Certain Aspects of Univalent Functions with Negative Coefficient Defined by Fractional Differential Operator

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ABSTRACT
This paper introduced a new subclass of univalent analytic functions and derived various properties like coefficient inequality, distortion theorem, radius of starlikeness and convexity, Hadamard product, extreme points, closure theorems for functions belonging to this classwith the help of fractional differential operator.

Keywords
Univalent functions, fractional derivative operator, Hadamard product.

1. INTRODUCTION
Let S_k denote the subclass of functions f(z) of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \qquad a_k \geq 1 \qquad (1.1)$$

which are analytic and univalent in the unit disc

U = $\{z: |z| < 1\}$ ofanalytic and univalent function

f(z) = z + $\sum_{k=2}^{\infty} a_k z^k$ of class S.

Let g(z) = z - $\sum_{k=2}^{\infty} b_k z^k \in S_k$, $b_k \geq 1$ (1.2)

which also analytic and univalent in U

The following definitions which are used for working in the classes of analytic functions

Definition (i): A function f(z)∈ S_k is said to be starlike of order α, 0≤ $\alpha < 1$ if

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \qquad z \in U \qquad (1.3)$$

Definition (ii): A function f(z)∈ S_k is said to be convex of order α, 0≤ $\alpha < 1$ if

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f(z)}\right) > \alpha, \qquad z \in U \qquad (1.4)$$

Definition (iii): If f(z), g(z) ∈ S_k then their Hadamard product is f(z)∗ $g(z)$ = z-$\sum_{k=2}^{\infty} a_k b_k z^k \in S_k$ (1.5)

for f(z) = z - $\sum_{k=2}^{\infty} a_k z^k$, g(z) = z − $\sum_{k=2}^{\infty} b_k z^k$

Definations(iv):(Fractional integral operator)

The fractional integral of order α, is defined for a function f(z) by

$$D_Z^{-\delta} f(z) = \frac{1}{\Gamma(\delta)} \int_0^z \frac{f(\psi)}{(z-\psi)^{1-\delta}} d\psi, \quad \delta > 0 \qquad (1.6)$$

where f(z) is an analytic function in a simply connected region of the z-plane containing the origin, and the multiplicity of $(z-\psi)^{1-\delta}$ is removed by requiring $\log(z-\psi)$ to be real when $z - \psi > 0$.

Definations(v):(Fractional derivative operator)

The fractional derivatives of order α, is defined for a function f(z) by

$$D_Z^{\delta} f(z) = \frac{1}{\Gamma(1-\delta)} \frac{d}{dz} \int_0^z \frac{f(\psi)}{(z-\psi)^{\delta}} d\psi, \quad 0 \leq \delta < 1 \qquad (1.7)$$

where f(z) is constrained, and the multiplicity of $(z-\psi)^{-\delta}$ is removed as in definition (iv).

Definations(vi):(Extended fractional derivative operator)

Under the hypothesis of definition (v) The fractional derivative of order n+ δ is defined, for a function f(z), by $D_Z^{n+\delta} f(z) = \frac{d^n}{dz^n} D_Z^{\delta} f(z)$ where $0 \leq \delta < 1; n \in N_0$

Using the above definition Srivastava and owa [2] introduced the operator

$$\Omega^{\lambda} f(z) = \Gamma(2-\lambda) z^{\lambda} D_Z^{\lambda} f(z)$$

$$\Omega^{\lambda} f(z) = z - \sum_{k=2}^{\infty} \Phi(k,\lambda)\, a_k z^k$$

(1.8)

where

$$\Phi(k,\lambda) = \frac{\Gamma(k+1)\Gamma(2-\lambda)}{\Gamma(k+1-\lambda)}$$

Using the definition ,S.M.Khairnar and Meena More [1]

that a function f(z) is in the class S($\alpha, \beta, \gamma, \mu$) if and only if,

$$\left| \frac{\boldsymbol{z}\frac{\Omega^{\lambda+1} f(z)}{\Omega^{\lambda} \boldsymbol{f(z)}} - \mathbf{1}}{\mathbf{2}\boldsymbol{\gamma}\left(\boldsymbol{z}\frac{\Omega^{\lambda+1} f(z)}{\Omega^{\lambda} \boldsymbol{f(z)}} - \boldsymbol{\alpha}\right) - \boldsymbol{\mu}\left(\boldsymbol{z}\frac{\Omega^{\lambda+1} f(z)}{\Omega^{\lambda} \boldsymbol{f(z)}} - \mathbf{1}\right)} \right| < \beta$$

for$|z| < 1$(1.9)

Where 0< $\beta \leq 1, \frac{1}{2} \leq \gamma \leq 1$, $0 \leq \alpha \leq \frac{1}{2}$, $\frac{1}{2} < \mu \leq 1$.

In this paper all the investigated results are motivated by S.M.Khairnar and S.M.Rajas[1],G.Murugusundaramoorthy And R. Themangani[2], M.Darus [3],

2. COEFFICIENT ESTIMATES
Theorem1.:Letf(z) = z - $\sum_{k=2}^{\infty} a_k z^k \in S_k$.Then f(z)∈ S($\alpha, \beta, \gamma, \mu$) if and only if

$$\sum_{k=2}^{\infty} [1 - k + 2\beta\gamma(k-\alpha) - \beta\mu(k-1)]\, \Phi(k,\lambda) a_k$$

$$\leq 2\beta\gamma(1-\alpha) \qquad (2.1)$$

where0≤ $\alpha < \frac{1}{2}$, 0<$\beta \leq 1, \frac{1}{2} \leq \gamma \leq 1$,

$$\Phi(k,\lambda) = \frac{\Gamma(k+1)\Gamma(2-\lambda)}{\Gamma(k+1-\lambda)}$$

The result is sharp for the function

$$f(z) = z - \frac{2\beta\gamma(1-\alpha)}{[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]\Phi(k\lambda)} z^k$$

Proof: Assume that the inequality (1.9) holds true and

$|z| = 1$.Then obtain

$$\left|z\Omega^{\lambda+1}f(z) - \Omega^{\lambda}\boldsymbol{f(z)}\right|$$

$$-\beta\left|\boldsymbol{2\gamma}\{z\Omega^{\lambda+1}f(z) - \alpha\Omega^{\lambda}\boldsymbol{f(z)}\} - \boldsymbol{\mu}\{z\Omega^{\lambda+1}f(z) - \Omega^{\lambda}\boldsymbol{f(z)}\}\right|$$

$$\leq \left|\sum_{k=2}^{\infty}(1-k)\Phi(k,\lambda)\, a_k z^k\right| - \beta\left|2\gamma\left[(1-\alpha)z + \sum_{k=2}^{\infty}(\alpha-k)\Phi(k,\lambda)\, a_k z^k\right] + \sum_{k=2}^{\infty}(k-1)\mu\Phi(k,\lambda)\, a_k z^k\right|$$

$$\leq \sum_{k=2}^{\infty}[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]\,\Phi(k,\lambda)a_k$$

$$\leq 2\beta\gamma(1-\alpha)$$

by hypothesis. Hence by maximum modulus principle,

f(z)∈ S$(\alpha,\beta,\gamma,\mu)$

Conversely, let f(z)∈ S$(\alpha,\beta,\gamma,\mu)$.

Then

$$\left|\frac{\boldsymbol{z}\frac{\Omega^{\lambda+1}f(z)}{\Omega^{\lambda}\boldsymbol{f(z)}} - \boldsymbol{1}}{\boldsymbol{2\gamma}\left(\boldsymbol{z}\frac{\Omega^{\lambda+1}f(z)}{\Omega^{\lambda}\boldsymbol{f(z)}} - \boldsymbol{\alpha}\right) - \boldsymbol{\mu}\left(\boldsymbol{z}\frac{\Omega^{\lambda+1}f(z)}{\Omega^{\lambda}\boldsymbol{f(z)}} - \boldsymbol{1}\right)}\right| < \beta$$

,z∈ $\boldsymbol{U}$

i.e $\left|\frac{\sum_{n=2}^{\infty}(1-k)\Phi(k,\lambda)a_k z^k}{|2\gamma[(1-\alpha)z+\sum_{k=2}^{\infty}(\alpha-k)\Phi(k,\lambda)a_k z^k]+\sum_{k=2}^{\infty}(k-1)\mu\Phi(k,\lambda)a_k z^k|}\right| < \beta$ (2.2)

Since $|Ref(z)| \leq |f(z)|$,$forallz$ we have

$\left|\boldsymbol{Re}\left\{\frac{\sum_{k=2}^{\infty}(1-k)\Phi(k,\lambda)a_k z^k}{|2\gamma[(1-\alpha)z+\sum_{k=2}^{\infty}(\alpha-k)\Phi(k,\lambda)a_k z^k]+\sum_{k=2}^{\infty}(k-1)\mu\Phi(k,\lambda)a_k z^k|}\right\}\right| < \beta$ (2.3)

Since $\frac{z\Omega^{\lambda+1}f(z)}{\Omega^{\lambda}f(z)}$ is real and upon clearing the denominator of the above expression,we choose the value of z on real axis and allowing z→ 1 through real values.

$$\frac{\sum_{k=2}^{\infty}(1-k)\Phi(k,\lambda)\, a_k z^k}{\left|2\gamma[(1-\alpha)z + \sum_{k=2}^{\infty}(\alpha-k)\Phi(k,\lambda)\, a_k z^k] + \sum_{k=2}^{\infty}(k-1)\mu\Phi(k,\lambda)\, a_k z^k\right|} \leq \boldsymbol{\beta}$$

$$\sum_{k=2}^{\infty}[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]\,\Phi(k,\lambda)a_k$$

$\leq 2\beta\gamma(1-\alpha)$(2.4)

which obviously is required assertion (2.1)

Finally, sharpness follows if we take

$$f(z) = z - \frac{2\beta\gamma(1-\alpha)}{[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]\Phi(k\lambda)} z^k$$

, k = 2,3,4,…(2.5)

Corollary: If f(z)∈ S$(\alpha,\beta,\gamma,\mu)$ then

$$a_k \leq \frac{2\beta\gamma(1-\alpha)}{[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]\Phi(k,\lambda)}$$

, k= 2,3,4,…… (2.6)

3. GROWTH AND DISTORTIONTHEOREM

Theorem2.: if the function f(z)∈ S$(\alpha,\beta,\gamma,\mu)$ then

$$|z| - \frac{2\beta\gamma(1-\alpha)}{[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]\Phi(k,\lambda)}|z|^2 \leq |f(z)| \leq |z| + \frac{2\beta\gamma(1-\alpha)}{[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]\Phi(k,\lambda)}|z|^2$$

The result is sharp for

$$f(z) = z - \frac{2\beta\gamma(1-\alpha)}{[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]\Phi(k,\lambda)} z^2$$

Proof:f(z) = z –$\sum_{k=2}^{\infty} a_k z^k$

$$|f(z)| \leq |z| + \sum_{k=2}^{\infty} a_k |z|^k$$

$$\leq |z| + \frac{2\beta\gamma(1-\alpha)}{[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]\Phi(k,\lambda)}|z|^2 \quad (3.1)$$

Similarly,

$$|f(z)| \geq |z| - \sum_{k=2}^{\infty} a_k |z|^k$$

$$\geq |z| - \frac{2\beta\gamma(1-\alpha)}{[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]\Phi(k,\lambda)}|z|^2 \quad (3.2)$$

Combining (3.1) and (3.2) ,we get

$$|z| - \frac{2\beta\gamma(1-\alpha)}{[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]\Phi(k,\lambda)}|z|^2 \leq |f(z)| \leq |z| + \frac{2\beta\gamma(1-\alpha)}{[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]\Phi(k,\lambda)}|z|^2$$

Theorem:If the function f(z)∈ S$(\alpha,\beta,\gamma,\mu)$ then

$$1 - \frac{4\beta\gamma(1-\alpha)}{[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]\Phi(k,\lambda)}|z| \leq |f'(z)| \leq 1 + \frac{4\beta\gamma(1-\alpha)}{[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]\Phi(k,\lambda)}|z|$$

The result is sharp for

$$f(z) = z - \frac{2\beta\gamma(1-\alpha)}{[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]\Phi(k,\lambda)} z^2$$

4. RADIUS OF STARLIKENESS AND CONVEXITY

Theorem3.: If the function f(z)∈ $S(\alpha,\beta,\gamma,\mu)$ then f(z) is a starlike of order $\alpha, 0 \leq \alpha < 1$ in $|z| < R$ where

R

$$= \begin{matrix} inf \\ k \end{matrix} \left\{ \frac{(1-\alpha)[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]\Phi(k,\lambda)}{2\beta\gamma(1-\alpha)(k-\alpha)} \right\}^{\frac{1}{k-1}}$$

The estimate is sharp for the function

$$f(z) = z - \frac{2\beta\gamma(1-\alpha)}{[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]\Phi(k,\lambda)} z^k$$

Proof: By definition (i)

R$\left(\frac{Zf'(z)}{f(z)}\right)>\alpha$

That is if $\left|\frac{Zf'(z)}{f(z)} - 1\right| \le 1-\alpha$(4.1)

$$\left|\frac{Zf'(z) - f(z)}{f(z)}\right| \le 1-\alpha$$

$$\left|\frac{z - \sum_{k=2}^{\infty} a_k n z^k - z + \sum_{k=2}^{\infty} a_k z^k}{z - \sum_{n=0}^{\infty} a_n z^n}\right| \le 1-\alpha$$

$$\left|\frac{-\sum_{n=2}^{\infty}(k-1)a_k z^k}{z - \sum_{k=2}^{\infty} a_k z^k}\right| \le \frac{\sum_{k=2}^{\infty}(k-1)|a_k||z|^{k-1}}{1 - \sum_{k=2}^{\infty}|a_k||z|^{k-1}} \le 1-\alpha$$

$$\sum_{k=2}^{\infty}(k-1)|a_k||z|^{k-1} \le (1-\alpha)\left(1 - \sum_{k=2}^{\infty}|a_k||z|^{k-1}\right)$$

$$\frac{\sum_{k=2}^{\infty}(k-\alpha)|a_k||z|^{k-1}}{(1-\alpha)} \le 1$$

(4.2)

By (2.6),

$$a_k \le \frac{2\beta\gamma(1-\alpha)}{[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]\Phi(k,\lambda)}$$

(4.3)

Using (4.2) and (4.3),

$$|z|^{k-1} \le \frac{(1-\alpha)[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]\Phi(k,\lambda)}{2\beta\gamma(1-\alpha)(k-\alpha)}$$

Thus

$|z| < R = \begin{matrix} inf \\ k \end{matrix} \left\{\frac{(1-\alpha)[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]\Phi(k,\lambda)}{2\beta\gamma(1-\alpha)(n-\alpha)}\right\}^{\frac{1}{k-1}}$, k= 2,3,------(4.4)

Theorem4.: Let f(z)∈ $S(\alpha,\beta,\gamma,\mu)$ then f(z) is convex of order $\alpha, 0 \le \alpha < 1$ in $|z| < R$ where R

$$= \begin{matrix} inf \\ k \end{matrix} \left\{ \frac{(1-\alpha)[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]\Phi(k,\ \lambda)}{2\beta\gamma(1-\alpha)k(k-\alpha)} \right\}^{\frac{1}{k-1}}$$

,The estimate is sharp for the function

$$f(z) = z - \frac{2\beta\gamma(1-\alpha)}{[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]\Phi(k,\lambda)} z^k$$

Proof: Let f(z)∈ $S(\alpha,\beta,\gamma,\mu)$ is convex of order $\alpha, 0 \le \alpha < 1$ if

$$Re\left[1 + \frac{zf''(z)}{f'(z)}\right] > \alpha$$

That is

if

$$\left|\frac{zf''(z)}{f'(z)}\right| < 1 \qquad (4.5)$$

which simplifies to

$\sum_{k=2}^{\infty} \frac{k(k-\alpha)a_k|z|^{k-1}}{(1-\alpha)} \le 1$(4.6)

By equation (2.6) we have

$$a_k \le \frac{2\beta\gamma(1-\alpha)}{[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]\Phi(k,,\lambda)}$$

(4.7)

Usingequation(4.6) and (4.7) ,

$$|z|^{k-1} \le \frac{(1-\alpha)[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]\Phi(k,\lambda)}{2\beta\gamma(1-\alpha)k(k-\alpha)}$$

(4.8)

Thus$|z| < R$

$$= \begin{matrix} inf \\ k \end{matrix} \left\{ \frac{(1-\alpha)[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]\Phi(k,\ \lambda)}{2\beta\gamma(1-\alpha)k(k-\alpha)} \right\}^{\frac{1}{k-1}}$$

5. EXTREME POINTS

Theorem5.: Let f_1(z) = z,

f_k (z) = z - $\frac{2\beta\gamma(1-\alpha)}{[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]\Phi(k,\lambda)} z^k$, then

f(z)∈ $S(\alpha,\beta,\gamma,\mu)$ iff it can be expressed In the form of f(z)∈ $S(\alpha,\beta,\gamma,\mu)$

Proof: Suppose

f(z) = $\sum_{k=2}^{\infty} \lambda_k f_k$(z)

$$= \sum_{k=2}^{\infty} \lambda_k \left(z - \frac{2\beta\gamma(1-\alpha)}{[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]\Phi(k,\lambda)} z^k\right)$$

= z $-\sum_{k=2}^{\infty} \lambda_k \frac{2\beta\gamma(1-\alpha)}{[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]\Phi(k,\lambda)} z^k$(5.1)

Now f(z)∈ $S(\alpha,\beta,\gamma,\mu)$ since

$$\sum_{k=2}^{\infty} \frac{[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]\Phi(k,\lambda)}{2\beta\gamma(1-\alpha)}. \\ X \frac{2\beta\gamma(1-\alpha)}{[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]\Phi(k,\lambda)} \lambda_k$$

= $\sum_{k=2}^{\infty} \lambda_k$ = 1-$\lambda_1 \le 1$

Conversely, suppose that f(z)∈ $S(\alpha,\beta,\gamma,\mu)$ then by theorem 1.

$$a_k \le \frac{2\beta\gamma(1-\alpha)}{[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]\Phi(k,\lambda)}$$

Setting

$$\lambda_k = \frac{[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]\Phi(k,\lambda)}{2\beta\gamma(1-\alpha)} a_k$$

, k= 2,3,….

And $\lambda_1 = 1 - \sum_{k=2}^{\infty} \lambda_k$ we notice that

f(z) = $\sum_{k=1}^{\infty} \lambda_k f_k$(z).

Hence the result.

6. HADAMARD PRODUCT

Theorem 6.: Let f(z),g(z)∈ $S(\alpha,\beta,\gamma,\mu)$ then

$(f* g)(z) = z - \sum_{k=2}^{\infty} a_k b_k z^k \in S(\alpha,\beta,\gamma,\mu)$

for f(z) = z - $\sum_{k=2}^{\infty} a_k z^k$, g(z) = z $- \sum_{k=2}^{\infty} b_k z^k$

where

$$\varphi \ge \frac{2\beta^2\gamma(1-\alpha)(1-n)}{[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]^2\Phi(k,\lambda)+2\beta^2\gamma(1-\alpha)[k-1-2\gamma(k-\alpha)]}$$

Proof: Let f(z),g(z)∈ $S(\alpha,\beta,\gamma,\mu)$ then

$$\sum_{k=2}^{\infty} \frac{[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]\Phi(k,\lambda)}{2\beta\gamma(1-\alpha)} a_k \le 1 \quad (6.1)$$

$$\sum_{k=2}^{\infty} \frac{[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]\Phi(k,\lambda)}{2\beta\gamma(1-\alpha)} b_k \le 1$$

(6.2)To find smallest number φ such that

$$\sum_{k=2}^{\infty} \frac{[1-k+2\varphi\gamma(k-\alpha)-\varphi\mu(k-1)]\Phi(k,\lambda)}{2\varphi\gamma(1-\alpha)} a_k b_k \le 1 \quad (6.3)$$

By Cauchy Schwarz inequality,

$$\sum_{k=2}^{\infty} \frac{[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]\Phi(k,\lambda)}{2\beta\gamma(1-\alpha)} \sqrt{a_k b_k} \le 1 \quad (6.4)$$

Thus it is enough to show that

$$\sum_{k=2}^{\infty} \frac{[1-k+2\varphi\gamma(k-\alpha)-\varphi\mu(k-1)]\Phi(k,\lambda)}{2\varphi\gamma(1-\alpha)} a_k b_k \le \sum_{k=2}^{\infty} \frac{[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]\Phi(k,\lambda)}{2\beta\gamma(1-\alpha)} \sqrt{a_k b_k}$$

That is

$$\sqrt{a_k b_k} \le \frac{\varphi[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]}{\beta[1-k+2\varphi\gamma(k-\alpha)-\varphi\mu(k-1)]} \quad (6.5)$$

From (6.4)

$$\sqrt{a_k b_k} \le \frac{2\beta\gamma(1-\alpha)}{[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]\Phi(k,\lambda)}$$

(6.6)

Therefore, in view of the (6.5) and (6.6) it is enough to show that

$$\frac{2\beta\gamma(1-\alpha)}{[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]\Phi(k,\lambda)} \le \frac{\varphi[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]}{\beta[1-k+2\varphi\gamma(k-\alpha)-\varphi\mu(k-1)]}$$

Which simplifies to

$$\varphi \ge \frac{2\beta^2\gamma(1-\alpha)(1-k)}{[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]^2\Phi(k,\lambda)+2\beta^2\gamma(1-\alpha)[k-1-2\gamma(k-\alpha)]}$$

7. CLOSURE THEOREMS

Theorem 7.: Let $f_j \in S(\alpha,\beta,\gamma,\mu)$,j=1,2,…………m then

$$g(z) = \sum_{j=1}^{m} c_j f_j(z) \in S(\alpha,\beta,\gamma,\mu)$$

Where $\sum_{j=1}^{m} c_j = 1$ and $f_j(z) = z - \sum_{k=2}^{\infty} a_k z^k$.

Proof:

$$g(z) = \sum_{j=1}^{m} c_j \left(z - \sum_{k=2}^{\infty} a_k z^k\right)$$

$$= z\sum_{j=1}^{m} c_j - \sum_{j=1}^{m}\sum_{k=2}^{\infty} c_j a_{k,j} z^k$$

$$= z - \sum_{k=2}^{\infty}\left(\sum_{j=1}^{m} c_j a_{k,j}\right) z^k$$

(7.1)

$$= z - \sum_{k=2}^{\infty} e_k z^k$$

(7.2)

$$where\ e_k = \sum_{j=1}^{m} c_j a_{k,j}$$

Since $f_j \in S(\alpha,\beta,\gamma,\mu)$ by Theorem 1,

$$\sum_{k=2}^{\infty} \frac{[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]\Phi(k,\lambda)}{2\beta\gamma(1-\alpha)} a_{k,j} \le 1$$

(7.3)

In view of (7.2),g(z) ∈ $S(\alpha,\beta,\gamma,\mu)$

$$\sum_{k=2}^{\infty} \frac{[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]\Phi(k,\lambda)}{2\beta\gamma(1-\alpha)} e_n \le 1$$

Now

$$\sum_{k=2}^{\infty} \frac{[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]\Phi(k,\lambda)}{2\beta\gamma(1-\alpha)} e_n$$

$$= \sum_{k=2}^{\infty} \frac{[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]\Phi(k,\lambda)}{2\beta\gamma(1-\alpha)} \sum_{j=1}^{m} c_j a_{k,j}$$

$$= \sum_{j=1}^{m} c_j \sum_{k=2}^{\infty} \frac{[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]\Phi(k,\lambda)}{2\beta\gamma(1-\alpha)} a_{k,j}$$

$$\leq \sum_{j=1}^{m} c_j$$

using (7.3)

=1.

Therefore g(z) ∈ $S(\alpha,\beta,\gamma,\mu)$

Theorem 8.: Let f(z),g(z) ∈ $S(\alpha,\beta,\gamma,\mu)$ then

$$h(z) = z - \sum_{k=2}^{\infty}(a_k^2 + b_k^2)z^k$$

is in $S(\alpha,\varphi,\gamma,\mu)$

where

$$\varphi \geq \frac{4\beta^2\gamma(1-\alpha)(1-k)}{[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]^2\Phi(k,\lambda)+4\beta^2\gamma(1-\alpha)[k-1-2\gamma(k-\alpha)]}$$

Proof: Let f(z),g(z) ∈ $S(\alpha,\beta,\gamma,\mu)$ and so

$$\sum_{k=2}^{\infty}\left[\frac{[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]\Phi(k,\lambda)}{2\beta\gamma(1-\alpha)}\right]^2 a_k^2$$

$$\leq \left[\frac{[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]\Phi(k,\lambda)}{2\beta\gamma(1-\alpha)}a_k\right]^2 \leq 1$$

(8.1)

And

$$\sum_{k=2}^{\infty}\left[\frac{[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]\Phi(k,\lambda)}{2\beta\gamma(1-\alpha)}\right]^2 b_k^2$$

$$\leq \left[\frac{[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]\Phi(k,\lambda)}{2\beta\gamma(1-\alpha)}b_k\right]^2 \leq 1$$

(8.2)

Adding (8.1) and (8.2)

$$\sum_{k=2}^{\infty}\frac{1}{2}\left[\frac{[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]\Phi(k,\lambda)}{2\beta\gamma(1-\alpha)}\right]^2 .\left(a_k^2+b_k^2\right) \leq 1 \quad (8.3)$$

I must show that h(z) ∈ $S(\alpha,\varphi,\gamma,\mu)$, that is

$$\sum_{k=2}^{\infty}\frac{1}{2}\left[\frac{[1-k+2\varphi\gamma(k-\alpha)-\varphi\mu(k-1)]\Phi(k,\lambda)}{2\varphi\gamma(1-\alpha)}\right]^2 .\left(a_k^2+b_k^2\right) \quad (8.4)$$

In view of (8.3) and (8.4) it is enough to show that

$$\frac{[1-k+2\varphi\gamma(k-\alpha)-\varphi\mu(k-1)]\Phi(k,\lambda)}{2\varphi\gamma(1-\alpha)} \leq \frac{1}{2}\left[\frac{[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]\Phi(k,\lambda)}{2\beta\gamma(1-\alpha)}\right]^2$$

which simplifies to

$$\varphi \geq \frac{4\beta^2\gamma(1-\alpha)(1-k)}{[1-k+2\beta\gamma(k-\alpha)-\beta\mu(k-1)]^2\Phi(k,\lambda)+4\beta^2\gamma(1-\alpha)[k-1-2\gamma(k-\alpha)]}$$

8. CONCLUSION

This paper derived the basic properties likecoefficient inequality, distortion theorem, radius of starlikeness and convexity, Hadamard product, extreme points, closure theorems of univalent and analytic functions with negative coefficient belonging to the class$S(\alpha,\beta,\gamma,\mu)$with the help of fractional differential operator which are new result,can be used in engineering field.

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