A New Technique to Solve Higher Order Ordinary Differential equations

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ABSTRACT
Modified Adomian decomposition method has been used intensively to solve linear and nonlinear singular boundary and initial value problems. It has been proved to be very efficient in generating series solutions of the problem under consideration under the assumption that such series solution exits. The method is illustrated by some examples of higher order ordinary equations systems and series solutions are obtained. The solutions have been compared with those obtained by exact solutions. We use modified Adomian decomposition method to solving singular boundary value problems and singular initial value problem of higher-order ordinary differential equations. The numerical results obtained by this way have been compared with the exact solution to show that the Adomian method is a powerful method for the solution of linear and nonlinear differential equations.

Keywords
Adomian decomposition method, Taylor series, Initial boundary value problem, Singular boundary value problems, Higher-order ordinary differential equation.

1. INTRODUCTION
The Adomian decomposition method is quantitative rather than qualitative, analytic, requiring neither linearization nor perturbation and continuous with no resort to discretization. It consists of splitting the given equation into linear and nonlinear parts, inverting the highest-order derivative operator contained in the linear operator on both sides, identifying the initial and boundary conditions and the terms involving the independent variables alone as initial approximation, decomposing the unknown function into a series whose components are to be determined, decomposing the nonlinear function in terms of special polynomials called Adomian's polynomials, and finding the successive terms of the series solution by recurrent relation using Adomian polynomials. A large amount of literature developed concerning Adomian decomposition method [1, 2, 4, 5], and the related modification [10, 11] to investigate various scientific models. It is the aim of this paper to introduce a new reliable modification of Adomian decomposition method .the decomposition method is simple and easy to use and produces reliable results with few iterations used.

2. MODIFIED ADOMIAN DECOMPOSITION METHOD
The singular initial value problem of n+1 order ordinary differential equation represent as

$$y\,(n+1) + \left(\frac{p}{x}\right) y^n + q\,(x, y) = r(x), \qquad (1)$$

$$y0 = N, \left(\frac{dy}{dx}\right)_0 = M$$

Where q(x, y) is a real function r(x) is given function and N, M are constants.

Now we introduces a new differential operator

$$L = x\text{-}1\ \frac{d^p y}{dx^p}\ xy \quad (2)$$

So equation (1) will becomes

$$Ly = r(x) - q\,(x, y) \qquad (3)$$

The inverse operator L-1 is therefore considered an n+1 fold integral operator,

$$L^{-1}(w) = x^{-1}\int_0^x \int_0^x x\,(w)\,dxdx$$

$$\text{here } w = \left(y^{(n+1)} + \frac{p}{x} y^n\right) \qquad (4)$$

Applying L-1 of (4) of w from Equation (1), we find

$$L^{-1}\left(y^{(n+1)} + \frac{p}{x} y^n\right) = x^{-1}\int_0^x \int_0^x x\left(y^{(n+1)} + \frac{p}{x} y^n\right) dxdx$$

Take (p=2) and (n=1), then Equation (3) will becomes

$$= x^{-1}\int_0^x \int_0^x x\left(y^{(1+1)} + \frac{2}{x} y^1\right) dxdx \qquad (5)$$

$$= x^{-1}\int_0^x \left\{x\left(\frac{dy}{dx}\right) + y - y(0)\right\} dx = y - y(0) \quad \text{here } \frac{dy}{dx} = y^1 \qquad (6)$$

Again, to take L-1 on Equation (4)

$$Y(x) = B + L\text{-}1\ r(x) - L\text{-}1\ q(x, y) \quad (7)$$

Adomian polynomials decompose a function y(x) into a sum of components, The Adomian decomposition method introduce the solution y(x) and the nonlinear function q(x, y) by infinity series

$$y(x) = \sum_{m=0}^{\infty} y_m\,(x), \qquad (8)$$

For a nonlinear operator q(x, y) as

$$q(x, y) = \sum_{m=0}^{\infty} B_m \qquad (9)$$

There appears to be no well-defined method for constructing a definitive set of polynomials for arbitrary F, but rather slightly different approaches are used for different specific functions.

One possible set of polynomials is given by

B0 = F(v0) (say) y0 ,

B1 = (x-x1) $\left(\frac{dy}{dx}\right)_0$

B2 =(x-x2) $\left(\frac{dy}{dx}\right)_0 + \frac{(x-x_1)^2}{2!}\left(\frac{d^2y}{dx^2}\right)_0$

B3 =(x-x3) $\left(\frac{dy}{dx}\right)_0$ +(x-x1) (x-x2)$\left(\frac{d^2y}{dx^2}\right)_0 + \frac{(x-x_1)^2}{3!}\left(\frac{d^3y}{dx^3}\right)_0$ (10)

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can be used to construct Adomian polynomials, when F(v0) is a nonlinear function Put the value of equation (8) and (9) in equation (3), we get

$$\sum_{m=0}^{\infty} y_m(x) = N + L^{-1}r(x) - L^{-1}\sum_{m=0}^{\infty} B_m(x) \quad (11)$$

Using Adomian decomposition method, ym(x) can be determined as

y0(x) =N + L-1 r(x) (12)

y(j+1) (x) = -L-1 (B j) j≥0,

Put one by one j=1,2,3,4,…….. in above expression.

y0(x) = N + L-1r(x);

y1(x) = - L-1(B0);

y2(x) = - L-1 (B1);

y3(x) = - L-1(B2);

……………(13)

From (10) and (13), we can determine the components ym(x), and hence the series solution of y(x) in (8) can be immediately obtained.

For numerical purposes, the n-term approximate

$$\beta_m = \sum_{m=0}^{n-1} y_m(x) \quad (14)$$

Ultimately, we can determine the components ym(x), and result obtained in form of a series. For Numerical purposes, the m-term approximate can be used to approximate the exact solution.

A generalization of Equation (1) has been studied by Wazwaz [20]. We replace the standard

Coefficients of $\frac{dy}{dx}$ and y by $\frac{2n}{x}$ and $\frac{n(n-1)}{x^2}$ respectively, for real n; n ≥ 0

We propose the new differential operator, as below

L =x-n $\frac{d^2}{dx^2}$ x^n (15)

So, the problem can be written

Ly = r(x) - q (x, y)

The inverse operator L-1 is therefore considered an n+1 fold integral operator,

$$L^{-1}(.) = x^{-n}\int_0^x \int_0^x x^n\,(.)dxdx \quad (17)$$

Similarly, we can calculate singular boundary value problems as equation (1).just change in equations (2, 4), as below L=x-1 $\frac{d^n}{dx^p}$ x^{1+n+m} $\frac{d}{dx}x^{n+m}(.)$ and

$$L^{-1}(w) = x^{m-n}\int_0^x x^{m-n-1}\int_0^x\int_0^x \int_0^x x\,(.)dx\ldots\ldots dx \quad (18)$$

And get same exact solution as similar (14).

Example 1: Consider the nonlinear singular initial value problems $\frac{d^2y}{dx^2}+3\left[\frac{2}{x}\right]\frac{dy}{dx}+3\left[\frac{2}{x^2}\right]y + y2 =$ x4+20 (19)

$y0 = 0\ ,\left(\frac{dy}{dx}\right)_0 = 0$

Standard Adomian decomposition method: from (15,17),

L =x-6$\frac{d}{dx}x^6\frac{d}{dx}(.)$ L(.) = $x^{-6}\int_0^x \int_0^x x^6\,(.)dxdx$

In an operator form Equation (19), becomes

Ly = x4+20 -$\left[\frac{6}{x^2}\right]$y -y2 (20)

By applying L-1 to both sides of (20), we have

y = L-1 (x4+20) -L-1 $\left[\frac{6}{x^2}\right]$y - L-1 (y2) (21)

Proceeding as before we obtained the recursive relationship

y0 = $\frac{10}{7}x^2 + x^6\left[\frac{1}{66}\right]$,

ym+1= L-1 $\left[-\frac{1}{x^2}y_m\right]$ - L-1(Bm) (22)

The Adomian polynomials for the nonlinear term F(y) = y2 are computed as follows

B0 =y20,

B1=2y0y1,

B2 =2y2y0+ $\frac{{y_1}^2}{2}$

B4=2y3y0 + y1y2,

.... (23)

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Which are obtained by using formal algorithms in [19].Substituting (23) into (22), gives the components

$$y0 = \frac{10}{7}x^2 + x^6\left[\frac{1}{66}\right],$$

$$y1 = -L\text{-}1\left[\frac{1}{x^2}y_0\right] - L\text{-}1(B0)$$

$$= \frac{-5}{126}x^2 - \frac{3483}{112112}x^6 - \frac{1}{3465}x^{10} - \frac{1}{1158696}x^{14}$$

$$y2= -L\text{-}1\left[\frac{1}{x^2}y_1\right] + L\text{-}1(B1)$$

$$= \frac{5}{4536}x^5 + \frac{71169}{2833294464}x^6 + \frac{18528773}{2528715309120}x^{10} + \frac{811073}{15659303692032}x^{14} + \frac{793}{6538749053760}x^{18} + \frac{1}{7631487021312}x^{22}$$

It is to see that the standard Adomian decomposition method is divergent to solve this problem.

Modified Adomian decomposition: According to(17).We put n=3,

In an operator form, Equation (18) becomes

Ly = x4+20 - y2 (24)

Now, by applying L-1 to both sides of (24) we have

y = L-1 (x4+20) - L-1 (y2),

Therefore

$$y = x^2 + x^6\left[\frac{1}{72}\right] - L\text{-}1\ (y2),$$

by divided $x^2+x^6\left[\frac{1}{72}\right]$ into two parts and we obtain the recursive relationship $y0 = x^2$

$$ym+1 = x^6\left[\frac{1}{72}\right] - L\text{-}1\ (Bm),$$

ym+1= 0, m≥0

In view of(24) the exact solution is given by

$$y = x^2$$

So, the exact solution is easing obtained by proposed Adomian method.

Example 2: Consider the nonlinear singular initial value problems $\frac{d^2y}{dx^2} + \left[\frac{2}{x}\right]\frac{dy}{dx} + 8(e\ y) = -4(e\ y/2)$

$$y0 = 0\ , \left(\frac{dy}{dx}\right)_0 = 0$$

We obtain the operator form

Ly = −4 (2ey + e y/2 (25)

Applying L−1 on both sides of (25) we get

y = −4L−1 (2ey + e y/2)

Using the decomposition series for the linear function y(x) and the polynomial series for the nonlinear term, we obtain the recursive relationship

y0 = 0, y m+1(x) = −4L−1 (B m), k ≥ 0. (26)

The Adomian polynomials for the nonlinear term 2e y + e y/2 are computed as follows:

B0 =y0,

B1=2y0y1,

B2 =2y2y0+ $\frac{{y_1}^2}{2!}$

B3=2y3y0 + y1y2 + $\frac{{y_1}^3}{3!}$,

....

(27)

....

Substituting (27) into (26) we get

y0 = 0,

y1(x) = −4L−1 (B0), = -2x2

y2(x) = −4L−1 (B1), = x4

$y3(x) = -4L-1\ (B2), = -\frac{2}{3}x6$

$y4(x) = -4L-1\ (B3), = \frac{1}{2}x8$

$y5(x) = -4L-1\ (B4), = -\frac{2}{5}\ x10$
$y6(x) = -4L-1\ (B5), = \frac{1}{3}x12$ (28)

Then the solution in a series form is given by

$y(x)= -2x2 + x4 - \frac{2}{3}x6 + \frac{1}{2}x8 - \frac{2}{5}x10 - \frac{1}{3}x12 + \ldots$

Hence the exact solution has the form

$y(x) = -2\log(1+x2).$

Example3: $\frac{d^3y}{dx^3} - \left[\frac{2}{x}\right]\frac{d^2y}{dx^2} - y - y2 =$ -x6e2x +7x2ex+6xex-6ex $0 \le x \le 1$

$y\ (0) = 0, \left(\frac{dy}{dx}\right)_0 = 0, \quad y\ (1) = e$ (29)

Solution: We know that Taylor Series of higher finite n order,

$F(x) = y0 + (x-x0)\left(\frac{dy}{dx}\right)_0 + \frac{(x-x_0)^2}{2!}\left(\frac{d^2y}{dx^2}\right)_0 + \frac{(x-x_0)^3}{3!}\left(\frac{d^3y}{dx^3}\right)_0 + \frac{(x-x_0)^4}{4!}\left(\frac{d^4y}{dx^4}\right)_0 + \ldots\ldots\frac{(x-x_0)^n}{n!}\left(\frac{d^ny}{dx^n}\right)_0$

Here we solve given P(x) =-x6e2x +7x2ex+6xex-6ex up to (n=10) finite order,

q(x)≈ -6+10x2+9x3+4.25x4+1.36666x5-0.66666x6-1.93452380x7-1.98923611x8

-1.331812169x9-0.666478174x10

We put value of q(x) in (18) as a new differential operator

$L\ (q(x)) = x\text{-}1\ \frac{d^2}{dx^2}\ x5\ \frac{d}{dx}\ x\text{-}4(q(x))$ (30)

Therefore

$$L(q(x)) = x^4\int_0^x x^{-5}\int_0^x\int_0^x x(q(x))\,dxdxdx$$

From the given eq. (29), it will becomes

L(y) =P(x) + y +y2

Take both sides L-1 in above equation and Proceeding as before, we obtained the result, as below

y= e x4+ L-1 P(x) + L-1 (y) + L-1 (y2)

y0 = e x4+ L-1 P(x),

ym+1=L-1 ym + L-1 Bm here m ≥0 (31)

Here Bm's are Adomian Polynomials of non-linear term y2, written as follow,

B0 =y02

B1=2y0y1

B2 =2y2y0+y12

……………

………….. (32)

Put equation (32) into equation (31), we get

y0 = x3+1.0382041x4+0.5x5+0.15x6+0.0337302x7+0.0061012x8-0.001851852x9-0.003582451x10-0.00258342x11-0.001261185x12-0.0004746x13……………..

y1=0.0388054x4+0.016666x6+0.00823972x7+0.00223214x8+0.00319444x9+0.00390766x10+0.00270645x11+0.00126549x12+0.000044539x13

y2=0.000413381x4-0.000307979x7+0.0000462963x9-0.000128465x10-0.000101745x11 -0.00000215681x12+0.0000308777x13+……………..

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Similarly this process will be proceeds and solution in a series obtained as

y= y0 +y 1 + y2

=x3+0.999812x4+0.5x5+0.1666666x6+0.0416619x7+0.0083333x8+0.00138889x9

+0.000196747x10+……………

The exact solution of above result solve by Taylor series is y(x) = x ex with n finite order as below

y(x) =x3+x4+0.5x5+0.166666x6+0.04166666x7+0.008333x8+0.0013888x9+0.000198x10+…..

3. CONCLUSION

Our aim here is to compare the decomposition method with the classical methods for solving differential equations in order to obtain a better understanding of it. Because the special functions are extremely useful tools for obtaining closed form as well as series solutions to a variety of problems arising in science and engineering we tried to rebating the known results by the new method. We proposed an efficient modification of the standard Adomian decomposition method for solving singular initial value problem and boundary value problem in the higher order ordinary differential equations. The obtained results show that the rate of convergence of modified Adomian decomposition method is higher than standard Adomian decomposition method for these problems.

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