Study of the Linear and Non-Linear Differential Equation for Physical System

Yogendra Vishwakarma
Assistant Professor
Viva Institute of Technology
Shirgoun Virar (E)

Ajazul Haque, PhD
Assistant Professor
Viva Institute of Technology
Shirgoun Virar (E) Khopoli

Deepak Gaikwad, PhD
Assistant Professor
K.M.C College

ABSTRACT
Present research paper "**Study of the Linear and Non-Linear Differential Equation for physical system"**is an attempt to improve and develop the skill in science and engineering for solving the real problems.Here studying the application of differential equation for various branch of science and technology such as applied mathematics, physics, chemistry and biology, how it can be applied in the field of engineering and population dynamics. This is an important and powerful tool for analyzing the relationship among various dynamicalparameters.

Keywords
Dynamicalparameters, real problems, physical system and non-linear.

1. INTRODUCTION
Most of the laws of physics, science, mechanics and classical mechanics are of the form of differential equations and these law totally devote on the differential equations of dependent variables i.e. the rate of change of dependent variables with respect to time .A differential equation plays a very important role for deriving the formulae and making the laws. An equation which involves some derivatives of the functionis called a differential equation. or simply an equation which contains a differential coefficient is called a differential equation.Study of differential equations are great thing there are many types of differential equations from which discussed only two subdivisions of them i.e. linear and non-linear differential equations.An equation which involves derivative of aunknown function only of a first order is called a linear differential equation. Otherwise, it will be known as non-linear differential equation. Thus, if $\frac{dy}{dx}$ or y' denotes the first order derivative of the unknown function y= f(x), then the equation $\frac{dy}{dx}$= y is linear, while the equation $\frac{dy}{dx} = y^2$ is non-linear. In other word, a differential equation is called linear if there is no terms like y^2, $(y')^3$, y.y', siny, logy or$e^y$for examples

$\frac{dy}{dx} + 2y$=2x, $\frac{d^2y}{dx^2} - 3\frac{dy}{dx}$ + y = 0 are linear differential equationsand

$\frac{d^2y}{dx^2} + y^2\frac{dy}{dx} + y = 0$,$(y')^2 + 2y = x$, y'' + logy = 0 are non-linear differential equations.

A solution to a differential equation is a function whose derivatives satisfy the equation. The order of a differential equation is the order of highest derivatives occurring in differential equation. The degree of the differential equation is the power of highest order derivatives when differential coefficients are rational and free from fractional power.

2. APPLICATIONS
Many engineering problems that are time dependent are often described in terms of differential equations with conditions imposed at single point. The following list is a few motivational examples that encountering in many fields of engineering,science and technology, where and how it can be apply in the field of engineeringproblems and population dynamics are shown below:

1. Example on Simple pendulum's
2. Energy conservation
3. Radioactive decay in nuclear physics
4. RC-Circuit and Coupled L-R electrical circuits
5. Newton's law of cooling in thermodynamics
6. Newton's second law in dynamics
7. Motion of a particle under a variable force field

Some of the motivational examples also encountering in population dynamics such as growth of population in the country and growth of bacteria in the fields etc.

3. MOTIVATIONAL EXAMPLES
Assume that a simple pendulum consists of a weight suspended on a string , with gravity the only force acting on the weight. If α is the angle the pendulum's string makes with a vertical line, then horizontal force on the weight toward the vertical is directly proportional to sine angle α.

i.e.$\frac{d^2\alpha}{dt^2} \propto \sin\alpha$

If remove proportionality constant ,it has to multiply some constant known as proportionality constant say $C_0$where C_0> o therefore,we have $\frac{d^2\alpha}{dt^2} = - C_0 \sin\alpha$

hence$\frac{d^2\alpha}{dt^2} + C_0 \sin\alpha = 0$ (1.1)

This is a non-linear second order differential Equation and this differential equation can not be solved exactly for the function $\alpha(t)$.

However , if angle of suspension is very small i.e. if sine of α is nearly equal to α ,then the differential Equation (1.1) converts into the linear second order differential equation

$\frac{d^2\alpha}{dt^2} + C_0\alpha = 0$ and this can be solved as follows:

$\therefore D^2\alpha + C_0\alpha = 0$,where D =$\frac{d}{dt}$

$\therefore (D^2 + C_0)\,\alpha = 0$

$\therefore D^2 + C_0 = 0$

$\therefore$ D=+ $\sqrt{C_0}$i and D=- $\sqrt{C_0}$i

Hence, $\alpha(t)$ =$acos\sqrt{c_0}$ t +b sin$\sqrt{c_0}$ t

which is a required solution of the linear second order differential equation.

4. ENERGY CONSERVATIVE

Consider a particle of mass 'm' is moving along a straight line. If the force acting at a point x, then F(x) is conservative which means that there exists a function V(x), known as the potential energy (P.E), such that F(x) =-$\frac{dV}{dx}$ where F is only a function of x and assume thatthere is no friction. From the lower class physics, since we know that the kinetic energy (K.E) of a particle of mass 'm' is given by

K.E = $\frac{1}{2}$ m v^2 and the velocity of the particle (v) = $\frac{dx}{dt}$ i.e. the rate of change w.r.t time t .

But total energy of the particle of mass 'm' is given by

E = K.E + P.E

$\therefore$ E = $\frac{1}{2}$ m v^2 + V(x)

$\therefore$E = $\frac{1}{2}$ m $(\frac{dx}{dt})^2$ + V(x)

$\therefore$ m $(\frac{dx}{dt})^2$ = 2(E - V(x))

$\therefore$ $(\frac{dx}{dt})^2$ = $\frac{2}{m}$(E - V(x))(1.2)

This is a non-linear differential equation of first order which can be solved by variables of separation method as we can see below:

$\therefore$ $\frac{dx}{dt} = \pm\sqrt{\frac{2}{m}}(E - V(x))$

$\therefore$ $\frac{dx}{\pm\sqrt{\frac{2}{m}(E - V(x))}}$ = dt

Now, variables are separated and it canbe integrated. therefore ,

$\pm\int \frac{dx}{\sqrt{\frac{2}{m}(E - V(x))}}$ = t - t_0 which is a required solution of non-linear differential equation(1.2).

Note that square root is there, so minus sign will be suitable for constant t_0.

5. RADIOACTIVE DECAY

Carbon 14 is a radioactive isotope of stable carbon 12. If Q(t) denotes the amount of carbon C14 at time t, then Q is known to satisfy the differential equation.

$\frac{dQ}{dt} = -\lambda Q$ where λ is constant.

Now, $\frac{dQ}{dt} = -\lambda Q$

$\therefore$ $\frac{dQ}{Q} = -\lambda dt$

$\therefore$ log Q=$-\lambda t + c$', where c' is an integrating constant.

$\therefore$ Q=$e^{-\lambda t + C'}$.

Thus, Q= $Ce^{-\lambda t}$, where C= $e^{c'}$.

This shows that the Radioactive Decay decreases exponentially at time t.

6. RC- CIRCUIT

In an RC circuit, the applied e.m.f is a constant E. Given that $\frac{dQ}{dt} = i$ where Q is the charge in the capacitor, i is the current in the circuit, R the resistance and C the capacitance the equation for the circuit is $Ri + \frac{Q}{C} = E$. Given initial charge is zero.

Now, given thatthe equation for the circuit is $Ri + \frac{Q}{C} = E$

$\therefore$ $R.\frac{dQ}{dt} + \frac{Q}{C} = E$

$\therefore$ $\frac{dQ}{dt} + \frac{Q}{CR} = \frac{E}{R}$

$\therefore$ DQ + $\frac{Q}{CR} = \frac{E}{R}$, where D =$\frac{d}{dt}$

$\therefore$ (D + $\frac{1}{CR}$) Q = $\frac{E}{R}$

The auxiliary equation is given by (D + $\frac{1}{CR}$) = 0.

D = $-\frac{1}{CR}$

The complimentary function is Q(t) = k $e^{-t/CR}$ where k is an arbitrary constant .

The particular integral is given by

P.I = $\frac{1}{D+\frac{1}{CR}}\frac{E}{R} = e^{-t/CR}\int e^{t/CR}.\frac{E}{R}$ .dt = $e^{-t/CR}.\frac{E}{R}\frac{e^{t/CR}}{\frac{1}{CR}}$ = CE.

Therefore, the charge in the capacitor is given by

Q(t) = k $e^{-t/CR}$ + CE.

But , initial charge is zero i.e. Q(0) = 0.

0 = Q(0) = k + CE

$\therefore$ k = - CE

Hence, the charge in the capacitor is given by

Q(t) = - CE $e^{-t/CR}$ + CE.

i.e. Q(t) = CE (1 - $e^{-t/CR}$) which is required charge in an RC-Circuit.

7. NEWTON'S LAW OF COOLING IN THERMODYNAMICS

Suppose an object has temperature T(t) at time t.Newton's law of cooling states that

T' = - k (T - E) ,where E is the environment temperature and k is the constant

Now, T' = - k (T - E)

$\therefore$ $\frac{dT}{dt}$ = - k (T - E)

$\therefore$ $\frac{dT}{(T - E)}$ = - k dt

By integrating,

log(T - E) = - k t + c where c is an integrating constant.

$\therefore$ (T - E) = e^{-kt+c}

$\therefore$T = E + $t_0 e^{-kt}$ which is a required temperature.

8. APPLICATION TO POPULATION DYNAMICS

The rate of change of unemployed of the country in certain year t directly proportional to the number of unemployed people.

i.e. $\frac{dU}{dt} \propto$ U, where U(t) is the number of unemployed in the country in year t.

If remove proportionality content,it has to multiply some constant therefore ,

$\frac{dU}{dt} = \lambda U$

Where λ is a proportionality constant and U(t) is the number of unemployed in the country.

$\frac{dU}{dt} - \lambda U = 0$

$\therefore$ DU-λ U =0 , where D =$\frac{d}{dt}$

$\therefore$ (D-λ) U =0

So, D=λ

Hence, U(t) = C $e^{\lambda t}$ where C is a constant to be determined.

Initially , U(o) = 1 hence C= 1

$\therefore$U(t) = $e^{\lambda t}$

This shows that unemployment of the country grows exponentially.

The population of a certain organisms at time t is assumed to satisfy the first order linear differential equation

$\frac{dP}{dt} = \lambda P(1 - \frac{P}{E})$, where P(t) is the number of population at time t,λ and E are positive constants.

Given differential equation $\frac{dP}{dt} = \lambda P(1 - \frac{P}{E})$(1.3)

$\therefore \frac{dP}{P(1 - \frac{P}{E})} = \lambda\, dt$

$\int \frac{dP}{P(1 - \frac{P}{E})} = \lambda t + t_0$ where t_0 is an integrating constant

Now, consider $\frac{1}{P(1 - \frac{P}{E})} = \frac{A}{P} + \frac{B}{(1 - \frac{P}{E})}$

$\therefore 1 = A\,(1 - \frac{P}{E}) + B.P$

when P = E, B = $\frac{1}{E}$ and when P = 0, A= 1.

$\therefore \int \left(\frac{1}{P} + \frac{\frac{1}{E}}{(1 - \frac{P}{E})}\right) dP = \lambda t + t_0$

$\therefore \int \left(\frac{1}{P} + \frac{1}{(E - P)}\right) dP = \lambda t + t_0$

$\therefore$ log P - log (E - P) = $\lambda t + t_0$

$\therefore \log\frac{P}{(E - P)} = \lambda t + t_0$

$\therefore \frac{P}{(E - P)} = e^{\lambda t + t0}$

$\therefore P = Ee^{\lambda t + t0} - Pe^{\lambda t + t0}$

$\therefore P + Pe^{\lambda t + t0} = Ee^{\lambda t + t0}$

$\therefore P(1 + e^{\lambda t + t0}) = Ee^{\lambda t + t0}$ as required solution of linear differential equation (1.3).

9. GROWTH OF BACTERIA

A certain species of bacteria grows according to

$\frac{dN}{dt} = \lambda N$ with N(0) = N_0 (1.4)

where N(t) is the amount of bacteria at time t, λ is a positive constant is the growth rate and N_0 is the initial amount when time t = 0.

Now $\frac{dN}{dt} = \lambda N$

This is a first order linear differential equation and it can be solved by variables of separation method, hence $\frac{dN}{N} = \lambda$t.By integrating on both side, we have log N =λt + c',

therefore, N = $ce^{\lambda t}$ where c = $e^{c'}$.

But N_0 = N(0) = c e^0 = c i.e. c = N_0.

Hence, N = $N_0\, e^{\lambda t}$ which is the required solution of the linear differentialequation(1.4).

10. CONCLUSION

From the above motivational examples one can see that a linear differential equation and non-linear differential equation has wide range of applications for solving engineering problems,population dynamics as well as other branch of scientific problems of the physical system.So this is an important and powerful tool for analyzing the relationship among various dynamical parameters in engineering field.

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