geodesic. We define $I[u, v]$ to the set (interval) of all vertices lying on some $u - v$ geodesic of G and for a nonempty subset S of $V(G)$, $I[S] = \bigcup_{u,v\in S} I[u, v]$.
A set S of vertices of G is called a geodetic set in G if $I[S] = V(G)$, and a geodetic set of minimum cardinality is a minimum geodetic set. The cardinality of a minimum geodetic set in G is called the geodetic number of G, and we denote it by $g(G)$.
Split geodetic number of a graph was studied by in [5]. A geodetic set S of a graph $G = (V, E)$ is a split geodetic set if the induced subgraph $\langle V - S\rangle$ is disconnected. The split geodetic number $g_s(G)$ of G is the minimum cardinality of a split geodetic set. Strong split geodetic number of a graph was studied by in [1]. A set S' of vertices of $G = (V, E)$ is called the strong split geodetic set if the induced subgraph $\langle V - S'\rangle$ is totally disconnected and a strong split geodetic set of minimum cardinality is the strong split geodetic number of G and is denoted by $g_{ss}(G)$. Non split geodetic number of a graph was studied by in [6]. A geodetic set S of a graph $G = (V, E)$ is a non split geodetic set if the induced subgraph $\langle V - S\rangle$ is connected. The non split geodetic number $g_{ns}(G)$ of G is the minimum cardinality of a non split geodetic set.
The strong product of graphs G_1 and G_2, denoted by $G_1 \boxtimes G_2$, has vertex set $V(G_1) \times V(G_2)$, where two distinct vertices (x_1, y_1) and (x_2, y_2) are adjacent with respect to the strong product if

(a) $x_1 = x_2$ and $y_1y_2 \in E(G_2)$ or
(b) $y_1 = y_2$ and $x_1x_2 \in E(G_1)$ or
(c) $x_1x_2 \in E(G_1)$ and $y_1y_2 \in E(G_2)$.
For any undefined term in this paper, see [3] and [4].

2. PRELIMINARY NOTES

We need the following results to prove further results.

THEOREM 2.1. [2] *Every geodetic set of a graph contains its extreme vertices.*

THEOREM 2.2. [5] *For the wheel* $W_n = K_1 + C_{n-1}$ $(n \geq 6)$,

$$g_s(W_n) = \begin{cases} \frac{n+2}{2} & \text{if n is even} \\ \frac{n+1}{2} & \text{if n is odd.} \end{cases}$$

THEOREM 2.3. [1] *For the wheel* $W_n = K_1 + C_{n-1}$ $(n \geq 6)$,

$$g_{ss}(W_n) = \begin{cases} \frac{n+2}{2} & \text{if n is even} \\ \frac{n+1}{2} & \text{if n is odd.} \end{cases}$$

PROPOSITION 2.4. *For any graph* G, $g_s(G) \leq g_{ss}(G)$.

3. MAIN RESULTS

THEOREM 3.1. *For any path* P_n *of order* $n > 5$, $g(K_2 \boxtimes P_n) = 4$.

Proof. Let $K_2 \boxtimes P_n$ be formed from two copies of G_1 and G_2 of P_n. Let $U = \{u_1, u_2, ..., u_n\} \in V(G_1)$, $W = \{w_1, w_2, ..., w_n\} \in V(G_2)$ and $V = U \cup W$. Let $S = \{u_1, u_n, w_1, w_n\}$ be the geodetic set of $K_2 \boxtimes P_n$, where $d(u_1, u_n) = diam(K_2 \boxtimes P_n) = d(w_1, w_n)$, which covers all the vertices of $K_2 \boxtimes P_n$. If possible let $P = \{u_1, u_n, w_1\}$, $|P| < |S|$ be set of vertices, for any $w_i \in V(K_2 \boxtimes P_n)$, $w_i \notin I[P]$ hence P is not a geodetic set. Thus S is the minimum geodetic set, there fore $g(K_2 \boxtimes P_n) = 4$.

THEOREM 3.2. *For any path* P_n *of order* $n > 5$, $g_s(K_2 \boxtimes P_n) = 6$.

Proof. Let $K_2 \boxtimes P_n$ be formed from two copies of G_1 and G_2 of P_n. Let $U = \{u_1, u_2, ..., u_n\} \in V(G_1)$, $W = \{w_1, w_2, ..., w_n\} \in V(G_2)$ and $V = U \cup W$. Let $S = H_1 \cup H_2$, where $H_1 = \{u_1, u_n, w_1, w_n\} \subseteq V(K_2 \boxtimes P_n)$ which covers all the vertices of $K_2 \boxtimes P_n$ and $H_2 = (u_i, w_i) \in E(K_2 \boxtimes P_n) \subseteq V - H_1$, u_i and w_i are the vertices having maximum degree i.e $deg(u_i) = deg(w_i) = 5$. Now S be the set of vertices which covers all the vertices of $K_2 \boxtimes P_n$. Such that $V - S$ has more then one component. Then by the above argument S is the minimal split geodetic set of $K_2 \boxtimes P_n$. Clearly it follows that $|S| = |H_1 \cup H_2| = 4 + 2 = 6$. There fore $g_s(K_2 \boxtimes P_n) = 6$.

THEOREM 3.3. *For any path* P_n *of order* $n \geq 5$

$$g_{ss}(K_2 \boxtimes P_n) \begin{cases} = 4 + n - 2 + \lfloor \frac{n}{3} \rfloor & n = 5, 6, 7 \\ \geq 4 + n - 2 + \lceil \frac{n}{3} \rceil & n \geq 8. \end{cases}$$

Proof. Let $K_2 \boxtimes P_n$ be formed from two copies of G_1 and G_2 of P_n. Let $U = \{u_1, u_2, ..., u_n\} \in V(G_1)$, $W = \{w_1, w_2, ..., w_n\} \in V(G_2)$ and $V = U \cup W$.
We have the following results.
Case i. Let $n = 5, 6, 7$.
Consider $S = H_1 \cup H_2 \cup H_3$, where $H_1 = \{u_1, u_n, w_1, w_n\} \subseteq V(K_2 \boxtimes P_n)$, which covers all the vertices of $K_2 \boxtimes P_n$, $H_2 = \{w_2, w_3, ..., w_{n-1}\} \subseteq V - H_1$, $|H_2| = n - 2$ and $H_3 = \{w_3, w_5, ..., w_i\} \subseteq V - H_1$, $|H_3| = \lfloor \frac{n}{3} \rfloor$. Now S be the set of vertices which covers all the vertices of $K_2 \boxtimes P_n$, such that $V - S$ is totally disconnected. Then by the above argument S is a minimal strong split geodetic set of $K_2 \boxtimes P_n$. Clearly $|S| = |H_1 \cup H_2 \cup H_3| = 4+n-2+\lfloor \frac{n}{3} \rfloor$. There fore $g_{ss}(K_2 \boxtimes P_n) = 4+n-2+\lfloor \frac{n}{3} \rfloor$.
Case ii. Let $n \geq 8$.
Consider $S = H_1 \cup H_2 \cup H_3$, where $H_1 = \{u_1, u_n, w_1, w_n\} \subseteq V(K_2 \boxtimes P_n)$, which covers all the vertices of $K_2 \boxtimes P_n$, $H_2 = \{w_2, w_3, ..., w_{n-1}\} \subseteq V - H_1$, $|H_2| = n - 2$ and $H_3 = \{w_3, w_5, ..., w_i\} \subseteq V - H_1$, $|H_3| \geq \lceil \frac{n}{3} \rceil$. Now S be the set of vertices which covers all the vertices of $K_2 \boxtimes P_n$, such that $V - S$ is totally disconnected. Then by the above argument S is a minimal strong split geodetic set of $K_2 \boxtimes P_n$. Clearly $|S| = |H_1 \cup H_2 \cup H_3| \geq 4+n-2+\lceil \frac{n}{3} \rceil$. There fore $g_{ss}(K_2 \boxtimes P_n) \geq 4+n-2+\lceil \frac{n}{3} \rceil$.

THEOREM 3.4. *For any path* P_n *of order* $n > 5$, $g_{ns}(K_2 \boxtimes P_n) = 4$.

Proof. Let $K_2 \boxtimes P_n$ be formed from two copies of G_1 and G_2 of P_n. Let $U = \{u_1, u_2, ..., u_n\} \in V(G_1)$, $W = \{w_1, w_2, ..., w_n\} \in V(G_2)$ and $V = U \cup W$. Let $S = \{u_1, u_n, w_1, w_n\}$ be the non split geodetic set of $K_2 \boxtimes P_n$, where $d(u_1, u_n) = diam(K_2 \boxtimes P_n) = d(w_1, w_n)$, which covers all the vertices of $K_2 \boxtimes P_n$ such that $V - S$ is connected. If possible let $P = \{u_1, u_n, w_1\}$, $|P| < |S|$ be set of vertices, for any $w_i \in V(K_2 \boxtimes P_n)$, $w_i \notin I[P]$ hence P is not a geodetic set. Thus S is the minimum non split geodetic set, there fore $g_{ns}(K_2 \boxtimes P_n) = 4$.

THEOREM 3.5. *For any cycle* C_n *of order* $n > 3$,

$$g(K_2 \boxtimes C_n) = \begin{cases} 4 & \text{if n is even} \\ 6 & \text{if n is odd.} \end{cases}$$

Proof. Let $K_2 \boxtimes C_n$ be formed from two copies of G_1 and G_2 of C_n. Let $U = \{u_1, u_2, ..., u_n\} \in V(G_1)$, $W = \{w_1, w_2, ..., w_n\} \in V(G_2)$ and $V = U \cup W$.
We have the following results.
Case i. Let n be even.
Consider $S = \{u_i, u_j, w_i, w_j\}$ be the set of vertices such that $d(u_i, u_j) = diam(K_2 \boxtimes C_n) = d(w_i, w_j)$ and $\{(u_i, w_i), (u_j, w_j)\} \in E(K_2 \boxtimes C_n)$, which covers all the vertices of $K_2 \boxtimes C_n$. Let $P = \{u_i, u_j, w_i\}$, $|P| < |S|$, for any $w_k \in V(K_2 \boxtimes C_n)$, $w_k \notin I[P]$. Hence P is not a geodetic set. Thus S is the minimum geodetic set of $K_2 \boxtimes C_n$. There fore $g(K_2 \boxtimes C_n) = 4$.
Case ii. Let n be odd.
Consider $S = \{u_i, u_j, u_k, w_i, w_j\}$ be the set of vertices such that $d(u_i, u_j) = d(u_j, u_k) = diam(K_2 \boxtimes C_n) = d(w_i, w_j) = d(w_j, w_k)$ and $\{(u_i, w_i), (u_j, w_j), (u_k, w_k)\} \in E(K_2 \boxtimes C_n)$, which covers all the vertices of $K_2 \boxtimes C_n$. Let $P = \{u_i, u_j, u_k, w_i, w_j\}$, $|P| < |S|$, for any $w_l \in V(K_2 \boxtimes C_n)$, $w_l \notin I[P]$. Hence P is not a geodetic set. Thus S is the minimum geodetic set of $K_2 \boxtimes C_n$. There fore $g(K_2 \boxtimes C_n) = 6$.

THEOREM 3.6. *For any cycle* C_n *of order* $n > 3$,

$$g_s(K_2 \boxtimes C_n) = \begin{cases} 4 & \text{if n is even} \\ 6 & \text{if n is odd.} \end{cases}$$

Proof. Let $K_2 \boxtimes C_n$ be formed from two copies of G_1 and G_2 of C_n. Let $U = \{u_1, u_2, ..., u_n\} \in V(G_1)$, $W = \{w_1, w_2, ..., w_n\} \in V(G_2)$ and $V = U \cup W$.
We have the following results.
Case i. Let n be even.
Consider $S = \{u_i, u_j, w_i, w_j\}$ be the split geodetic set, where $d(u_i, u_j) = diam(K_2 \boxtimes C_n) = d(w_i, w_j)$ and $\{(u_i, w_i), (u_j, w_j)\} \in E(K_2 \boxtimes C_n)$, which covers all the vertices of $K_2 \boxtimes C_n$, such that $V - S$ is disconnected. Let $P = \{u_i, u_j, w_i\}$, $|P| < |S|$, for any $w_k \in V(K_2 \boxtimes C_n)$, $w_k \notin I[P]$. Hence P is not a geodetic set. Thus S is the minimal split geodetic set of $K_2 \boxtimes C_n$. There fore $g_s(K_2 \boxtimes C_n) = 4$.
Case ii. Let n be odd.

Consider $S = \{u_i, u_j, u_k, w_i, w_j\}$ be the split geodetic set, where $d(u_i, u_j) = d(u_j, u_k) = diam(K_2 \boxtimes C_n) = d(w_i, w_j) = d(w_j, w_k)$ and $\{(u_i, w_i), (u_j, w_j), (u_k, w_k)\} \in E(K_2 \boxtimes C_n)$, which covers all the vertices of $K_2 \boxtimes C_n$, such that $V - S$ is disconnected. Let $P = \{u_i, u_j, u_k, w_i, w_j\}$, $|P| < |S|$, for any $w_l \in V(K_2 \boxtimes C_n)$, $w_l \notin I[P]$. Hence P is not a geodetic set. Thus S is the minimal split geodetic set of $K_2 \boxtimes C_n$. There fore $g_s(K_2 \boxtimes C_n) = 6$.

THEOREM 3.7. *For any cycle C_n of order $n > 3$, $g_{ss}(K_2 \boxtimes C_n) = 2n - 4$.*

Proof. Let $K_2 \boxtimes C_n$ be formed from two copies of G_1 and G_2 of C_n. Let $U = \{u_1, u_2, ..., u_n\} \in V(G_1)$, $W = \{w_1, w_2, ..., w_n\} \in V(G_2)$ and $V = U \cup W$.
We have the following results.
Case i. Let n be even.
Let $H_1 = \{u_i, u_j, w_i, w_j\}$ be the minimum set of vertices which covers all the vertices of $K_2 \boxtimes C_n$ by case i of Theorem 3.5 . Now $S = H_1 \cup H_2$, where $H_2 \in V - H_1$, $V - H_1$ has two identical components and $|H_2| = 2(n - 4)$. Thus $I(S) = V(K_2 \boxtimes C_n)$, clearly $V - S$ has independent set. Then by the above argument S is a minimal strong split geodetic set of $K_2 \boxtimes C_n$. Hence $|S| = |H_1 \cup H_2| = 4 + 2(n-4) = 2n-4$. There fore $g_{ss}(K_2 \boxtimes C_n) = 2n - 4$.
Case ii. Let n be odd.
Let $H_1 = \{u_i, u_j, u_k, w_i, w_j, w_k\}$ be the minimum set of vertices which covers all the vertices of $K_2 \boxtimes C_n$ by case ii of Theorem 3.5 . Now $S = H_1 \cup H_2$, where $H_2 \in V - H_1$, $V - H_1$ has two identical components and $|H_2| = 2(n - 5)$. Thus $I(S) = V(K_2 \boxtimes C_n)$, clearly $V - S$ has independent set. Then by the above argument S is a minimal strong split geodetic set of $K_2 \boxtimes C_n$. Hence $|S| = |H_1 \cup H_2| = 6 + 2(n - 5) = 2n - 4$. There fore $g_{ss}(K_2 \boxtimes C_n) = 2n - 4$.

THEOREM 3.8. *G' be the graph obtained by adding an end-edge (x, y) to a cycle $C_n = G$ of order $n > 3$, with $x \in G$ and $y \notin G$. Then*

$$g(K_2 \boxtimes G') = \begin{cases} 4 & \text{for even cycle} \\ 6 & \text{for odd cycle.} \end{cases}$$

Proof. Let $K_2 \boxtimes G'$ be formed from two copies of G'_1 and G'_2 of G'. Let $U = \{u_1, u_2, ..., u_n\} \in V(G'_1)$, $W = \{w_1, w_2, ..., w_n\} \in V(G'_2)$ and $V = U \cup W$.
We have the following results.
Case i. For even cycle.
Let $S = \{u_i, u_j, w_i, w_j\}$ be the set of vertices such that u_i and w_i are the vertices formed from end-vertex of G', $(u_i, w_i) \in E(K_2 \boxtimes G')$ and u_j, w_j are the vertices formed from the antipodal vertex of G corresponding to the vertex adjacent to end vertex, $(u_j, w_j) \in E(K_2 \boxtimes G')$. Clearly $I(S) = V(K_2 \boxtimes G')$. Thus S is the minimal geodetic set. There fore $g(K_2 \boxtimes G') = 4$.
Case ii. For odd cycle.
Let $S = \{u_i, u_j, u_k, w_i, w_j, w_k\}$ be the set of vertices such that u_i and w_i are the vertices formed from end-vertex of G', $(u_i, w_i) \in E(K_2 \boxtimes G')$ and $d(u_i, u_j) = 2$, $d(u_j, u_k) = \lfloor \frac{n}{2} \rfloor$ similarly $d(w_i, w_j) = 2$, $d(w_j, w_k) = \lfloor \frac{n}{2} \rfloor$ and $\{(u_j, w_j), (u_k, w_k)\} \in E(K_2 \boxtimes G')$. Clearly $I(S) = V(K_2 \boxtimes G')$. Thus S is the minimal geodetic set. There fore $g(K_2 \boxtimes G') = 6$.

THEOREM 3.9. *G' be the graph obtained by adding an end-edge (x, y) to a cycle $C_n = G$ of order $n > 3$, with $x \in G$ and $y \notin G$. Then*

$$g_s(K_2 \boxtimes G') = \begin{cases} 7 & \text{for even cycle} \\ 6 & \text{for odd cycle.} \end{cases}$$

Proof. Let $K_2 \boxtimes G'$ be formed from two copies of G'_1 and G'_2 of G'. Let $U = \{u_1, u_2, ..., u_n\} \in V(G'_1)$, $W = \{w_1, w_2, ..., w_n\} \in V(G'_2)$ and $V = U \cup W$.
We have the following results.
Case i. For even cycle.
Let $H_1 = \{u_i, u_j, w_i, w_j\}$ be the minimum set of vertices which covers all the vertices of $K_2 \boxtimes G'$ by case i of Theorem 3.8. Consider $S = H_1 \cup H_2$, where $H_2 = \{u_k, w_k, w_l\} \subseteq V - H_1$ such that $V - S$ has more then one component. Then by the above argument S is minimal split geodetic set of $K_2 \boxtimes G'$. Hence $|S| = |H_1 \cup H_2| = 4 + 3 = 7$. There fore $g_s(K_2 \boxtimes G') = 7$.
Case ii. For odd cycle.
Let $S = \{u_i, u_j, u_k, w_i, w_j, w_k\}$ be the minimal geodetic set by case ii of Theorem 3.8. Since $V - S$ has two components S it self is the minimal split geodetic set of $K_2 \boxtimes G'$.There fore $g_s(K_2 \boxtimes G') = 6$.

THEOREM 3.10. *G' be the graph obtained by adding an end-edge (x, y) to a cycle $C_n = G$ of order $n > 3$, with $x \in G$ and $y \notin G$. Then*

$$g_{ss}(K_2 \boxtimes G') = \begin{cases} \frac{3n+4}{2} & \text{for even cycle} \\ 6 + n - 2 + \lfloor \frac{n}{3} \rfloor & \text{for n=5} \\ 6 + n & \text{for n=7} \\ 6 + n - 2 + \lceil \frac{n}{3} \rceil & \text{for odd cycle.} \end{cases}$$

Proof. Let $K_2 \boxtimes G'$ be formed from two copies of G'_1 and G'_2 of G'. Let $U = \{u_1, u_2, ..., u_n\} \in V(G'_1)$, $W = \{w_1, w_2, ..., w_n\} \in V(G'_2)$ and $V = U \cup W$.
We have the following results.
Case i. For even cycle.
Consider $S = H_1 \cup H_2$ be the minimal strong split geodetic set of $K_2 \boxtimes G'$, where $H_1 = \{u_i, u_j, w_i, w_j, u_k, w_k, w_l\}$, $I(H_1) = V(K_2 \boxtimes G')$ by case i of Theorem 3.9 and $H_2 \subseteq V - H_1$, $|H_2| = \frac{3n-10}{2}$, such that $V - S$ is totally disconnected. Hence $|S| = |H_1 \cup H_2| = 7 + \frac{3n-10}{2} = \frac{3n+4}{2}$. There fore $g_{ss}(K_2 \boxtimes G') = \frac{3n+4}{2}$.
Case ii. For $n = 5$.
Consider $S = H_1 \cup H_2$ be the minimal strong split geodetic set of $K_2 \boxtimes G'$, where $H_1 = \{u_i, u_j, u_k, w_i, w_j, w_k\}$ be the set of vertices which covers all the vertices of $K_2 \boxtimes G'$ by case ii of Theorem 3.8 and $H_2 \subseteq V - H_1$, $|H_2| = n - 2 + \lfloor \frac{n}{3} \rfloor$, such that $V - S$ is totally disconnected. Hence $|S| = |H_1 \cup H_2| = 6 + n - 2 + \lfloor \frac{n}{3} \rfloor$. There fore $g_{ss}(K_2 \boxtimes G') = 6 + n - 2 + \lfloor \frac{n}{3} \rfloor$.
Case iii. For $n = 7$.
Consider $S = H_1 \cup H_2$ be the minimal strong split geodetic set of $K_2 \boxtimes G'$, where $H_1 = \{u_i, u_j, u_k, w_i, w_j, w_k\}$ be the set of vertices which covers all the vertices of $K_2 \boxtimes G'$ by case ii of Theorem 3.8 and $H_2 \subseteq V - H_1$, $|H_2| = n$, such that $V - S$ is to-

tally disconnected. Hence $|S| = |H_1 \cup H_2| = 6 + n$. There fore $g_{ss}(K_2 \boxtimes G^{'}) = 6 + n$.
Case iv. For odd cycle ($n > 7$).
Consider $S = H_1 \cup H_2$ be the minimal strong split geodetic set of $K_2 \boxtimes G^{'}$, where $H_1 = \{u_i, u_j, u_k, w_i, w_j, w_k\}$ be the set of vertices which covers all the vertices of $K_2 \boxtimes G^{'}$ by case ii of Theorem 3.8 and $H_2 \subseteq V - H_1$, $|H_2| = n - 2 + \lceil \frac{n}{3} \rceil$, such that $V - S$ is totally disconnected. Hence $|S| = |H_1 \cup H_2| = 6 + n - 2 + \lceil \frac{n}{3} \rceil$. There fore $g_{ss}(K_2 \boxtimes G^{'}) = 6 + n - 2 + \lceil \frac{n}{3} \rceil$.

THEOREM 3.11. *For the wheel $W_n = K_1 + C_{n-1}$ ($n \geq 6$),*

$$g(K_2 \boxtimes W_n) = \begin{cases} n & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Let $K_2 \boxtimes W_n$ be formed from two copies of G_1 and G_2 of W_n. Let $U = \{u_1, u_2, ..., u_n\} \in V(G_1)$, $W = \{w_1, w_2, ..., w_n\} \in V(G_2)$ and $V = U \cup W$.
We have the following results.
Case i. Let n be even.
Consider the geodesic $P : \{u_1, u_2, u_6, w_2, w_6, u_3\}$, $Q : \{u_3, u_4, u_6, w_4, w_6, u5\}$,..., $R : \{u_{n-3}, u_{n-2}, u_n, w_{n-2}, w_n, u_{n-1}\}$ and the geodesics, $H = \{w_1 - w_3, w_3 - w_5, ..., w_{n-3} - w_{n-1}\}$ It is clear that the vertices $u_2, u_4, u_6, ..., u_{n-2}$,$w_2, w_4, w_6, ..., w_{n-2}$ lies on the geodesics P,Q,R and H. Thus the set $S = \{u_1, u_3, ..., u_{n-1}$,$w_1, w_3, ..., w_{n-1}\}$ is the minimum geodetic set $K_2 \boxtimes W_n$. There fore $g(K_2 \boxtimes W_n) = |S| = n$.
Case ii. Let n be odd.
Consider the geodesic $P : \{u_1, u_2, u_7, w_2, w_7, u_3\}$, $Q : \{u_3, u_4, u_7, w_4, w_7, u5\}$,..., $R : \{u_{n-4}, u_{n-3}, u_n, w_{n-3}, w_n, u_{n-2}\}$ and the geodesics, $H = \{w_1 - w_3, w_3 - w_5, ..., w_{n-4} - w_{n-2}\}$ It is clear that the vertices $u_2, u_4, u_6, ..., u_{n-1}$,$w_2, w_4, w_6, ..., w_{n-1}$ lies on the geodesics P,Q,R and H. Thus the set $S = \{u_1, u_3, ..., u_{n-2}$,$w_1, w_3, ..., w_{n-2}\}$ is the minimum geodetic set $K_2 \boxtimes W_n$. There fore $g(K_2 \boxtimes W_n) = |S| = n - 1$.

THEOREM 3.12. *For the wheel $W_n = K_1 + C_{n-1}$ ($n \geq 6$),*

$$g_s(K_2 \boxtimes W_n) = \begin{cases} n+2 & \text{if } n \text{ is even} \\ n+1 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Let $K_2 \boxtimes W_n$ be formed from two copies of G_1 and G_2 of W_n. Let $U = \{u_1, u_2, ..., u_n\} \in V(G_1)$, $W = \{w_1, w_2, ..., w_n\} \in V(G_2)$ and $V = U \cup W$.
We have the following results.
Case i. Let n be even.
Consider $S = H_1 \cup H_2$ where $H_1 = \{u_1, u_3, ..., u_{n-1}$,$w_1, w_3, ..., w_{n-1}\}$ be the minimum geodetic set by case i of Theorem 3.11 and $H_2 = \{v_i, w_i\} \subseteq V(K_2 \boxtimes W_n)$, formed from the vertex K_1 of W_n. Now S is the minimal split geodetic set of $K_2 \boxtimes W_n$, since $V - S$ has $\frac{n-2}{2}$ times K_2 components. Hence $|S| = |H_1 \cup H_2| = n + 2$. There fore $g_s(K_2 \boxtimes W_n) = n + 2$.
Case ii. Let n be odd.
Consider $S = H_1 \cup H_2$ where $H_1 = \{u_1, u_3, ..., u_{n-2}$,$w_1, w_3, ..., w_{n-2}\}$ be the minimum geodetic set by case ii of Theorem 3.11 and $H_2 = \{v_i, w_i\} \subseteq V(K_2 \boxtimes W_n)$, formed from the vertex K_1 of W_n. Now S is the minimal split geodetic set of $K_2 \boxtimes W_n$, since $V - S$ has $\frac{n-1}{2}$ times K_2 components. Hence $|S| = |H_1 \cup H_2| = n - 1 + 2 = n + 1$. There fore $g_s(K_2 \boxtimes W_n) = n + 1$.

THEOREM 3.13. *For the wheel $W_n = K_1 + C_{n-1}$ ($n \geq 6$),*

$$g_{ss}(K_2 \boxtimes W_n) = \begin{cases} \frac{3n+2}{2} & \text{if } n \text{ is even} \\ \frac{3n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Let $K_2 \boxtimes W_n$ be formed from two copies of G_1 and G_2 of W_n. Let $U = \{u_1, u_2, ..., u_n\} \in V(G_1)$, $W = \{w_1, w_2, ..., w_n\} \in V(G_2)$ and $V = U \cup W$.
We have the following results.
Case i. Let n be even.
Consider $H = H_1 \cup H_2$ where $H_1 = \{u_1, u_3, ..., u_{n-1}$,$w_1, w_3, ..., w_{n-1}\}$ be the minimum geodetic set by case i of Theorem 3.11 and $H_2 = \{v_i, w_i\} \subseteq V(K_2 \boxtimes W_n)$, formed from the vertex K_1 of W_n, such that $V - H$ has $\frac{n-2}{2}$ times K_2 components. Let $S = H \cup H_3$, where $H_3 \subseteq V - H$ consists of one vertex from each K_2 components, $|H_3| = \frac{n-2}{2}$. Now S is the minimal strong split geodetic set of $K_2 \boxtimes W_n$ since $V - S$ has isolated vertices. Hence $|S| = |H \cup H_3| = |H_1 \cup H_2 \cup H_3| = n + 2 + \frac{n-2}{2} = \frac{3n+2}{2}$. There fore $g_{ss}(K_2 \boxtimes W_n) = \frac{3n+2}{2}$.
Case ii. Let n be odd.
Consider $H = H_1 \cup H_2$ where $H_1 = \{u_1, u_3, ..., u_{n-2}$,$w_1, w_3, ..., w_{n-2}\}$ be the minimum geodetic set by case ii of Theorem 3.11 and $H_2 = \{v_i, w_i\} \subseteq V(K_2 \boxtimes W_n)$, formed from the vertex K_1 of W_n, such that $V - H$ has $\frac{n-1}{2}$ times K_2 components. Let $S = H \cup H_3$, where $H_3 \subseteq V - H$ consists of one vertex from each K_2 components, $|H_3| = \frac{n-1}{2}$. Now S is the minimal strong split geodetic set of $K_2 \boxtimes W_n$ since $V - S$ has isolated vertices. Hence $|S| = |H \cup H_3| = |H_1 \cup H_2 \cup H_3|$ $= n - 1 + 2 + \frac{n-1}{2} = \frac{3n+1}{2}$. There fore $g_{ss}(K_2 \boxtimes W_n) = \frac{3n+1}{2}$.

THEOREM 3.14. *For the wheel $W_n = K_1 + C_{n-1}$ ($n \geq 6$),*

$$g_{ns}(K_2 \boxtimes W_n) = \begin{cases} n & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Let $K_2 \boxtimes W_n$ be formed from two copies of G_1 and G_2 of W_n. Let $U = \{u_1, u_2, ..., u_n\} \in V(G_1)$, $W = \{w_1, w_2, ..., w_n\} \in V(G_2)$ and $V = U \cup W$.
We have the following results.
Case i. Let n be even.
Consider the geodesic $P : \{u_1, u_2, u_6, w_2, w_6, u_3\}$, $Q : \{u_3, u_4, u_6, w_4, w_6, u5\}$,..., $R : \{u_{n-3}, u_{n-2}, u_n, w_{n-2}, w_n, u_{n-1}\}$ and the geodesics, $H = \{w_1 - w_3, w_3 - w_5, ..., w_{n-3} - w_{n-1}\}$ It is clear that the vertices $u_2, u_4, u_6, ..., u_{n-2}$,$w_2, w_4, w_6, ..., w_{n-2}$ lies on the geodesics P,Q,R and H. Thus the set $S = \{u_1, u_3, ..., u_{n-1}$,$w_1, w_3, ..., w_{n-1}\}$ is the minimum non split geodetic set $K_2 \boxtimes W_n$, since $V - S$ is connected. There fore $g_{ns}(K_2 \boxtimes W_n) = |S| = n$.
Case ii. Let n be odd.
Consider the geodesic $P : \{u_1, u_2, u_7, w_2, w_7, u_3\}$, $Q : \{u_3, u_4, u_7, w_4, w_7, u5\}$,..., $R : \{u_{n-4}, u_{n-3}, u_n, w_{n-3}, w_n, u_{n-2}\}$ and the geodesics, $H = \{w_1 - w_3, w_3 - w_5, ..., w_{n-4} - w_{n-2}\}$ It is clear that the vertices $u_2, u_4, u_6, ..., u_{n-1}$,$w_2, w_4, w_6, ..., w_{n-1}$ lies on the geodesics P,Q,R and H. Thus the set $S = \{u_1, u_3, ..., u_{n-2}$,$w_1, w_3, ..., w_{n-2}\}$ is the minimum

non split geodetic set $K_2 \boxtimes W_n$, since $V - S$ is connected. There fore $g_{ns}(K_2 \boxtimes W_n) = |S| = n - 1$.

THEOREM 3.15. *For any tree T with each internal vertex connected to single end-edge, having k end-edges. Then $g(K_2 \boxtimes T) = 2k$.*

Proof. Let $K_2 \boxtimes T$ be formed from two copies of G_1 and G_2 of T. Let $U = \{u_1, u_2, ..., u_n\} \in V(G_1)$, $W = \{w_1, w_2, ..., w_n\} \in V(G_2)$ and $V = U \cup W$. Let $S = \{u_1, u_2, ..., u_k, w_1, w_2, ..., w_k\}$ be the set of vertices of $K_2 \boxtimes T$ formed from the set of end-edges of T, such that $I(S) = V(K_2 \boxtimes T)$. Thus S is the minimal geodetic set of $K_2 \boxtimes T$. There fore $g(K_2 \boxtimes T) = |S| = 2k$.

THEOREM 3.16. *For any tree T with each internal vertex connected to single end-edge, having k end-edges. Then $g_s(K_2 \boxtimes T) = 2k + 2$.*

Proof. Let $K_2 \boxtimes T$ be formed from two copies of G_1 and G_2 of T. Let $U = \{u_1, u_2, ..., u_n\} \in V(G_1)$, $W = \{w_1, w_2, ..., w_n\} \in V(G_2)$ and $V = U \cup W$. Consider $S = H_1 \cup H_2$, where $H_1 = \{u_1, u_2, ..., u_k, w_1, w_2, ..., w_k\} \subseteq V(K_2 \boxtimes T)$ formed from the set of end-edges of T and $H_2 = \{u_i, w_i\} \subseteq V - H_1$ having maximum degree, $I(S) = V(K_2 \boxtimes T)$ and $V - S$ is disconnected. Thus by the above argument S is the minimal split geodetic set of $K_2 \boxtimes T$. There fore $g_s(K_2 \boxtimes T) = |S| = |H_1 \cup H_2| = 2k + 2$.

THEOREM 3.17. *For any tree T with each internal vertex connected to single end-edge, having k end-edges. Then*

$$g_{ss}(K_2 \boxtimes T) = \begin{cases} \frac{7k}{2} & \text{if k is even} \\ \frac{7k-1}{2} & \text{if k is odd.} \end{cases}$$

Proof. Let $K_2 \boxtimes T$ be formed from two copies of G_1 and G_2 of T. Let $U = \{u_1, u_2, ..., u_n\} \in V(G_1)$, $W = \{w_1, w_2, ..., w_n\} \in V(G_2)$ and $V = U \cup W$.
We have the following results.
Case i. Let k be even.
Consider $S = H_1 \cup H_2$, be the strong split geodetic set of $K_2 \boxtimes T$, where $H_1 = \{u_1, u_2, ..., u_k, w_1, w_2, ..., w_k\} \subseteq V(K_2 \boxtimes T)$ formed from the set of end-edges of T, $I(H_1) = V(K_2 \boxtimes T)$ and $H_2 = \{u_i, w_i, u_j, w_j, ...\} \subseteq V - H_1$, $|H_2| = \frac{3k}{2}$. Thus by the above argument S is the minimal strong split geodetic set of $K_2 \boxtimes T$. There fore $g_{ss}(K_2 \boxtimes T) = |S| = |H_1 \cup H_2| = 2k + \frac{3k}{2} = \frac{7k}{2}$.
Case ii. Let k be odd.
Consider $S = H_1 \cup H_2$, be the strong split geodetic set of $K_2 \boxtimes T$, where $H_1 = \{u_1, u_2, ..., u_k, w_1, w_2, ..., w_k\} \subseteq V(K_2 \boxtimes T)$ formed from the set of end-edges of T, $I(H_1) = V(K_2 \boxtimes T)$ and $H_2 = \{u_i, w_i, u_j, w_j, ...\} \subseteq V - H_1$, $|H_2| = \frac{3k-1}{2}$. Thus by the above argument S is the minimal strong split geodetic set of $K_2 \boxtimes T$. There fore $g_{ss}(K_2 \boxtimes T) = |S| = |H_1 \cup H_2| = 2k + \frac{3k-1}{2} = \frac{7k-1}{2}$.

4. COMPOSITION OF GRAPHS

The composition $G = G_1[G_2]$ has $V = V_1 \times V_2$ as its point set, and $u = (u_1, u_2)$ is adjacent with $v = (v_1, v_2)$ whenever [u_1 adj v_1] or [$u_1 = v_1$ and u_2 adj v_2].

THEOREM 4.1. *For any cycle C_n of order $n > 3$,*

$$g(K_2[C_n]) = \begin{cases} 2 & \text{for n =4} \\ 3 & \text{for n =5, 6} \\ 4 & \text{for n ¿ 6.} \end{cases}$$

Proof. Let $K_2[C_n]$ be formed from two copies of G_1 and G_2 of C_n. Let $U = \{u_1, u_2, ..., u_n\} \in V(G_1)$, $W = \{w_1, w_2, ..., w_n\} \in V(G_2)$ and $V = U \cup W$.
We have the following results.
Case i. Let $n = 4$.
Consider $S = \{v_1, v_3\}$ be the set which covers all the vertices of $K_2[C_n]$, where $\{v_1, v_3\} \notin E(G_1)$. Which forms a minimum geodetic set of $K_2[C_n]$, therefore $g(K_2[C_n]) = 2$.
Case ii. Let $n = 5, 6$.
Consider $S = \{v_1, v_3, v_5\}$ be the set which covers all the vertices of $K_2[C_n]$, where $\{(v_1, v_3), (v_3, v_5)\} \notin E(G_1)$ and $d(v_1, v_3) = d(v_3, v_5) = diam(K_2[C_n])$. If possible let $P = \{v_1, v_3\} \in V(K_2[C_n])$, $|P| < |S|$ be a set, for any $v_i \notin I[P]$. Thus S ia minimal geodetic set $K_2[C_n]$. Therefore $g(K_2[C_n]) = 3$.
Case iii. Let $n > 6$.
Consider $S = \{v_i, v_j, w_i, w_j\}$ be the minimal geodetic set of $K_2[C_n]$, where $\{(v_i, v_j), (w_i, w_j)\} \notin E(K_2[C_n])$ and $d(v_i, v_j) = d(w_i, w_j) = diam(K_2[C_n])$, which covers all the vertices of $K_2[C_n]$. Thus $g(K_2[C_n]) = 4$.

THEOREM 4.2. *For any cycle C_n of order $n > 3$,*

$$g_s(K_2[C_n]) = \begin{cases} \frac{2n+4}{2} & \text{for n =4,5,6} \\ n + 2 & \text{for n¿6.} \end{cases}$$

THEOREM 4.3. *For any cycle C_n of order $n > 3$,*

$$g_{ss}(K_2[C_n]) = \begin{cases} \frac{3n}{2} & \text{if n is even} \\ \frac{3n+1}{2} & \text{if n is odd.} \end{cases}$$

THEOREM 4.4. *G' be the graph obtained by adding an end-edge (x, y) to a cycle $C_n = G$ of order $n > 3$, with $x \in G$ and $y \notin G$. Then $g(K_2[G']) = 4$.*

THEOREM 4.5. *G' be the graph obtained by adding an end-edge (x, y) to a cycle $C_n = G$ of order $n > 3$, with $x \in G$ and $y \notin G$. Then $g_s(K_2[G']) = n + 3$*

THEOREM 4.6. *G' be the graph obtained by adding an end-edge (x, y) to a cycle $C_n = G$ of order $n > 3$, with $x \in G$ and $y \notin G$. Then*

$$g_{ss}(K_2[G']) = \begin{cases} \frac{3n+2}{2} & \text{for even cycle} \\ \frac{3n+3}{2} & \text{for odd cycle.} \end{cases}$$

5. JOIN OF GRAPHS

The join of two graphs G_1 and G_2, written as $G_1 + G_2$, is defined as the union of G_1 and G_2 together with all edges (u, v) for which $u \in V(G_1)$ and $v \in V(G_2)$. Two vertices of a graph G are said to be joined in G if the edge (u, v) is contained in the edge set of G.

THEOREM 5.1. *For any cycle C_n of order $n > 3$,*

$$g(K_2 + C_n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

THEOREM 5.2. *For any cycle C_n of order $n > 3$,*

$$g_s(K_2 + C_n) = g_{ss}(K_2 + C_n) = \begin{cases} \frac{n+4}{2} & \text{if } n \text{ is even} \\ \frac{n+5}{2} & \text{if } n \text{ is odd.} \end{cases}$$

THEOREM 5.3. *G' be the graph obtained by adding an end-edge (x, y) to a cycle $C_n = G$ of order $n > 3$, with $x \in G$ and $y \notin G$. Then*

$$g(K_2 + G') = \begin{cases} \frac{n+2}{2} & \text{for even cycle} \\ \frac{n+3}{2} & \text{for odd cycle.} \end{cases}$$

THEOREM 5.4. *G' be the graph obtained by adding an end-edge (x, y) to a cycle $C_n = G$ of order $n > 3$, with $x \in G$ and $y \notin G$. Then*

$$g_s(K_2 + G') = g_{ss}(K_2 + G') = \begin{cases} \frac{n+6}{2} & \text{for even cycle} \\ \frac{n+7}{2} & \text{for odd cycle.} \end{cases}$$

6. CONCLUSION

In this paper we have establish many results on split geodetic number, nonsplit geodetic number, strong split geodetic number of strong product of graph and some observation on split geodetic number, nonsplit geodetic number, strong split geodetic number of composition graphs and join of graphs.

7. ACKNOWLEDGEMENT

Our thanks to the referee's for the help in revising the manuscript.

8. REFERENCES

[1] Ashalatha K.S.,Venkanagouda. M. Goudar, Venkatesha., 2014. Strong split geodetic number of a graph, International Journal of Computer Applications., 89(4) (2014), 1-4

[2] G. Chartrand, F. Harary, and P.Zhang, 2002. On the geodetic number of a graph.Networks.39,(2002),1-6.

[3] G. Chartrand and P.Zhang, 2006. Introduction to Graph Theory, Tata McGraw Hill Pub.Co.Ltd.

[4] F.Harary, 1969. Graph Theory,Addison-Wesely,Reading,MA.

[5] Venkanagouda M.Goudar, K.S.Ashalatha, Venkatesha, 2014. Split Geodetic Number of a Graph, Advances and Applications in Discrete Mathematics. 13(1) (2014), 9-22.

[6] Venkanagouda. M. Goudar, Tejaswini K. M, Venkatesha., Non split geodetic number of a graph, Indian Journal of Pure and Applied Mathematics. (Communicated).