Weakly Compatible Maps in Complex Valued G- Metric Spaces

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ABSTRACT
In this paper, we introduce the notion of complex valued G-metric spaces and prove a common fixed point theorem for weakly compatible maps in this newly defined spaces.

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Complex valued G-metric space, weakly compatible.

1. INTRODUCTION
The study of fixed points of mappings satisfying certain contractive conditions have been at the center of rigorous research activity. Recently, Mustafa and Sims [8,9] have shown that most of the results concerning Dhage's D-metric spaces are invalid, therefore they introduced an improved version of the generalized metric space structure which they called G-metric spaces.

In 2006, Mustafa and Sims [9] introduced the concept of G-metric spaces as follows:

Definition 1.1. Let X be a non-empty set, and let G: $X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties:

(G1) G(x, y, z) =0 if x = y = z,

(G2) $0 < G(x, y, z)$ for all x, y ∈ X with x ≠ y,

(G3) $G(x, x, y) \leq G(x, y, z)$ for all x, y, z ∈ X with y ≠ z,

(G4) G(x, y, z) = G(x, z, y) = G(y, z, x) =... (symmetry in all three variables), (G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all x, y, z, a ∈ X(rectangle inequality).

Then the function G is called a generalized metric or, more specially a G-metric on X, and the pair (X, G) is called a G- metric space.

The idea of complex metric space was initiated by Azam et.al.[1] to exploit the idea of complex valued normed spaces and complex valued Hilbert spaces.

Definition 1.2. Let ℂ be the set of complex numbers andz_1, z_2 ∈ ℂ. Define a partial order $\precsim$ on ℂ as follows:

$z_1 \precsim z_2$ if and only if Re $(z_1) \leq$ Re (z_2) and Im $(z_1) \leq$ Im (z_2)

That is $z_1 \precsim z_2$ if one of the following holds

(C1): Re $(z_1) =$ Re (z_2) and Im $(z_1) =$ Im (z_2)

(C2): Re $(z_1) <$ Re (z_2) and Im $(z_1) =$ Im (z_2)

(C3): Re $(z_1) =$ Re (z_2) and Im $(z_1) <$ Im (z_2)

(C4): Re $(z_1) <$ Re (z_2) and Im $(z_1) <$ Im (z_2)

In particular, we will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (C2), (C3) and (C4) is satisfied and we will write $z_1 \prec z_2$ if only (C4) is satisfied.

Remark 1.3. We obtained that the following statements hold:

(i) a, b ∈ R and $a \leq b \Longrightarrow az \precsim bz$ for all z ∈ ℂ

(ii) $0 \precsim z_1 \precnsim z_2 \Longrightarrow |z_1| < |z_2|$

(iii) $z_1 \precsim z_2$ and $z_2 \prec z_3 \Longrightarrow z_1 \prec z_3$.

Now we introduce the notion of complex valued G-metric space akin to the notion of complex valued metric spaces [1] as follows:

Definition 1.4. Let X be a non-empty set. Let G: $X \times X \times X \to \mathbb{C}$ be a function satisfying the following properties:

(CG1) G(x, y, z) =0 if x = y = z,

(CG2) $0 \prec G(x, y, z)$ for all x, y ∈ X with x ≠ y,

(CG3) $G(x, x, y) \precsim G(x, y, z)$ for all x, y, z ∈ X with y ≠ z,

(CG4) G(x, y, z) = G(x, z, y) = G(y, z, x) =... (symmetry in all three variables) (CG5) $G(x, y, z) \precsim G(x, a, a) + G(a, y, z)$ for all x, y, z, a ∈ X.

Then the function G is called a complex valued generalized metric or more specially, a complex valued G-metric on X, and the pair (X, G) is called a complex valued G- metric space.

2. THE COMPLEX VALUED G-METRIC TOPOLOGY
A point x ∈ X is called *interior point* of a set A⊆ X, whenever there exists $0 \prec r \in \mathbb{C}$ such that

$B_G(x, r) = \{ y \in X: G(x, y, y) \prec r\} \subseteq A$.

A point x ∈ X is called *limit point* of a set A whenever there exists $0 \prec r \in \mathbb{C}$,

$B_G(x, r) \cap (A/X) \neq \emptyset$.

A is called *open* whenever each element of A is an interior point of A. A subset B⊆ X is called *closed* whenever each limit point of B belongs to B.

Proposition 2.1. Let (X, G) be complex valued G-metric space, then for any $x_0 \in X$ and r > 0, we have

(1) If $G(x_0, x, y) \prec r$, then $x, y \in B_G(x_0, r)$

(2) If $y \in B_G(x_0, r)$, then there exists a $\delta > 0$ such that $B_G(y, \delta) \subseteq B_G(x_0, r)$.

Proposition 2.2 Let (X, G) be complex valued G-metric space, then for all $x_0 \in X$ and r > 0, we have,

$$B_G\left(x_0, \frac{1}{3}r\right) \subseteq B_{d_G}(x_0, r) \subseteq B_G(x_0, r).$$

Where $d_G(x, y) = G(x, y, y) + G(x, x, y)$.

3. CONVERGENCE, CONTINUITY AND COMPLETENESS IN COMPLEX VALUED G-METRIC SPACES

Definition 3.1. Let (X, G) be a complex valued G-metric space, let $\{x_n\}$ be a sequence of points of X, we say that $\{x_n\}$ is complex valued G-convergent to x if for any $\epsilon > 0$, there exists $k \in N$ such that $G(x, x_n, x_m) \prec \epsilon$, for all n, m ≥ k. We refer to x as the limit of the sequence $\{x_n\}$ and we write $x_n \xrightarrow{(G)} x$.

Proposition 3.1. Let (X, G) be complex valued G-metric space, then for a sequence $\{x_n\} \subseteq X$ and point x ∈ X, the following are equivalent:

(1) $\{x_n\}$ *is complex valued G − convergent to x*

(2) $|G(x_n, x_n, x)| \to 0$ as $n \to \infty$

(3) $|G(x_n, x, x)| \to 0$ as $n \to \infty$

(4) $|G(x_m, x_n, x)| \to 0$ as $n, m \to \infty$

Definition 3.2. Let (X, G) and (X', G') be two complex valued G-metric spaces. Then a function f: $X \to X'$ is complex valued G-continuous at a point $x_0 \in X$ if $f^{-1}(B_{G'}(f(x_0), r)) \in \tau(G)$, for all r > 0. We say f is complex valued G-continuous if it complex valued G-continuous at all points of X; that is, continuous as a function from X with the $\tau(G)$- topology to X' with $\tau(G')$- topology.

Since complex valued G-metric topologies are metric topologies we have:

Proposition 3.2. Let (X, G) and (X', G') be two complex valued G-metric spaces. Then a function f : $X \to X'$ is complex valued G-continuous at a point x ∈ X if and only if it is complex valued G-sequentially continuous at x: that is whenever $\{x_n\}$ is complex valued G-convergent to x we have $(f\{x_n\})$ *is* complex valued G-convergent to f(x).

Proposition 3.3. Let (X, G) be a complex valued G-metric spaces, then the function G(x,y,z) is jointly continuous in all three of its variables.

Proof. Suppose $\{x_k\}$, $\{y_m\}$, and $\{z_n\}$, are complex valued G-convergent to x, y and z respectively. Then, by (CG5) we have,

$$G(x, y, z) \precsim G(y, y_m, y_m) + G(y_m, x, z)$$

$$G(z, x, y_m) \precsim G(x, x_k, x_k) + G(x_k, y_m, z)$$

and

$$G(z, x_k, y_m) \precsim G(z, z_n, z_n) + G(z_n, y_m, x_k),$$

so,

$$G(x, y, z) - G(x_k, y_m, z_n) \precsim G(y, y_m, y_m) + G(x, x_k, x_k) + G(z, z_n, z_n).$$

Similarly,

$$G(x_k, y_m, z_n) - G(x, y, z) \precsim G(x_k, x, x) + G(y_m, y, y) + G(z_n, z, z).$$

But then combining these using (3) of proposition 4.1 we have,

$$|G(x_k, y_m, z_n) - G(x, y, z)| \le 2(G(x, x_k, x_k) + G(y, y_m, y_m) + G(z, z_n, z_n)),$$

$|G(x_k, y_m, z_n) - G(x, y, z)| \to 0$, as k, m, n, →∞ and the result follows by proposition 3.2.

Definition 3.2. Let (X, G) be a complex valued G-metric space, a sequence $\{x_n\}$ is complex valued G- Cauchy if given $\epsilon > 0$, there exists $k \in N$ such that $G(x_n, x_m, x_l) \prec \epsilon$ for all n, m, $l \ge$ k.

Definition 3.3. A complex valued G-metric space (X, G) is said to be complex valued G-complete if every complex valued G-Cauchy sequence is complex valued G-convergent in (X, G).

Proposition 3.4. Let (X, G) be a complex valued G-metric space. Then the following are equivalent:

(1) The sequence $\{x_n\}$ is a complex valued G-Cauchy in X.

(2) For every $\epsilon > 0$, there exists $k \in N$ such that $G(x_n, x_m, x_m) \prec \epsilon$, for all n, m ≥ k .

(3) $\{x_n\}$ is a Cauchy sequence in the complex valued metric space (X, d_G).

Proposition 3.5. Let (X, G) be a complex valued G-metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is complex valued G- convergent to x if and only if $|G(x, x_n, x_m)| \to 0$ as n, m → ∞.

Proof. Suppose that $\{x_n\}$ is complex valued G- convergent to x. For a given real number $\epsilon > 0$, let

$$c = \frac{\epsilon}{\sqrt{2}} + i\frac{\epsilon}{\sqrt{2}}.$$

Then $0 \prec r \in \mathbb{C}$ and there is a natural number k, such that $G(x, x_n, x_m) \prec \epsilon$ for all n, m ≥k.

Therefore,

$|G(x, x_n, x_m)| < |c| = \epsilon$ for all n, m ≥k.

It follows that $|G(x, x_n, x_m)| \to 0$ as n, m → ∞.

Conversely, suppose that $|G(x, x_n, x_m)| \to 0$ as n, m → ∞. Then given r ∈ ℂ with 0 ≺ c, there exists a real number $\delta > 0$, such that for z ∈ ℂ

$$|z| < \delta \Rightarrow z \prec c.$$

For this δ, there is a natural number k such that

$|G(x, x_n, x_m)| < \delta$ for all n, m ≥k.

This means that $G(x, x_n, x_m) \prec \epsilon$ for all n, m ≥k. Hence $\{x_n\}$ is complex valued G- convergent to x.

Proposition 3.6. Let (X, G) be a complex valued G-metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is complex valued G- Cauchy sequence if and only if $|G(x_n, x_m, x_l)| \to 0$ as n, m → ∞.

Proof. Suppose that $\{x_n\}$ is complex valued G- Cauchy sequence. For a given real number $\epsilon > 0$, let

$$c = \frac{\epsilon}{\sqrt{2}} + i\frac{\epsilon}{\sqrt{2}}.$$

Then 0 ≺ r ∈ ℂ and there is a natural number k, such that $G(x_n, x_m, x_l) \prec \epsilon$ for all n, m ≥k.

Therefore,

$|G(x_n, x_m, x_l)| < |c| = \epsilon$ for all n, m ≥k.

It follows that $|G(x_n, x_m, x_l)| \to 0$ as n, m → ∞.

Conversely, suppose that $|G(x_n, x_m, x_l)| \to 0$ as n, m → ∞. Then given c ∈ ℂ with 0 ≺ c, there exists a real number $\delta > 0$, such that for z ∈ ℂ

$$|z| < \delta \Rightarrow z \prec c.$$

For this δ, there is a natural number k such that

$|G(x_n, x_m, x_l)| < \delta$ for all n, m ≥k.

This means that $G(x_n, x_m, x_l) \prec \epsilon$ for all n, m ≥ k. Hence $\{x_n\}$ is complex valued G- Cauchy sequence.

4. PROPERTIES OF COMPLEX VALUED G-METRIC SPACES

Proposition 4.1. Let (X, G) be a complex valued G-metric space. Then for any x, y, z, a in X it follows that:

(i) If G(x, y, z) =0 if x = y = z

(ii) $G(x, y, z) \precsim G(x, x, y) + G(x, x, z)$

(iii) $G(x, y, y) \precsim 2G(y, x, x)$

(iv) $G(x, y, z) \precsim G(x, a, z) + G(a, y, z)$

(v) $G(x, y, z) \precsim 2/3(G(x, y, a) + G(x, a, z) + G(a, y, z))$

(vi) $G(x, y, z) \precsim (G(x, a, a) + G(y, a, a) + G(z, a, a))$.

Proposition 4.2. Let (X, G) be a complex valued G-metric space. Then the following are equivalent:

(i) (X, G) is symmetric.

(ii) $G(x, y, y) \precsim G(x, y, a)$, for all x, y, a ∈ X.

(iii) $G(x, y, z) \precsim G(x, y, a) + G(z, y, b)$ for all x, y, a, b ∈ X.

In 1998, Jungck [7] introduced the concept of weakly compatibility as follows:

Definition 4.3 Two self mappings S and T are said to be weakly compatible if they commute at their coincidence points.

5. MAIN RESULT

Now we prove our main result for a pair of self mappings:

Theorem 5.1. Let (X, G) be a complete complex valued G-metric space. Let S, T: X→ X be self mappings satisfying the following conditions:

(2.1) S(X) ⊆ T(X),

(2.2) any one of the subspace S(X) or T(X) is complete,

(2.3) G(Sx, Sy, Sz) ≾ k G(Tx, Ty, Tz) for all x, y, z ∈ X, where 0 ≤ k< 1,

(2.4) S and T are weakly compatible self maps.

Then S and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point in X. By (2.1), one can choose a point x_1 in X such that $Sx_0 = Tx_1$. In general choose x_{n+1} such that

$$y_n = Sx_n = Tx_{n+1}.$$

Now, we prove $\{y_n\}$ is a complex valued G- Cauchy sequence in X.

Putting $x = x_n$, $y = x_{n+1}$, $z = x_{n+1}$ in (2.1), we have

$$G(Sx_n, Sx_{n+1}, Sx_{n+1}) \precsim kG(Tx_n, Tx_{n+1}, Tx_{n+1}) = kG(Sx_{n-1}, Sx_n, Sx_n)$$

Continuing in the same way, we have

$$G(Sx_n, Sx_{n+1}, Sx_{n+1}) \precsim k^n G(Sx_0, Sx_1, Sx_1)$$

This implies that $G(y_n, y_{n+1}, y_{n+1}) \precsim k^n G(y_0, y_1, y_1)$

Then, for all n, m ∈ N, n < m, we have by (CG5)

$G(y_n, y_m, y_m) \precsim G(y_n, y_{n+1}, y_{n+1}) + G(y_{n+1}, y_{n+2}, y_{n+2} + G(y_{n+2}, y_{n+3}, y_{n+3}) + \ldots + G(y_{m-1}, y_m, y_m) \precsim (k^n +$

$k^{n+1}+k^{n+2}+\cdots+k^{m-1}$) G (y_0, y_1, y_1) $\precsim$ $\frac{k^n}{1-k}$ G (y_0, y_1, y_1),

Therefore,

$$|G(y_n, y_m, y_m)| \le \left(\frac{k^n}{1-k}\right)|G(y_0, y_1, y_1)|$$

Since $k \in [0, 1]$, if we taking limit as n, m$\to\infty$, then $\frac{k^n}{1-k}$ $|G(y_0, y_1, y_1)| \to 0$,

i.e., $G(y_0, y_1, y_1) \to 0$

For n, m, $l \in$ N (CG5) implies that

$G(y_n, y_m, y_l) \precsim G(y_n, y_m, y_m) + G(y_l, y_m, y_m)$,

Therefore, $|G(y_n, y_m, y_l)| \le |G(y_n, y_m, y_m)| + |G(y_l, y_m, y_m)|$

Taking limit as n, m, $l \to \infty$, we get $|G(y_n, y_m, y_l)| \to 0$ i.e., $G(y_n, y_m, y_l) \to 0$. So $\{y_n\}$ is complex valued G-Cauchy sequence. Since either S(X) or T(X) is complete. Without loss of generality, we assume that T(X) is complete subspace of X, then the subsequence of $\{y_n\}$ must get a limit in T(X) (say) z .Then Tu=z for some u$\in$ X, as $\{y_n\}$ is a complex valued G-Cauchy sequence containing a convergent subsequence, therefore the sequence $\{y_n\}$also convergent implying thereby the convergence of subsequence of the convergent sequence. Next we show that Su = z. On setting x= u, y = x_n and z = x_n,in (2.3), we have

$G(Su, Sx_n, Sx_n) \precsim kG(Tu, Tx_n, Tx_n)$

Taking limit as n$\to\infty$, we have $G(Su, z, z) \precsim kG(Tu, z, z)$

Therefore, $|G(Su, z, z)| \le k|G(Tu, z, z)|$ implies that Su=z.

Therefore, Su =Tu =z. i.e., u is coincidence point of S and T. Since S and T are weakly compatible, it follows that STu = TSu i.e., Sz =Tz.

We now show that Sz =z. Suppose that S(z) ≠z, therefore $0 \prec G(Sz, z, z)$ implies that $|G(Sz, z, z)| > 0$.

Putting x = z, y = u, z = u in (2.3), we have

$G(Sz, Su, Su) \precsim kG(Tz, Tu, Tu)$ = kG(Sz, z, z)

i.e., $|G(Sz, z, z)| \le k|G(Sz, z, z)| < |G(Sz, z, z)|$ which is a contradiction, therefore

Sz = z . Thus Sz =Tz =z i.e., z is a common fixed point of S and T.

Uniqueness: To prove uniqueness, suppose that w≠z be another common fixed point of S and T. Then $0 \prec G(z, w, w)$ implies that $|G(z, w, w)| > 0$.

Putting x = z, y = u, z = u in (2.3), we have

$G(z, w, w) = G(Sz, Sw, Sw) \precsim kG(Tz, Tw, Tw)$ = kG(z, w, w)

i.e., $|G(z, w, w)| \le k|G(z, w, w)| < |G(Sz, z, z)|$, which is a contradiction, therefore

z = w. Thus Sz =Tz =z i.e., z is a unique common fixed point of S and T.

Example 5.1. Let X = [-1, 1] and let G: $X \times X \times X \to \mathbb{C}$ be complex valued G-metric space defined as follows:

G(x, y, z) = $|x-y| + |y-z| + |z-x|$, for all x, y, z $\in$ X. Then (X, G) is complex valued G-metric space. Define S, T: X$\to$ X as Sx = $\frac{x}{2}$ and Tx = $\frac{x}{6}$.

Here we note that, (2.1) S(X) $\subseteq$ T(X), (2.2) Both S(X) and T(X) are complete,

(2.3) G(Sx, Sy, Sz) $\precsim$ k G(Tx, Ty, Tz) holds for all x, y, z $\in$ X, $1/3 \le k < 1$, (2.4) S and T are weakly compatible because S and T commute at their coincidence point i.e., at x=0 and x=0 is the unique common fixed point of S and T.

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