Fixed Points of Non Self Maps

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ABSTRACT
The purpose of this paper is to present some fixed point theorems for non self maps in d_p-complete topological spaces which extend the results of Linda Marie Saliga.

Keywords
d_p- complete topological spaces, *d*-complete topological spaces and non self maps.

1. INTRODUCTION
Troy L. Hicks [5] has introduced *d-complete* topological spaces, attributing the basic ideas of these spaces to Kasahara ([8], [9]) and Iseki [7] as follows:

1.1 Definition: A topological space (*X*, *t*) is said to be *d-complete* if there is a mapping $d : X \times X \to [0,\infty)$ such that (i) $d(x, y) = 0 \Leftrightarrow x = y$ and (ii) $< x_n >$ is a sequence in *X* such that $\sum_{n=1}^{\infty} d(x_n, x_{n+1})$ is convergent implies that $< x_n >$ converges in (*X*, *t*).

In this paper we introduce d_p- *complete* topological spaces as a generalization of *d*-complete topological spaces for any integer $p \ge 2$. For a non-empty set X, let X^p be its *p*-fold cartesian product.

1.2 Definition: A topological space (*X*, *t*) is said to be d_p *complete* if there is a mapping $d_p : X^p \to [0, \infty)$ such that (i) $d_p(x_1, x_2, \ldots, x_p) = 0 \Leftrightarrow x_1 = x_2 = \ldots = x_p$ and (ii) $< x_n >$ is a sequence in *X* with $\lim_{n\to\infty} d_p(x_n, x_{n+1}, x_{n+2}, \ldots, x_{n+p-1}) = 0$ implies that $< x_n >$ converges to some point in (*X*, *t*). A d_p - complete topological space is denoted by (*X*, *t*, d_p)

1.3 Remark: The function *d* in the Definition 1.1 and the function d_2 (the case *p* = 2) in Definition 1.2 are both defined on $X \times X$ and satisfy condition (i) of the definitions which are identical. Since the convergence of an infinite series $\sum_{n=1}^{\infty} \alpha_n$ of real numbers implies that $\lim_{n\to\infty} \alpha_n = 0$, but not conversely; it follows that every *d*-complete topological space is d_2 - complete, but not conversely. Therefore the class of d_2 - complete topological spaces is wider than the class of *d*-complete spaces and hence a separate study of fixed point theorems of self-maps on d_2 - complete topological spaces is meaningful.
The purpose of this paper is to establish certain fixed point theorems of non self-maps of d_p - complete topological spaces for $p \ge 2$.

2. PRELIMINARIES
Let *X* be a non-empty set. A mapping $d_p : X^p \to [0, \infty)$ is called a *p-non-negative* on *X* provided $d_p(x_1, x_2, \ldots, x_p) = 0 \Leftrightarrow x_1 = x_2 = \ldots = x_p$.

2.1 Definition: Suppose (*X*, *t*) is a topological space and d_p is a *p*-non negative on *X*. A sequence $< x_n >$ in *X* is said to be a d_p - *Cauchy sequence* if $d_p(x_n, x_{n+1}, \ldots, x_{n+p-1}) \to 0$ as $n \to \infty$.
In view of Definition 2.1, a topological space (*X*, *t*) is d_p - complete if there is a *p*-non- negative d_p on *X* such that every d_p - Cauchy sequence in *X* converges to some point in (*X*, *t*).
If *T* is a self map of a non-empty set *X* and $x \in X$, then the *orbit of x*, $O_T(x)$ is given by $O_T(x) = \{x, Tx, T^2x, \ldots\}$. If *T* is a self map of a topological space *X*, then a mapping $G : X \to [0,\infty)$ is said to be *T-orbitally lower semi-continuous* (resp. *T-orbitally continuous*) at $x^* \in X$ if $< x_n >$ is a sequence in $O_T(x)$ for some $x \in X$ with $x_n \to x^*$ as $n \to \infty$ then $G(x^*) \le \liminf_{n\to\infty} G(x_n)$ (resp. $G(x^*) = \lim_{n\to\infty} G(x_n)$). A self map *T* of topological space *X* is said to be *w-continuous* at $x \in X$ if $x_n \to x$ as $n \to \infty$ implies $Tx_n \to Tx$ as $n \to \infty$.
If d_p is a *p*-non-negative on a non-empty set *X*, and $T : X \to X$ then we write, for simplicity of notation, that

(2.2) $G_p(x) := d_p(x, Tx, T^2x, \ldots, T^{p-1}x)$ for $x \in X$

Clearly we have

(2.3) $G_p(x) = 0$ if and only if *x* is a fixed point of *T* .

3. MAIN RESULTS
3.1 Theorem: Suppose (X,t,d_p) is a d_p-completeHausdorff topological space, C is a closed subset of X and T : C → X is an open mapping with C⊂T(C). Suppose d_p (x_1, x_2,…., x_p) ≤ k(d_p(Tx_1,Tx_2,…,Tx_p)) for all

$x_1,x_2,\ldots,x_p\in C$, where $k: [0,\infty) \to [0,\infty)$ is a non-decreasing function with $k(0) = 0$. Then T has a fixed point iff there exists an $x_0\in C$ with $k^n(d_p(T^{p-1}x_0, T^{p-2}x_0, \ldots, Tx_0,x_0)) \to 0$ as $n \to \infty$.

Proof : If $x,y\in C$ are such that $Tx = Ty$, then the inequality of the theorem gives $d_p(x,y,y,\ldots,y) \le k(d_p(Tx,Ty,Ty,\ldots, Ty)) = k(0) = 0$ which gives $x = y$. Hence T is one-one. Thus $T : C\to T(C)$ is a bijective map and hence $T^{-1} : T(C)\to C$ exists. Since T is open (by hypothesis), it follows that T^{-1} is a continuous function and hence T^{-1} is a w-continuous function.
Now first suppose that T has a fixed point $z\in C$.
Then $Tz = z$ so that $k^n (d_p(T^{p-1}z, T^{p-2}z,\ldots, Tz, z)) = 0$ giving that $\lim_{n\to\infty} k^n (d_p(T^{p-1}z, T^{p-2}z,\ldots, Tz, z)) = 0$.
Let T_1 denote the restriction of T^{-1} to C, so that $T_1 : C\to C$ and
(3.2) $d_p(T_1x_1, T_1x_2,\ldots, T_1x_p) \le k(d_p(x_1,x_2,\ldots, x_p))$ for all $x_1,x_2,\ldots, x_p\in C$, since $TT_1(x) = x$ for all $x\in C$. Let $x_1\in C$ be arbitrary and $x_2 = T_1x_1$, $x_3 = T_1^2x_1,\ldots, x_p = T_1^{p-1}x_1$.
Then (3.2) gives
$d_p(T_1x_1, T_1^2x_1,\ldots, T_1^px_1) \le k (d_p(x_1,T_1x_1,T_1^2x_1,\ldots, T_1^{p-1}x_1))$
for all $x_1\in C$ which gives, in particular, that

$$d_p(T_1x_0, T_1^2x_0,\ldots, T_1^px_0) \le k(d_p(x_0,T_1x_0,\ldots,T_1^{p-1}x_0)) \le k^2(d_p(Tx_0, TT_1x_0,\ldots, TT_1^{p-1}x_0)) = k^2(d_p(Tx_0, x_0, T_1x_0,\ldots, T_1^{p-2} x_0)).$$

Now by induction, we get

$$d_p(T_1^{n-p+1}x_0,T_1^{n-p+2}x_0,\ldots,T_1^nx_0)\le k^n(d_p(T^{p-1}x_0,T^{p-2}x_0,\ldots,Tx_0, {}_0)).$$

Hence $d_p(T_1^{n-p+1}x_0, T_1^{n-p+2}x_0,\ldots, T_1^nx_0) \to 0$ as $n\to\infty$ which gives that $(T_1^nx_0)$ is a d_p– Cauchy sequence in X.
Since X is d_p– complete, the sequence $(T_1^nx_0)$ converges to some point z. That is, $T_1^nx_0 \to z$ as $n\to\infty$.
Note that $z \in C$ since C is closed.
Now $T_1(T_1^nx_0) \to T_1z$ as $n \to \infty$ since T_1 is w-continuous.
But $T_1^{n+1}x_0 \to z$ as $n \to \infty$ and since limits are unique in X, we get that $T_1z = z$.
Now $T(T_1z) = Tz$ gives $z = Tz$ since $T(T_1z) = z$ and hence T has a fixed point.

3.3 Remark: It may be noted that in view of Remark 1.3, the result proved by Linda Marie Saliga ([10], Theorem 1,pp.103,104) is a particular case of Theorem 3.1 .

3.4 Corollary: Suppose $T : C\to X$ where C is a closed subset of a d_p–complete Hausdorff p-symmetrizable topological space with $C\subset T(C)$. Suppose $d_p(x_1,x_2,\ldots, x_p) \le [d_p(Tx_1,Tx_2,\ldots, Tx_p)]^s$ where $s >1$ for all $x_1,x_2,\ldots x_p\in C$. If there exists $x_0\in C$ such that $d_p (T^{p-1}x_0, T^{p-2}x_0, \ldots, Tx_0,x_0) < 1$, then T has a fixed point.

Proof Let x,y be in C with $x \ne y$.
Then $0<d_p (x,y,y,\ldots, y) \le [d_p(Tx,Ty,\ldots, Ty)]^s$ which gives $Tx \ne Ty$.
Thus, T is one – one and hence $T^{-1} : T(C) \to C$ exists.
Now the inequality of the theorem gives
$d_p(T^{-1}x_1,T^{-1}x_2,\ldots, T^{-1}x_p) \le [d_p(x_1,x_2,\ldots, x_p)]^s$ which implies that T^{-1} is continuous. Hence T must be an open map.
Let $x_0\in C$ be such that $d_p (T^{p-1}x_0, T^{p-2}x_0,\ldots, Tx_0, x_0) <1$.
If $d_p(T^{p-1}x_0, T^{p-2}x_0,\ldots, Tx_0, x_0) = 0$, then x_0 is a fixed point of T.
Suppose $0<d_p (T^{p-1}x_0, T^{p-2}x_0,\ldots, Tx_0, x_0) <1$.
Let $k(t) = t^s$ and $t = d_p (T^{p-1}x_0, T^{p-2}x_0, \ldots, Tx_0, x_0)$.
Note that $(\alpha t)^s<\alpha t^s$ if $0<\alpha<1$. Since $t^s < t$, there is an $\beta\in(0,1)$such that $t^s = \beta t$.
We claim that
(3.5) $t^{ns} \le \beta^n t$ for all natural numbers n.
For n = 1, (3.5) holds. Now assume that (3.5) holds for n = k. That is, $t^{ks} \le \beta^k t$.
Then $t^{(k+1)s} = t^{ks}\, t^s \le \beta^k t\, \beta t$, since $t^s = \beta t$
$= \beta^{k+1} t^2$
$\le \beta^{k+1} t$, since $0 < t < 1$.
Hence, by induction, we get (3.5).
Therefore, $k^n(d_p(T^{p-1}x_0, T^{p-2}x_0,\ldots, Tx_0, x_0))$
$= [d_p (T^{p-1}x_0, T^{p-2}x_0,\ldots, Tx_0,x_0)]^{ns}$
$= t^{ns} \le \beta^n t \to 0$ as $n \to \infty$, since $0 < \beta < 1$ and
hence the theorem follows from Theorem 3.1.

It may be noted that the existence of a fixed point for T on a closed subset of a d_p-complete Hausdorff topological space (X,t,d_p) is not ensured, if the inequality of the Theorem 3.1 is replaced by any one of the following :
$d_p(Tx_1,Tx_2,\ldots, Tx_p) \ge k(d_p(x_1,x_2,\ldots, x_p))$ for all $x_1,x_2, \ldots,x_p\in C$.
or
$d_p(x_1,x_2,\ldots, x_p) \ge k(d_p(Tx_1,Tx_2,\ldots, Tx_p))$ for all $x_1,x_2,\ldots, x_p\in C$.
or
$k(d_p(x_1,x_2,\ldots, x_p)) \ge d_p(Tx_1,Tx_2,\ldots, Tx_p)$ for all $x_1,x_2,\ldots, x_p\in C$.
L.M. Saliga has provided counter examples in the case p=2 ([10], Examples 2,3,4, p.105,106).

3.6 Theorem: Let (X,t,d_p) be a d_p - complete Hausdorff topological space, C be a closed subset of X and $T : C \to X$ with $C\subset T(C)$. Suppose there exists a function $k : [0, \infty) \to [0, \infty)$ such that $k(d_p(Tx_1,Tx_2,\ldots, Tx_p)) \ge d_p(x_1,x_2,\ldots, x_p)$ for all $x_1, x_2,\ldots x_p\in C$, where k is a non-decreasing function with $k(0) =0$ and there exists an $x_0\in C$ such that $k^n (d_p(T^{p-1}x_0, T^{p-2}x_0,\ldots, Tx_0,x_0))\to 0$ as $n\to\infty$. If $d_p(T^{p-1}x, T^{p-2}x,\ldots, Tx,x)$ is lower semi-continuous on C, then T has a fixed point.

Proof: If x and y are in C with $x\ne y$. Then $0<d_p(x,y,y,\ldots, y) \le k (d_p(Tx,Ty,\ldots, Ty))$ gives that $Tx \ne Ty$. Hence T is one – one and T^{-1} exists.Let T_1 be the restriction of T^{-1} to C. That is , $T_1 = T^{-1} |C$. Now $T_1 : C\to C$ and for $x \in C$, we have

$$d_p (x,T_1x, T_1^2x, \ldots, T_1^{p-1}x) \le k (d_p (Tx,x, T_1x,\ldots,T_1^{p-2}x)) \le k^2(d_p(T^2x,Tx,x,T_1x,\ldots,T_1^{p-3}x)).$$

Hence, by induction, we get
(3.7) $d_p(T_1^{n-p+1}x, T_1^{n-p+2}x,\ldots, T_1^nx) \le k^n (d_p(T^{p-1}x, T^{p-2}x,\ldots, Tx))$.
If there exists $x_0\in C$ such that $k^n (d_p(T^{p-1}x_0, T^{p-2}x_0, \ldots, Tx_0)) \to 0$ as $n\to\infty$, then $d_p(T_1^{n-p+1}x_0, T_1^{n-p+2}x_0, \ldots, T_1^nx_0) \to 0$ as $n \to \infty$, by (3.7),which gives that $(T_1^nx_0)$ is d_p-Cauchy. Since X is d_p–complete,there exists $z\in x$ such that $T_1^nx_0 \to z$ as $n \to \infty$.
Note that $z\in C$ since $T_1^nx_0\in C$ for all n and C is closed.
Now $d_p(T^{p-1}x, T^{p-2}x,\ldots, Tx,x)$ is lower semi-continuous on C gives $d_p(T^{p-1}z, T^{p-2}z,\ldots, Tz,z)\le\liminf d_p(T_1^{n-p+1}x_0,T_1^{n-p+2}x_0,\ldots, T_1^nx_0) \to 0$ as $n \to \infty$ giving that $Tz = z$.

3.7 Remark: Note that the result proved by Linda Marie Saliga ([10], Theorem 2, pp.105) is a particular case of Theorem 3.6 .

3.8 Theorem: Let C be a compact subset of a Hausdorff topological space (X,t) and d_p be a p-non-negative on X. Suppose $T : C\to X$ with $C\subset T(C)$, T and $G_p(x)$ are both continuous, and $G_p(Tx) > G_p(x)$ for all $x\in T^{-1}(C)$ with $x \ne Tx$. Then T has a fixed point in C.

Proof: Since C is a compact subset of a Hausdorff topological space, we get that C is closed and since $T : C \to X$

is continous, so $T^{-1}(C)$ is closed. Hence $T^{-1}(C)$ is compact since $T^{-1}(C) \subset C$.Also, $G_p(x)$ is continuous so it attains its minimum on $T^{-1}(C)$, say at z. That is,

(3.9) $G_p(z) \le G_p(x)$ for all $x \in T^{-1}(C)$

Now $z \in C \subset T(C)$ so there exists $y \in T^{-1}(C)$ such that Ty = z.

If $y \neq z$, then $G_p(z) = G_p(Ty) > G_p(y)$ which is a contradiction to (3.9).

Thus y = z = Ty is a fixed point of T.

3.9 Remark: Note that the result proved by Linda Marie Saliga ([10], Theorem 3, pp.106) is a particular case of Theorem 3.8 .

3.10 Theorem : Let C be a compact subset of a Hausdorff topological space (X,t) and d_p be a p-non-negative on X. Suppose $T : C \rightarrow X$ with $C \subset T(C)$, T and $G_p(x)$ are both continuous, $f : [0,\infty) \rightarrow [0,\infty)$ is continuous and $f(t) > 0$ for $t \neq 0$. If we know that $G_p(Tx) \le \lambda\ f(G_p(x))$ where $0<\lambda<1$, for all $x \in T^{-1}(C)$ implies T has a fixed point then $G_p(Tx)<f(G_p(x))$ for all $x \in T^{-1}(C)$ such that $f(G_p(x)) \neq 0$ gives a fixed point.

Proof: Since C is a compact subset of a Hausdorff space, it is closed and since T is continuous, $T^{-1}(C)$ is closed and hence is compact since $T^{-1}(C) \subset C$.

Suppose $G_p(x) \neq 0$ for all $x \in T^{-1}(C)$. Then $G_p(x) > 0$ so that $f(G_p(x)) > 0$ for all $x \in T^{-1}(C)$.

Now define p(x) on $T^{-1}(C)$ by

$p(x) = \dfrac{G_p(Tx)}{f(G_p(x))}$. Then p is continuous since T, f and $G_p(x)$ are continuous. Therefore p attains its maximum on $T^{-1}(C)$, say at z. That is, $p(x) \le p(z)$ for all $x \in T^{-1}(C)$.

Now $p(x) \le p(z) < 1$ so $G_p(Tx) \le p(z)\ f(G_p(x))$ and T must have a fixed point.

3.11 Remark: Note that the result proved by Linda Marie Saliga ([10], Theorem 4, pp.106) is a particular case of Theorem 3.10 .

3.12 Theorem: Let C be a compact subset of a Hausdorff toplogical space (X,t) and d_p be a p-non-negative on X. Suppose $T : C \to X$ with $C \subset T(C)$, T and $G_p(x)$ are both continuous, $f : [0, \infty) \to [0, \infty)$ is continuous and $f(t) > 0$ for $t \neq 0$. If we know that $G_p(Tx) \ge \lambda\ f(G_p(x))$ where $\lambda > 1$, for all $x \in T^{-1}(C)$ implies T has a fixed point, then $G_p(Tx) > f(G_p(x))$ for all $x \in T^{-1}(C)$ such that $f(G_p(x)) \neq 0$ gives a fixed point.

Proof : Since C is a compact subset of a Hausdorff space, we get that C is closed and since T is continuous, $T^{-1}(C)$ is closed. Hence $T^{-1}(C)$ is compact, since $T^{-1}(C) \subset C$.

Suppose $G_p(x) \neq 0$ for all $x \in T^{-1}(C)$. Then $G_p(x) > 0$ and $f(G_p(x)) > 0$.

Now, define $p(x) = \dfrac{G_p(Tx)}{f(G_p(x))}$.

Then p is continuous, since T, f and $G_p(x)$ are continuous and hence p attains its minimum on $T^{-1}(C)$, say at z. That is, $p(z) \le p(x)$ for all $x \in T^{-1}(C)$.

Now $p(x) \ge p(z) > 1$ so $G_p(Tx) \ge p(z)\ f(G_p(x))$ and T must have a fixed point.

3.13 Remark: Note that the result proved by Linda Marie Saliga ([10], Theorem 5, pp.106,107) is a particular case of Theorem 3.12 .

4. REFERENCES

[1] J. Achari, *Results on fixed point theorems*, Maths. Vesnik 2 (15) (30) (1978), 219-221.

[2] Ciric, B. Ljubomir, *A certain class of maps and fixed point theorems*, Publ. L'Inst. Math. (Beograd) 20 (1976), 73-77.

[3] B. Fisher, *Fixed point and constant mappings on metric spaces*, Rend. Accad. Lincei 61(1976), 329 – 332.

[4] K.M. Ghosh, *An extension of contractive mappings*, JASSY 23(1977), 39-42.

[5] Troy. L. Hicks, *Fixed point theorems for d-complete topological spaces I*, Internet. J. Math & Math. Sci. 15 (1992), 435-440.

[6] Troy. L. Hicks and B.E. Rhoades, *Fixed point theorems for d-complete topological spaces II*, Math. Japonica 37, No. 5(1992), 847-853.

[7] K. Iseki, *An approach to fixed point theorems*, Math. Seminar Notes, 3(1975), 193-202.

[8] S. Kasahara, *On some generalizations of the Banach contraction theorem*, Math. Seminar Notes, 3(1975), 161-169.

[9] S. Kasahara, *Some fixed point and coincidence theorems in L-Spaces*, Math. Seminar Notes, 3(1975), 181-187.

[10] Linda Marie Saliga, Fixed point theorems for non-selfmaps in d-complete topological spaces, Intenat. J. Math & Maths. Sci. 19.1 (1996), 103-110.