# Some Identities for Even and Odd Fibonacci-Like and Lucas Numbers

Bijendra Singh[1] Omprakash Sikhwal[2] Shikha Bhatnagar[2]

School of Studies in Mathematics, Vikram University Ujjain, India[1]
Department of Mathematics, Mandsaur Institute of Technology, Mandsaur, India[2]

## ABSTRACT
The Fibonacci sequence are well known and widely investigated. The Fibonacci and Lucas sequences have enjoyed a rich history. To this day, interest remains in the relation of such sequences to many fields. In this paper, we obtain some identities for common factor of Fibonacci-Like and Lucas numbers. The new identities for even and odd both numbers are obtained.

## Mathematics Subject Classification
11B39, 11B37, 11B99

## Keywords
Fibonacci numbers, Lucas numbers, Fibonacci-Like numbers

## 1. INTRODUCTION
Mathematics can be considered as the underlying order of the universe, and the Fibonacci numbers is one of the most fascinating discovery made in the mathematical world. Among numerical sequences, the Fibonacci sequence has achieved a kind of celebrity status and has been studied extensively in number theory, applied mathematics, physics, computer science, and biology [2].

The Fibonacci sequence in each next term is the sum of previous two terms with two specific initial values, is a source of many nice and interesting identities. A similar interpretation also exists for Lucas sequence. The Fibonacci numbers have been studied both for their applications and the mathematical beauty of rich and interesting identities that they satisfy. The Fibonacci numbers given by the recurrence relation,

$$F_{n+1} = F_n + F_{n-1}, \text{ where } n \geq 1 \qquad (1.1)$$

with initial conditions $F_0 = 0,\ F_1 = 1$.

The Lucas numbers are given by the recurrence relation:

$$L_{n+1} = L_n + L_{n-1}, \text{ where } n \geq 1 \qquad (1.2)$$

with initial conditions $L_0 = 0,\ L_1 = 1$.

The Fibonacci-Like numbers [1] is defined by the recurrence relation:

$$S_{n+1} = S_n + S_{n-1}, \text{ where } n \geq 1 \qquad (1.3)$$

with initial conditions $S_0 = 2,\ S_1 = 2$.

The associated initial conditions $S_0$ and $S_1$ are the sum of initial conditions of Fibonacci and Lucas numbers respectively.

i.e. $S_0 = F_0 + L_0$ and $S_1 = F_1 + L_1$.

The general form of Lucas numbers can be written by Binet's formula given as

$$L_n = \frac{(1+\sqrt{5})^n + (1-\sqrt{5})^n}{2^n}, \text{ where } n \geq 1 \qquad (1.4)$$

The Binet's formula for Fibonacci-Like numbers is given by

$$S_n = \frac{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}}{2^n\sqrt{5}}, \text{ where } n \geq 1 \qquad (1.5)$$

The value of Fibonacci-Like and Lucas number are presented in the following table:

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | ... |
|---|---|---|---|---|---|---|---|---|---|---|
| $S_n$ | 2 | 2 | 4 | 6 | 10 | 16 | 26 | 42 | 68 | ... |
| $L_n$ | 2 | 1 | 3 | 4 | 7 | 11 | 18 | 29 | 47 | ... |

There are a lot of identities about Fibonacci and Lucas numbers. In [3], $F_{4n}+1,\ F_{4n+1}+1, F_{4n+2}+1,\ F_{4n+3}+1,\ F_{4n+1}-1$ $L_{4n+1}-1, L_{4n+1}+1, F_{4n+3}-1,\ L_{4n+3}-1,\ L_{4n+3}+1$ are defined in the form of the relation between Fibonacci and Lucas numbers. In [4], the identities $F_{2n}$ and $F_{2n}+1$ are presented. In this paper we obtain some identities for even and odd Fibonacci-Like and Lucas numbers.

## 2. SOME IDENTITIES FOR FIBONACCI-LIKE AND LUCAS NUMBERS

**Theorem**(1). $S_{2n}L_{2n+1} = S_{4n+1}$, where $n \geq 1$.

Proof. $S_{2n}L_{2n+1} =$

$$\left[\frac{\left(1+\sqrt{5}\right)^{2n+1} - \left(1-\sqrt{5}\right)^{2n+1}}{2^{2n}\sqrt{5}}\right] \cdot \left[\frac{(1+\sqrt{5})^{2n+1} + (1-\sqrt{5})^{2n+1}}{2^{2n+1}}\right],$$

$$= \left[\frac{(1+\sqrt{5})^{4n+2} - (1-\sqrt{5})^{4n+2}}{2^{4n+1}\sqrt{5}}\right]$$

$$= S_{4n+1}.$$

**Theorem**(2). $S_{2n-1}L_{2n} = S_{4n-1}$, where $n \geq 1$.

Proof. $S_{2n-1}L_{2n} =$

$$\left[\frac{\left(1+\sqrt{5}\right)^{2n}-\left(1-\sqrt{5}\right)^{2n}}{2^{2n-1}\sqrt{5}}\right]\cdot\left[\frac{(1+\sqrt{5})^{2n}+(1-\sqrt{5})^{2n}}{2^{2n}}\right],$$

$$=\left[\frac{(1+\sqrt{5})^{4n}-(1-\sqrt{5})^{4n}}{2^{4n-1}\sqrt{5}}\right]$$

$$=S_{4n-1}.$$

**Theorem(3).** $S_{2n+1}L_{2n+2}=S_{4n+3}$, where $n\geq 1$.

Proof. $S_{2n+1}L_{2n+2} =$

$$\left[\frac{\left(1+\sqrt{5}\right)^{2n+2}-\left(1-\sqrt{5}\right)^{2n+2}}{2^{2n+1}\sqrt{5}}\right]\cdot\left[\frac{(1+\sqrt{5})^{2n+2}+(1-\sqrt{5})^{2n+2}}{2^{2n+2}}\right],$$

$$=\left[\frac{(1+\sqrt{5})^{4n+4}-(1-\sqrt{5})^{4n+4}}{2^{4n+3}\sqrt{5}}\right]$$

$$=S_{4n+3}.$$

**Theorem(4).** $S_{2n+1}L_{2n}=S_{4n+1}+2$, where $n\geq 1$.

Proof. $S_{2n+1}L_{2n} =$

$$\left[\frac{\left(1+\sqrt{5}\right)^{2n+2}-\left(1-\sqrt{5}\right)^{2n+2}}{2^{2n+1}\sqrt{5}}\right]\cdot\left[\frac{(1+\sqrt{5})^{2n}+(1-\sqrt{5})^{2n}}{2^{2n}}\right],$$

$$=\left[\frac{(1+\sqrt{5})^{4n+2}-(1-\sqrt{5})^{4n+2}}{2^{4n+1}\sqrt{5}}\right]+$$

$$\left[\frac{(1+\sqrt{5})^{2n+2}(1-\sqrt{5})^{2n}-(1-\sqrt{5})^{2n+2}(1+\sqrt{5})^{2n}}{2^{4n+1}\sqrt{5}}\right],$$

$$=S_{4n+1}+\frac{(1+\sqrt{5})^{2n}(1-\sqrt{5})^{2n}}{2^{4n+1}\sqrt{5}}\left[(1+\sqrt{5})^{2}-(1-\sqrt{5})^{2}\right],$$

$$=S_{4n+1}+\frac{1}{2\sqrt{5}}\left[\frac{\left(1+\sqrt{5}\right)(1-\sqrt{5})}{2^2}\right]^{2n}\left[4\sqrt{5}\right],$$

$$=S_{4n+1}+\frac{1}{2\sqrt{5}}(-1)^{2n}\left[4\sqrt{5}\right],$$

$$=S_{4n+1}+2.$$

**Theorem(5).** $S_{2n}L_{2n}=S_{4n}+2$, where $n\geq 1$.

Proof. $S_{2n}L_{2n} =$

$$\left[\frac{(1+\sqrt{5})^{2n+1}-(1-\sqrt{5})^{2n+1}}{2^{2n}\sqrt{5}}\right]\left[\frac{(1+\sqrt{5})^{2n}+(1-\sqrt{5})^{2n}}{2^{2n}}\right],$$

$$=\left[\frac{(1+\sqrt{5})^{4n+1}-(1-\sqrt{5})^{4n+1}}{2^{4n}\sqrt{5}}\right]+$$

$$\left[\frac{(1+\sqrt{5})^{2n+1}(1-\sqrt{5})^{2n}-(1-\sqrt{5})^{2n+1}(1+\sqrt{5})^{2n}}{2^{4n}\sqrt{5}}\right],$$

$$=S_{4n}+\frac{(1+\sqrt{5})^{2n}(1-\sqrt{5})^{2n}}{2^{4n+1}\sqrt{5}}\left[\left(1+\sqrt{5}\right)-(1-\sqrt{5})\right],$$

$$=S_{4n}+\frac{1}{\sqrt{5}}\left[\frac{\left(1+\sqrt{5}\right)(1-\sqrt{5})}{2^2}\right]^{2n}\left[2\sqrt{5}\right],$$

$$=S_{4n}+\frac{1}{\sqrt{5}}(-1)^{2n}\left[2\sqrt{5}\right],$$

$$=S_{4n}+2.$$

**Theorem(6).** $S_{2n+2}L_{2n}=S_{4n+2}+4$, where $n\geq 1$.

Proof. $S_{2n+2}L_{2n} =$

$$\left[\frac{(1+\sqrt{5})^{2n+3}-(1-\sqrt{5})^{2n+3}}{2^{2n+2}\sqrt{5}}\right]\left[\frac{(1+\sqrt{5})^{2n}+(1-\sqrt{5})^{2n}}{2^{2n}}\right],$$

$$=\left[\frac{(1+\sqrt{5})^{4n+3}-(1-\sqrt{5})^{4n+3}}{2^{4n+2}\sqrt{5}}\right]+$$

$$\left[\frac{(1+\sqrt{5})^{2n+3}(1-\sqrt{5})^{2n}-(1-\sqrt{5})^{2n+3}(1+\sqrt{5})^{2n}}{2^{4n+2}\sqrt{5}}\right],$$

$$=S_{4n+2}+\frac{(1+\sqrt{5})^{2n}(1-\sqrt{5})^{2n}}{2^{4n+2}\sqrt{5}}\left[(1+\sqrt{5})^{3}-(1-\sqrt{5})^{3}\right],$$

$$=S_{4n+2}+\frac{1}{4\sqrt{5}}\left[\frac{\left(1+\sqrt{5}\right)(1-\sqrt{5})}{2^2}\right]^{2n}\left[16\sqrt{5}\right],$$

$$=S_{4n+2}+4(-1)^{2n},$$

$$=S_{4n+2}+4.$$

**Theorem(7).** $S_{2n-1}L_{2n+1}=S_{4n}-2$, where $n\geq 1$.

Proof. $S_{2n-1}L_{2n+1} =$

$$\left[\frac{(1+\sqrt{5})^{2n}-(1-\sqrt{5})^{2n}}{2^{2n-1}\sqrt{5}}\right]\left[\frac{(1+\sqrt{5})^{2n+1}+(1-\sqrt{5})^{2n+1}}{2^{2n+1}}\right],$$

$$=\left[\frac{(1+\sqrt{5})^{4n+1}-(1-\sqrt{5})^{4n+1}}{2^{4n}\sqrt{5}}\right]+$$

$$\left[\frac{(1+\sqrt{5})^{2n}(1-\sqrt{5})^{2n+1}-(1-\sqrt{5})^{2n}(1+\sqrt{5})^{2n+1}}{2^{4n}\sqrt{5}}\right],$$

$$=S_{4n}+\frac{1}{\sqrt{5}}\left[\frac{\left(1+\sqrt{5}\right)(1-\sqrt{5})}{2^2}\right]^{2n}\left[\left(1-\sqrt{5}\right)-(1+\sqrt{5})\right],$$

$$=S_{4n}+\frac{1}{\sqrt{5}}(-1)^{2n}\left[-2\sqrt{5}\right],$$

$$=S_{4n}-2.$$

**Theorem**(8). $S_{2n-1}L_{2n-1} = S_{4n-2} - 2,$ where $n \geq 1.$

Proof. $S_{2n-1}L_{2n-1} =$

$$\left[\frac{(1+\sqrt{5})^{2n}-(1-\sqrt{5})^{2n}}{2^{2n-1}\sqrt{5}}\right]\left[\frac{(1+\sqrt{5})^{2n-1}+(1-\sqrt{5})^{2n-1}}{2^{2n-1}}\right],$$

$$=\left[\frac{(1+\sqrt{5})^{4n-1}-(1-\sqrt{5})^{4n-1}}{2^{4n-2}\sqrt{5}}\right]+$$

$$\left[\frac{(1+\sqrt{5})^{2n}(1-\sqrt{5})^{2n-1}-(1-\sqrt{5})^{2n}(1+\sqrt{5})^{2n-1}}{2^{4n-2}\sqrt{5}}\right],$$

$$=\mathrm{S}_{4n-2}+\frac{1}{2^{-2}\sqrt{5}}\left[\frac{(1+\sqrt{5})(1-\sqrt{5})}{2^2}\right]^{2n}\left[\frac{1}{(1-\sqrt{5})}-\frac{1}{(1+\sqrt{5})}\right],$$

$$=\mathrm{S}_{4n-2}+\frac{4}{\sqrt{5}}(-1)^{2n}\left[\frac{2\sqrt{5}}{-4}\right],$$

$$=\mathrm{S}_{4n-2}-2.$$

**Theorem**(9). $S_{2n+1}L_{2n+1} = S_{4n+2} - 2,$ where $n \geq 1.$

Proof. $S_{2n+1}L_{2n+1} =$

$$\left[\frac{(1+\sqrt{5})^{2n+2}-(1-\sqrt{5})^{2n+2}}{2^{2n+1}\sqrt{5}}\right]\left[\frac{(1+\sqrt{5})^{2n+1}+(1-\sqrt{5})^{2n+1}}{2^{2n+1}}\right],$$

$$=\left[\frac{\left(1+\sqrt{5}\right)^{4n+3}-\left(1-\sqrt{5}\right)^{4n+3}}{2^{4n+2}\sqrt{5}}\right]+$$

$$\left[\frac{\left(1+\sqrt{5}\right)^{2n+2}\left(1-\sqrt{5}\right)^{2n+1}-\left(1-\sqrt{5}\right)^{2n+2}\left(1+\sqrt{5}\right)^{2n+1}}{2^{4n+2}\sqrt{5}}\right],$$

$$=\mathrm{S}_{4n+2}+\frac{1}{4\sqrt{5}}\left(\frac{1}{2^2}\right)^{2n}\left(1+\sqrt{5}\right)^{2n+1}\left(1-\sqrt{5}\right)^{2n+1}\left[\left(1+\sqrt{5}\right)-\left(1-\sqrt{5}\right)\right],$$

$$=\mathrm{S}_{4n+2}+\frac{1}{4\sqrt{5}}\left[\frac{\left(1+\sqrt{5}\right)\left(1-\sqrt{5}\right)}{2^2}\right]^{2n}\left[\left(1+\sqrt{5}\right)\left(1-\sqrt{5}\right)\right]\left[2\sqrt{5}\right],$$

$$=\mathrm{S}_{4n+2}+\frac{1}{4\sqrt{5}}(-1)^{2n}\left[-4\right]\left[2\sqrt{5}\right],$$

$$=\mathrm{S}_{4n+2}-2.$$

**Theorem**(10). $S_{2n+2}L_{2n+1} = S_{4n+3} - 2,$ where $n \geq 1.$

Proof. $S_{2n+2}L_{2n+1} =$

$$\left[\frac{(1+\sqrt{5})^{2n+3}-(1-\sqrt{5})^{2n+3}}{2^{2n+2}\sqrt{5}}\right]\left[\frac{(1+\sqrt{5})^{2n+1}+(1-\sqrt{5})^{2n+1}}{2^{2n+1}}\right],$$

$$=\left[\frac{\left(1+\sqrt{5}\right)^{4n+4}-\left(1-\sqrt{5}\right)^{4n+4}}{2^{4n+3}\sqrt{5}}\right]+$$

$$\left[\frac{\left(1+\sqrt{5}\right)^{2n+3}\left(1-\sqrt{5}\right)^{2n+1}-\left(1-\sqrt{5}\right)^{2n+3}\left(1+\sqrt{5}\right)^{2n+1}}{2^{4n+3}\sqrt{5}}\right],$$

$$=\mathrm{S}_{4n+3}+\frac{(1+\sqrt{5})^{2n+1}(1-\sqrt{5})^{2n+1}}{2^{4n+3}}\left[(1+\sqrt{5})^2-(1-\sqrt{5})^2\right],$$

$$=\mathrm{S}_{4n+3}+\frac{1}{8\sqrt{5}}\left[\frac{\left(1+\sqrt{5}\right)\left(1-\sqrt{5}\right)}{2^2}\right]^{2n}\left[\left(1+\sqrt{5}\right)\left(1-\sqrt{5}\right)\right]\left[4\sqrt{5}\right],$$

$$=\mathrm{S}_{4n+3}+\frac{1}{8\sqrt{5}}(-1)^{2n}\left[-4\right]\left[4\sqrt{5}\right],$$

$$=\mathrm{S}_{4n+3}-2.$$

## 3. CONCLUSION

This paper describes identities for Fibonacci-Like and Lucas numbers by their Binet's formula. These identities can be used to develop new identities of Fibonacci and their associated sequences.

## 4. REFERENCES

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