STUDY OF TRANSVERSELY ISOTROPIC AND LAYERED HALF – SPACE TO SURFACE LOADS

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ABSTRACT
In this paper the method of vector functions is introduced in association with the propagator matrix method to solve the deformation of transversely isotropic and layered elastic materials under surface loads. It is shown that the equilibrium equations are reduced to the two sets of simultaneously linear differential equations which are called type I and type II. The general solutions and the layer matrices are then obtained from the two sets of differential equations.

Keywords
propagator matrix, isotropic, equilibrium.

1. INTRODUCTION
[1] [2] solved the general problem of three-dimensional deformation of a transversely isotropic and homogeneous half –space using the potential function method. However that method requires the solution of a system of simultaneous linear equations with an order proportional to the number of layers and the introduction of auxiliary variables and coordinates transformations for the three dimensional problems [4] solved the problem by assuming axially symmetric deformation of a transversely isotropic and layered half-space by surface loads using propagator matrix method [9] and the generalized love's strain potential.

The propagator matrix method is used to solve the problem of the static deformation of a transversely isotropic and layered elastic half-space under the action of general surface loads. The general solutions and the layer matrices are then obtained from the two sets of differential equations. By using the continuity conditions at the layer interfaces and the boundary conditions at the surfaces, the displacement and stress components at any point of the medium are obtained by multiplication of matrices. As the solution is obtained for different cases of characteristic roots determined by the elastic constants of the media, the present formulation avoids the complicated nature of the problem on the one hand [8] and on the other hand can be reduced directly to the solutions of the corresponding two-dimensional deformation [7] and axially symmetric deformation [4] and also to the solution of the corresponding isotropic case [3]

2. BASIC EQUATIONS
2.1 Stress strain relation
We choose the axis of symmetry of a homogeneous and transversely isotropic elastic medium as the Z- axis. The generalized Hooke's law in Cartesian coordinates (x, y, z) can be expressed as [10]

$$\sigma_{xx} = A_{11}e_{xx}+A_{12}e_{yy}+A_{13}e_{zz} \qquad \sigma_{yz} = 2A_{44}e_{yz}$$
$$\sigma_{yy} = A_{12}e_{xx}+A_{11}e_{yy}+A_{13}e_{zz} \qquad \sigma_{xz} = 2A_{44}e_{xz}$$
$$\sigma_{zz} = A_{13}e_{xx}+A_{13}e_{yy}+A_{33}e_{zz} \qquad \sigma_{xy} = 2A_{66}e_{xy} \quad (1.1)$$

Where $A_{66} = \frac{(A_{11}-A_{12})}{2}$ (1.2) σ_{xx},σ_{yy}etc.,are the components of stress ; e_{xx}, e_{yy}, etc are the components of strain. The parameters A_{11}, A_{12}, A_{13}, A_{33} and A_{44} are the five elastic constants of medium. In the case of an isotropic medium

$$A_{44} = A_{66} = \frac{A_{11}-A_{12}}{2} = \frac{E}{2(1-v)}$$
$$A_{12} = A_{13} = \frac{Ev}{(1+v)(1-2v)} A_{11} = A_{33} = \frac{E(1-v)}{[(1+v)(1-2v)]}$$
(1.3)

Where E is the Young's modulus and v is the Poisson's ratio.

2.2 Equilibrium equations
In the absence of body forces

$$\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} + \frac{\partial\sigma_{xz}}{\partial z} = 0$$
$$\frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\sigma_{yz}}{\partial z} = 0 \qquad (1.4)$$
$$\frac{\partial\sigma_{xz}}{\partial x} + \frac{\partial\sigma_{yz}}{\partial y} + \frac{\partial\sigma_{zz}}{\partial z} = 0$$

The strain-displacement relations

$$e_{xx} = \frac{\partial u_x}{\partial x} \qquad 2e_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$
$$e_{yy} = \frac{\partial u_y}{\partial y} \qquad 2e_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}$$
$$e_{zz} = \frac{\partial u_z}{\partial z} \qquad 2e_{xz} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \qquad (1.5)$$

Where (u_x, u_y, u_z) are the components of the displacement vector. We now introduce a system of vector functions [6]

$$L(x, y; \alpha, \beta) = i_z S(x, y; \alpha, \beta)$$
$$M(x, y; \alpha, \beta) = \text{grad } S = i_x \frac{\partial S}{\partial x} + i_y \frac{\partial S}{\partial y}$$
$$N(x, y; \alpha, \beta) = \text{curl } i_z S = i_x \frac{\partial S}{\partial y} - i_y \frac{\partial S}{\partial x} \qquad (1.6)$$

Where (i_x, i_y, i_z) are the unit vectors in (x, y, z) direction of Cartesian coordinates. The scalar function S(x, y, α, β) given as

$$S(x, y, \alpha, \beta) = \frac{\exp[-i(\alpha x+\beta y)]}{2\pi} \qquad (1.7)$$

Satisfy the Holmholtz equation

$$\frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} + \lambda^2 S = 0 \qquad (1.8)$$
$$\lambda^2 = \alpha^2 + \beta^2 \qquad (1.9)$$

In equations (1.6) - (1.8) α, β and λ are parameter variables. Owing to the orthogonality of the system (1.6), any vector functions may be expressed in terms of them. In particular, for the unknown displacement and 'surface' stress vectors , we may have

$$u(x, y, z) = \iint_{-\infty}^{+\infty} [U_L(z)L(x, y) + U_M(z)M(x, y) + U_N(z)N(x, y)] d\alpha d\beta$$

$$\mathbf{T}(x,y,z) = \sigma_{xz}\mathbf{i}_x + \sigma_{yz}\mathbf{i}_y + \sigma_{zz}\mathbf{i}_z \qquad (1.10)$$

$$= \iint_{-\infty}^{+\infty} [T_L(z)\mathbf{L}(x,y) + T_M(z)\mathbf{M}(x,y) + T_N(z)\mathbf{N}(x,y)]\, d\alpha d\beta \qquad (1.11)$$

In equations, (1.10)-(1.11) the dependences of vector functions L, M and N on the parameters α and β or λ have been omitted for simplicity.

It is of interest to note from (1.6) and (1.10) that while the displacement solutions expressed in terms of N have zero dilatation, the solution expressed in term of L and M gives zero z component of the curl of the displacement vector.

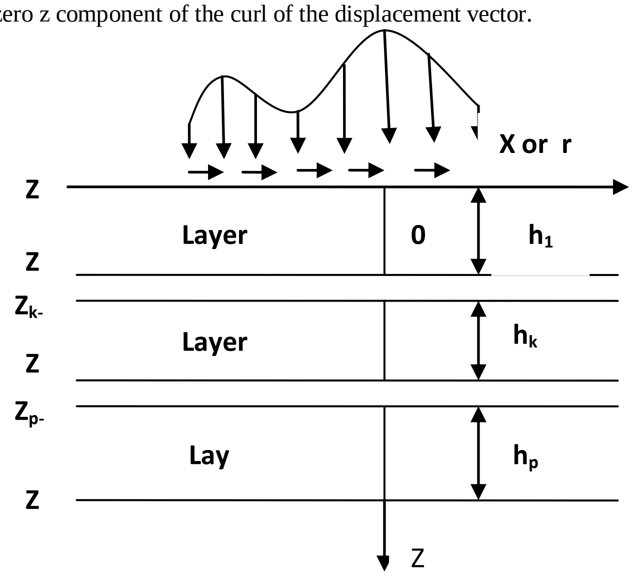

Fig 1.1 Scheme of a layered elastic system under surface loads

3. GENERAL SOLUTIONS & LAYER MATRICES

The problem that we analyse is shown schematically in Fig.1.General surface loading P(x, y) is applied to the surface z=0 of a layered elastic system, which is composed of parallel, homogeneous and transversely isotropic p layers lying over a homogeneous half-space. The layer interfaces are assumed to be in welded contact with the exception to the layer interface $z = z_p$. Since the layers are in welded contact, so continuity of displacements and stresses at the interface $z=z_k$ (k =1, 2,……., p-1) holds. Substituting equation (1.10) into (1.5) we get the following strains:

$$e_{xx} = U_M \frac{\partial^2 S}{\partial x^2} + U_N \frac{\partial^2 S}{\partial x \partial y}$$

$$e_{yy} = U_M \frac{\partial^2 S}{\partial y^2} + U_N \frac{\partial^2 S}{\partial x \partial y} \qquad (1.12)$$

$$e_{zz} = \frac{\partial U_L}{\partial z} S = \frac{dU_L}{dz} S$$

$$e_{xy} = \frac{1}{2}\left[2U_M \frac{\partial^2}{\partial x \partial y} + U_N\left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2}\right)\right] S(x,y)$$

$$e_{yz} = \frac{1}{2}\left[\frac{dU_M}{dz}\frac{\partial}{\partial y} - \frac{dU_N}{dz}\frac{\partial}{\partial x} + U_L \frac{\partial}{\partial y}\right] S(x,y)$$

$$e_{zx} = \frac{1}{2}\left[U_L \frac{\partial}{\partial x} + \frac{dU_M}{dz}\frac{\partial}{\partial x} + \frac{dU_N}{dz}\frac{\partial}{\partial y}\right] S(x,y)$$

Where

$$u_x = U_M \frac{\partial S}{\partial x} + U_N \frac{\partial S}{\partial y}$$

$$u_y = U_M \frac{\partial S}{\partial y} - U_N \frac{\partial S}{\partial y} \qquad (1.13)$$

$$u_z = U_L S$$

Using the stress-strain relations given in equation (1.1) in equation (1.12), we get the fallowing stresses:

$$\sigma_{xx} = \left[A_{11}\left(U_M \frac{\partial^2}{\partial x^2} + U_N \frac{\partial^2}{\partial x \partial y}\right) + A_{12}\left(U_M \frac{\partial^2}{\partial y^2} - U_M \frac{\partial^2}{\partial x \partial y}\right) + A_{13}\frac{dU_L}{dz}\right] S(x,y)$$

$$\sigma_{yy} = \left[A_{12}\left(U_M \frac{\partial^2}{\partial x^2} + U_N \frac{\partial^2}{\partial x \partial y}\right) + A_{11}\left(U_M \frac{\partial^2}{\partial x^2} - U_N \frac{\partial^2}{\partial x \partial y}\right) + A_{13}\frac{dU_L}{dz}\right] S(x,y)$$

$$\sigma_{zz} = \left[A_{13}U_M\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + A_{33}\frac{dU_L}{dz}\right] S(x,y)$$

$$\sigma_{xy} = A_{66}\left[2U_M \frac{\partial^2}{\partial x \partial y} + U_N\left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2}\right)\right] S(x,y)$$

$$\sigma_{yz} = A_{44}\left[\frac{dU_M}{dz}\frac{\partial}{\partial y} - \frac{dU_N}{dz}\frac{\partial}{\partial x} + U_L \frac{\partial}{\partial y}\right] S(x,y)$$

$$\sigma_{zx} = A_{44}\left[\frac{dU_M}{dz}\frac{\partial}{\partial y} + \frac{dU_N}{dz}\frac{\partial}{\partial x} + \frac{dU_N}{dz}\frac{\partial}{\partial y}\right] S(x,y) \qquad (1.14)$$

Except for special cases, we will omit the subscript k and the notations $\iint_{-\infty}^{\infty}[\ldots.]d\alpha d\beta$.

Also from equations (1.11) and (1.6) surface stress vector is given as:

$$\mathbf{T}(x,y,z) = \sigma_{xz}\mathbf{i}_x + \sigma_{yz}\mathbf{i}_y + \sigma_{zz}\mathbf{i}_z$$

Where

$$\sigma_{xz} = T_M \frac{\partial S}{\partial x} + T_N \frac{\partial S}{\partial y}$$

$$\sigma_{yz} = T_M \frac{\partial S}{\partial y} - T_N \frac{\partial S}{\partial x}$$

$$\sigma_{zz} = T_L(z) S(x,y) \qquad (1.15)$$

Comparing equation (1.14) and (1.15), we get

$$\left(T_M \frac{\partial}{\partial x} + T_N \frac{\partial}{\partial y}\right) S = \left[A_{44}\left(U_L \frac{\partial}{\partial x} + \frac{dU_M}{dz}\frac{\partial}{\partial x} + \frac{dU_N}{dz}\frac{\partial}{\partial y}\right)\right] S$$

$$\left(T_M \frac{\partial}{\partial y} - T_N \frac{\partial}{\partial x}\right) S = \left[A_{44}\left(U_L \frac{\partial}{\partial y} + \frac{dU_M}{dz}\frac{\partial}{\partial y} - \frac{dU_N}{dz}\frac{\partial}{\partial x}\right)\right] S$$

$$T_L S = \left[A_{13}U_M\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + A_{33}\frac{dU_L}{dz}\right] S \qquad (1.16)$$

Solving above equations, we obtain the fllowing relations between the expansion coefficients.

$$T_L = -\lambda^2 A_{13} U_M + A_{33}\frac{dU_L}{dz}$$

$$T_M = A_{44}\left(U_L + \frac{dU_M}{dz}\right)$$

$$T_N = A_{44}\frac{dU_N}{dz} \qquad (1.17)$$

Substituting the expressions of stresses from equations (1.14) into the equations of equilibrium (1.4) and by making use of (1.17), we obtain the other three relations between T_L, T_M, T_N

$$\frac{dT_N}{dz} - \lambda^2 U_N A_{66} = 0$$

$$\frac{dT_L}{dz} - \lambda^2 T_M = 0$$

$$-\lambda^2 U_M A_{11} + A_{13}\frac{dU_L}{dz} + \frac{dT_M}{dz} = 0 \qquad (1.18)$$

Equations (1.17) and (1.18) can be cast into two independent sets of simultaneous linear differential equations. They are called type I and type II respectively.

A.For type I

$$\frac{dU_L}{dz} = \lambda^2 U_M \frac{A_{13}}{A_{33}} + \frac{T_L}{A_{33}}$$
$$\frac{dU_M}{dz} = -U_L + \frac{T_M}{A_{44}}$$
$$\frac{dT_L}{dz} = \lambda^2 T_M$$
$$\frac{dT_M}{dz} = \lambda^2 U_M\left(\frac{A_{11}A_{33} - A_{13}^2}{A_{33}}\right) - \frac{A_{13}T_L}{A_{33}} \qquad (1.19)$$

B.For type II

$$\frac{dU_N}{dz} = \frac{1}{A_{44}} T_N$$
$$\frac{dT_N}{dz} = \lambda^2 A_{66} U_N \qquad (1.20)$$

For general solution for type I, we write equation (1.19) in the form

$$\frac{dU_L}{dz} = \frac{(\lambda U_M)\lambda A_{13}}{A_{33}} + \frac{\lambda\left(\frac{T_L}{\lambda}\right)}{A_{33}}$$
$$\frac{d(\lambda U_M)}{dz} = -\lambda U_L + \frac{\lambda T_M}{A_{44}}$$
$$\frac{d\left(\frac{T_L}{\lambda}\right)}{dz} = \lambda T_M$$
$$\frac{d(T_M)}{dz} = (\lambda U_M)\frac{\lambda(A_{11}A_{33} - A_{13}^2)}{A_{33}} - \frac{\lambda A_{13}}{A_{33}}\frac{T_L}{\lambda}$$

To solve the above system of simultaneous equations, writing them in matrix form as

$$\frac{d}{dz}[E(z)] = V[E(z)]$$

Where $\quad [E(z)] = \left[U_L(z) \quad \lambda U_M(z) \quad \frac{T_L(z)}{\lambda} \quad T_M(z)\right]^T$

(1.21)

And V is the coefficient matrix

$$V = \begin{bmatrix} 0 & \frac{\lambda A_{13}}{A_{33}} & \frac{\lambda}{A_{33}} & 0 \\ -\lambda & 0 & 0 & \frac{\lambda}{A_{44}} \\ 0 & 0 & 0 & \lambda \\ 0 & a\lambda & \frac{-\lambda A_{13}}{A_{33}} & 0 \end{bmatrix}$$

$$a = \frac{A_{11}A_{33} - A_{13}^2}{A_{33}}$$

We solve above system of differential equations by eigen value method [5].

The characteristic equation is

$$|V - XI| = 0$$

$$\text{Or } \lambda \begin{vmatrix} 0-x & \frac{A_{13}}{A_{33}} & \frac{1}{A_{33}} & 0 \\ -1 & 0-x & 0 & \frac{1}{A_{44}} \\ 0 & 0 & 0-x & 1 \\ 0 & a & \frac{-A_{13}}{A_{33}} & 0-x \end{vmatrix} = 0$$

Expanding the determinant, we get the characteristic equation

$$A_{33}A_{44}x^4 + x^2(A_{13}^2 + 2A_{13}A_{44} - A_{11}A_{33}) + A_{11}A_{44} = 0$$

This can be written as

$$(A_{44}x^2 - A_{11})(A_{33}x^2 - A_{44}) + (A_{13} + A_{44})^2 x^2 = 0$$

(1.22)

This is quadratic in x^2 and gives two values of x^2 and hence four values of x i.e. $\pm x_1$, $\pm x_2$ (say) known as characteristic roots. Thus solution of a system of differential equations depends on the different cases of characteristic roots.

Case I: When characteristic roots are distinct ($x_1 \neq x_2$)

For x_1 the corresponding eigen-vector can be obtained by solving the equations

$$[V - x_1 I][Z^1] = 0$$

Where

$$[Z^1] = [Z_{11}Z_{21}Z_{31}Z_{41}]^T = \left[c(x_1)d(x_1)\ {}^{1}/_{x_1}\ 1\right]^T$$

Where values of c(x) and d(x) are

$$c(x) = \frac{A_{11} + x^2A_{13}}{x^2(A_{11}A_{33} - A_{13}^2)}$$
$$d(x) = \frac{A_{13} + x^2A_{33}}{x(A_{11}A_{33} - A_{13}^2)} \qquad (1.23)$$

Another eigen-vector $[z^2]$ can be obtained by replacing x_1 by $-x_1$. The remaining two are obtained by replacing x_1 by x_2 in $[z^1]$ and $[z^2]$ respectively. This solution of equation (1.22) when eigen-values are not equal is given by

$$[E(z)] = A_1[z^1]e^{\lambda x_1 z} + B_1[z^2]e^{-\lambda x_1 z} + C_1[z^3]e^{\lambda x_2 z} + D_1[z^4]e^{-\lambda x_2 z} \qquad (1.24)$$

Where

$$[z^1] = [c(x_1)\ d(x_1)\ 1/x_1\ 1]^T$$
$$[z^2] = [c(x_1)\ -d(x_1)\ -1/x_1\ 1]^T$$

$[z^3]$ and $[z^4]$ are obtained by replacing x_1 by x_2 in $[z^1]$ and $[z^2]$ respectively.

Case II: When characteristic roots are equal ($x_1 = x_2$)

In this case first eigen-vector will be same as in case I i.e

$$[V - x_1 I][\overline{\alpha}] = 0$$

Where

$$[\overline{\alpha_1}] = [c(x_1)\ d(x_1)\ 1/x_1\ 1]^T$$

And the other vector will be

$$[V - x_1 I]\overline{\beta_1} = \overline{\alpha_1}$$

Or

$$\lambda \begin{bmatrix} -x_1 & {}^{A_{13}}/_{A_{33}} & {}^{1}/_{A_{33}} & 0 \\ -1 & -x_1 & 0 & {}^{1}/_{A_{44}} \\ 0 & 0 & -x_1 & 1 \\ 0 & a & -{}^{A_{13}}/_{A_{33}} & -x_1 \end{bmatrix} \begin{bmatrix} A_2 \\ B_2 \\ C_2 \\ D_2 \end{bmatrix} = \begin{bmatrix} c(x_1) \\ d(x_1) \\ {}^{1}/_{x_1} \\ 1 \end{bmatrix}$$

Solving the matrix multiplication, we get

$$-x_1A_2 + \frac{A_{13}}{A_{33}}B_2 + \frac{1}{A_{33}}C_2 = \frac{A_{11} + x^2A_{13}}{\lambda x_1^2(A_{11}A_{33} - A_{13}^2)}$$

$$-A_2 - x_1B_2 + \frac{1}{A_{44}}D_2 = \frac{A_{13} + x_1^2A_{33}}{\lambda x_1(A_{11}A_{33} - A_{13}^2)}$$

$$-x_1C_2 + D_2 = \frac{1}{\lambda x_1}$$

$$-aB_2 - \frac{A_{13}}{A_{33}}C_2 - x_1D_2 = \frac{1}{\lambda}$$

Solving this system of simultaneous equations for A_2, B_2, C_2 and D_2, we get

$$C_2 = \frac{D_2}{x_1} - \frac{1}{\lambda x_1^2}$$
$$B_2 = \frac{d'(x_1)}{\lambda} + d(x_1)D_2$$
$$A_2 = \frac{c'(x_1)}{\lambda} + c(x_1)D_2$$

Taking D_2=z, we have

$$\overline{\beta_1} = \begin{bmatrix} \frac{c'(x_1)}{\lambda} + c(x_1)z \\ \frac{d'(x_1)}{\lambda} + d(x_1)z \\ -\frac{x_1^{-2}}{\lambda} + x_1^{-1}z \end{bmatrix} \quad (1.25)$$

The other two are obtained by replacing x_1 by $-x_1$ in α_1 and β_1 respectively. Thus the solution of equation (1.22) when eigen-values are equal is given by

$$[E(z)] = A_1[\overline{\alpha_1}]e^{\lambda x_1 z} + B_1[\overline{\alpha_2}]e^{-\lambda x_1 z} + C_1[\overline{\beta_1}]e^{\lambda x_1 z} + D_1[\overline{\beta_2}]e^{-\lambda x_1 z} \quad (1.26)$$

Where

$\overline{\alpha_1} = [c(x_1) \quad d(x_1) \quad 1/x_1 \quad 1]^T$

$\overline{\beta_1}$ is given by equation (1.25), $\overline{\alpha_2}$ and $\overline{\beta_2}$ are obtained by replacing x_1 by $-x_1$ in $\overline{\alpha_1}$ and $\overline{\beta_1}$ respectively. From equations (1.24) and (1.26), we see that solution of type I can be written as

$$[E(z)] = [Z(z)][K] \quad (1.27)$$

Where the elements of solution matrix [Z(z)] are given in Table-1,

$$[K] = [A_1 \;\; B_1 \;\; C_1 \;\; D_1]^T \quad (1.28)$$

A_1, B_1, C_1 and D_1 are the arbitrary functions of λ. From equation (1.27), we have the relation for nth layer as,

$[E(z_n)] = [Z(z_n)][K]$

Similarly, for (n-1)th layer

$[E(z_{n-1})] = [Z(z_{n-1})][K]$

Eliminating [K] from above two equations, we get

$[E(z_{n-1})] = [Z(z_{n-1})][Z(z_n)]^{-1}[E(z_n)]$.

Finally, we get the propagating relation for a_k layer as

$$[E(z_{k-1})] = [a_k][E(z_k)] \quad (1.29)$$

Where $[a_k]$ is the propagator matrix of layer k.

For general solution for type II, we solve the simultaneous equations given by equation (1.20) and its solution is given as

$$U_N = A^L e^{\lambda s z} + B^L e^{-\lambda s z}$$

$$T_N = \bar{s}\lambda A^L e^{\lambda s z} - \bar{s}\lambda B^L e^{-\lambda s z} \quad (1.30)$$

Where

$s = \left(\frac{A_{66}}{A_{44}}\right)^{\frac{1}{2}}$, $\bar{s} = sA_{44}$.

This general solution can be written in the matrix form as

$$\begin{bmatrix} U_N \\ T_N \end{bmatrix} = \begin{bmatrix} e^{\lambda s z} & e^{-\lambda s z} \\ \bar{s}\lambda e^{\lambda s z} & -\bar{s}\lambda e^{-\lambda s z} \end{bmatrix} \begin{bmatrix} A^L \\ B^L \end{bmatrix}$$

$$\text{Or} \begin{bmatrix} U_N \\ T_N \end{bmatrix} = [Z^L] \begin{bmatrix} A^L \\ B^L \end{bmatrix} \quad (1.31)$$

Where A^L and B^L are the arbitrary functions of λ and the solution matrix

$$[Z^L] = \begin{bmatrix} \exp(\lambda s z) & \exp(-\lambda s z) \\ \lambda\bar{s}\exp(\lambda s z) & -\lambda\bar{s}\exp(-\lambda s z) \end{bmatrix} \quad (1.32)$$

Rewriting equation (1.31) as

$$\begin{bmatrix} U_N \\ T_N/\lambda \end{bmatrix} = \begin{bmatrix} e^{\lambda s z} & e^{-\lambda s z} \\ \bar{s}e^{\lambda s z} & -\bar{s}e^{-\lambda s z} \end{bmatrix} \begin{bmatrix} A^L \\ B^L \end{bmatrix} \quad (1.33)$$

Due to continuity of U_N, T_N at $(k-1)^{th}$ layer

$$\begin{bmatrix} U_N(z_{k-1}) \\ T_N(z_{k-1}/\lambda) \end{bmatrix} = [z`(z_{k-1})] \begin{bmatrix} A^L \\ B^L \end{bmatrix}$$

Similarly for k^{th} layer

$$\begin{bmatrix} U_N(z_k) \\ T_N(z_k/\lambda) \end{bmatrix} = [z`(z_k)] \begin{bmatrix} A^L \\ B^L \end{bmatrix}$$

Eliminating A^L, B^L we get

$$\begin{bmatrix} U_N(z_{k-1}) \\ T_N(z_{k-1}/\lambda) \end{bmatrix} = [z`(z_{k-1})][z`(z_k)]^{-1} \begin{bmatrix} U_N(z_k) \\ T_N(z_k)/\lambda \end{bmatrix}$$

where

$$[z'(z_k)]^{-1} = \frac{1}{2\bar{s}} \begin{bmatrix} \bar{s}e^{-\lambda s z_k} & e^{-\lambda s z_k} \\ \bar{s}e^{\lambda s z_k} & -e^{\lambda s z_k} \end{bmatrix}$$

Thus

$$\begin{bmatrix} U_N(z_{k-1}) \\ T_N(z_{k-1})/\lambda \end{bmatrix} = \begin{bmatrix} e^{\lambda s z_{k-1}} & e^{-\lambda s z_{k-1}} \\ \bar{s}e^{\lambda s z_{k-1}} & -\bar{s}e^{-\lambda s z_{k-1}} \end{bmatrix} \frac{1}{2\bar{s}} \begin{bmatrix} \bar{s}e^{-\lambda s z_k} & e^{-\lambda s z_k} \\ \bar{s}e^{\lambda s z_k} & -e^{\lambda s z_k} \end{bmatrix} \begin{bmatrix} U_N(z_k) \\ T_N(z_k)/\lambda \end{bmatrix}$$

Therefore, the propagating relation is

$$\begin{bmatrix} U_N(z_{k-1}) \\ T_N(z_{k-1})/\lambda \end{bmatrix} = [a_k^L] \begin{bmatrix} U_N(z_k) \\ T_N(z_k)/\lambda \end{bmatrix} \quad (1.34)$$

Where

$$[a_k^L] = \begin{bmatrix} ch(\lambda s_k h_k) & \left(-1/\bar{s}_k\right) sh(\lambda s_k h_k) \\ -\bar{s}_k sh(\lambda s_k h_k) & ch(\lambda s_k h_k) \end{bmatrix} \quad (1.35)$$

$[a_k^L]$ is the layer matrix or the propagator matrix of the layer k.

Table-1

The elements of the solution matrix [Z(z)]in equation (1.27) are:-

Case (1) when characteristic roots are not equal $(x_1 \neq x_2)$

$Z_{11} = C(X_1)e^{\lambda X_1 Z}$ $\quad Z_{12} = C(X_1)e^{-\lambda X_1 Z}$

$Z_{21} = d(X_1)e^{\lambda X_1 Z}$ $\quad Z_{22} = -d(X_1)e^{-\lambda X_1 Z}$

$Z_{31} = \frac{1}{X_1}e^{\lambda X_1 Z}$ $\quad Z_{32} = -\frac{1}{X_1}e^{-\lambda X_1 Z}$

$Z_{41} = e^{\lambda X_1 Z}$ $\quad Z_{42} = e^{-\lambda X_1 Z}$ $\quad$ (A1)

Where $(x_1 \& x_2)$ are the characteristic roots of the following equation

(A44x2-A11) (A33x2-A44) + (A13=A44)2□2 = 0

Zi3 & Zi4 are obtained from Zi1 & Zi2 respectively on replacing □1 by □2 (i=1, 2, 3, 4)

Case (2) when characteristic roots are equal (□1 = □2)

$$Z13 = \left(\frac{c'(X_1)}{\lambda} + C(X_1)Z\right)e^{\lambda x_1 z} \quad (A2)$$

$$Z23 = \left(\frac{d'(X_1)}{\lambda} + d(X_1)Z\right)e^{\lambda x_1 z} \quad (A3)$$

$$Z33 = \left(-\frac{1}{\lambda x_1^2} + \frac{Z}{x_1}\right)e^{\lambda x_1 z}$$

$$Z43 = Ze^{\lambda x_1 z}$$

While Zi1 and Zi2 are same as those in equation (A1), Zi4 are obtained from Zi3 on replacing □1 by –□1 (i = 1, 2, 3, 4) in equation (A2) and (A3). The dash denotes the derivation with respect to □1.

4. CONCLUSIONS

The method of vector functions is introduced in association with the propagator matrix method to solve the deformation of transversely isotropic and layered elastic materials under surface loads. The formulation is presented so that it can be used directly to perform practical calculations. As the solution is given in Cartesian system of vector functions, one can easily solve problems for different types of surface loading. It is shown that the formulation given is especially suitable for two-dimensional [7] and axially symmetric [4] deformation. Since the general solution and the propagator matrix for different cases of characteristic root are also given, which includes the isotropic case [3] the present formulation provides a complete solution of deformations by surface loads of transversely isotropic and layered elastic half-space.

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