On Solving Abel Integral Equations involving Fox H-Function

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ABSTRACT
The present paper deals with the solution of Abel integral equation involving Fox- H function. The method is based on approximations of fractional integrals and Caputo derivatives due to Jahanshahi et al. The approximation formula of Abel integral equation using numerical trapezoidal rule is also obtained. The paper is also illustrating the effectiveness of proposed approach in form of many particular examples. The results are mostly derived in a closed form in terms of the H-function, suitable for numerical computation. On account of general nature of H-function a number of results involving special functions can be obtained merely by specializing the parameters.

Keywords
Abel Integral Equations, Fox H-Function, Riemann–Liouville fractional derivatives, Caputo fractional derivatives.

1. INTRODUCTION
The solutions of Abel integral equations of first kind through fractional calculus have studied by many authors due to its vast scope in various physical and engineering problems , like semi-conductors, heat conduction and chemical reactions[1-2], nuclear physics, optics and astrophysics [3-4], velocity laws of stellar winds [5,6,7],a method based on Chebyshev polynomials is given in [8].Many numerical methods for solving abel integral equation have been developed over past few years like product integration methods [9,10],collocation methods[11], wavelets methods [12,13,14], Adomian decompositions methods[15] , fractional multistep methods [16,17],Plato [18] gives fractional multi steps for weakly singular integral equations .Li and Zhao solved abel integral equation with Mikusinski's operator of fractional order [19], Li and Tao solved the fractional integral equation transforming it into abel integral equation of second kind[20]. Recently Jahanshahi et al. [21] developed method for numerically solving Abel integral equation of first kind. The present paper investigates the approximate method to develop a solution for class of Abel integral equation of first kind involving Fox H function. The paper is arranged as follows. In section 2 we recall elementary definitions of fractional integration and fractional derivatives and their important properties. Section 3 gives some known approximation for fractional derivatives and integrals, the main results given in section 4 where we develop exact and approximate solution of Abel integral equation involving Fox H-function with particular examples.

2. DEFINITIONS
In this section, we give some basic definitions, relations and properties of fractional operators. We refer authors to [22-26].

Definition 2.1 Let $\alpha > 0$ with $n-1 < \alpha \leq n,\ n \in \mathbb{N}and< x < b$. The left and right Riemann–Liouville fractional integrals of order α of a given function $f$are defined by

$$ {}_aJ_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt \qquad (2.1)$$

and

$$ {}_xJ_b^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt \qquad (2.2)$$

respectively, where Γ is Euler's gamma function.

Definition 2.2 The left and right Riemann–Liouville fractional derivatives of order $\alpha > 0, n-1 < \alpha \leq n,\ n \in \mathbb{N}$ are defined by [22,26]

$$ {}_aD_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x (x-t)^{n-\alpha-1} f(t)dt \qquad (2.3)$$

and

$$ {}_xD_b^\alpha f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_x^b (t-x)^{n-\alpha-1} f(t)dt \qquad (2.4)$$

respectively.

Definition 2.3 The left and right Caputo fractional derivatives of order $\alpha > 0, n-1 < \alpha \leq n,\ n \in \mathbb{N}$, are defined as follows[26]

$$ {}_a^cD_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^{(n)}(t)dt \qquad (2.5)$$

and

$$ {}_x^cD_b^\alpha f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b (t-x)^{n-\alpha-1} f^{(n)}(t)dt \qquad (2.6)$$

respectively.

Definition 2.4 The Grunwald–Letnikov fractional derivatives are defined for $\alpha > 0$ as follows[26]

$$ D^\alpha f(x) = \lim_{h\to 0} h^{-\alpha} \sum_{r=0}^{\infty} (-1)^r \binom{\alpha}{r} f(x-rh) \qquad (2.7)$$

and

$$ D^{-\alpha} f(x) = \lim_{h\to 0} h^{\alpha} \sum_{r=0}^{\infty} \begin{bmatrix} \alpha \\ r \end{bmatrix} f(x-rh) \qquad (2.8)$$

where

$$ \begin{bmatrix} \alpha \\ r \end{bmatrix} = \frac{\alpha(\alpha+1)(\alpha+2)\ldots(\alpha+r-1)}{r!}.$$

Definition 2.5 The following relations between Caputo and Riemann–Liouville fractional derivatives are given as follows [26]

$$ {}_a^cD_x^\alpha f(x) = {}_aD_x^\alpha f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k-\alpha+1)} (x-a)^{k-\alpha} \quad (2.9)$$

and

$$ {}_x^cD_b^\alpha f(x) = {}_xD_b^\alpha f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{\Gamma(k-\alpha+1)} \times (b-x)^{k-\alpha} \quad (2.10)$$

Therefore it is easy to see, if $f \epsilon\ C^n[a,b]$ and $f^{(k)}(a) = 0$, $k = 0,1,\dots,n-1$, then

$$ {}_a^cD_x^\alpha f(x) = {}_aD_x^\alpha f $$

If $f^{(k)}(b) = 0,\ k = 0,1,\dots,n-1,$ then

$$ {}_x^cD_b^\alpha f = {}_xD_b^\alpha f. $$

Some basic properties of fractional integrals and derivatives are given as follows:

(i) All fractional operators are linear, that is, if L is an arbitrary fractional operator, then

$$ L(tf+sg) = tL(f) + sL(g) \quad (2.11)$$

(ii) For all functions $f,g \in C^n[a,b]$or $f,g \in L^p(a,b)$ and $t,s \in \mathbb{R}$; if $\alpha,\beta > 0$, then

$$ J^\beta = J^{\alpha+\beta}, \quad D^\alpha D^\beta = D^{\alpha+\beta} \quad (2.12)$$

(iii) If $f \in L^\infty(a,b)$ or $f \in C^n[a,b]$ and $\alpha > 0$, then

$$ {}_a^cD_x^\alpha f\ {}_aJ_x^\alpha f = f, {}_x^cD_b^\alpha f\ {}_xJ_b^\alpha f = f \quad (2.13)$$

(iv) If $f \in C^n[a,b]$ and $\alpha > 0$, then

$$ {}_aJ_x^\alpha\, {}_a^cD_x^\alpha f(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (x-a)^k \quad (2.14)$$

and

$$ {}_xJ_b^\alpha\, {}_x^cD_b^\alpha f(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b)}{k!} (b-x)^k \quad (2.15)$$

It is clearly seen that the Caputo fractional derivatives are the inverse operators for the Riemann–Liouville fractional integrals.

Definition 2.6 The H-function introduced and defined by Fox [23] via a Mellin-Barnes type integral as:

$$ H_{P,Q}^{M,N}(z) \equiv H_{P,Q}^{M,N}\left[z \middle| \begin{smallmatrix}(a_i,\alpha_i)_{1,P}\\(b_j,\beta_j)_{1,Q}\end{smallmatrix}\right] = \frac{1}{2\pi i}\int_{\mathcal{L}} \mathcal{H}_{P,Q}^{M,N}(s)\, z^{-s} ds \quad (2.16)$$

with

$$ \mathcal{H}_{P,Q}^{M,N}(s) \equiv \mathcal{H}_{P,Q}^{M,N}\left[\begin{smallmatrix}(a_i,\alpha_i)_{1,P}\\(b_j,\beta_j)_{1,Q}\end{smallmatrix}\middle| s\right] = \frac{\prod_{j=1}^{M}\Gamma(b_j+\beta_j s)\prod_{i=1}^{N}\Gamma(1-a_i-\alpha_i s)}{\prod_{i=N+1}^{P}\Gamma(a_i+\alpha_i s)\prod_{j=M+1}^{Q}\Gamma(1-b_j-\beta_j s)} \quad (2.17)$$

Sufficient convergence conditions on parameters, Asymptotic expansions and analytic continuations of the H-function have been discussed by Mathai and Saxena [23].

3. SOME KNOWN APPROXIMATION RESULTS

Diethelm [27-29] uses the product trapezoidal rule with respect to the weight function $(t_k-.)^{\alpha-1}$ to approximate Riemann–Liouville fractional integrals. More precisely, the approximation

$$ \int_{t_0}^{t_k}(t_k-u)^{\alpha-1}f(u)\,du \simeq \int_{t_0}^{t_k}(t_k-u)^{\alpha-1}\tilde{f}_k(u)du \quad (3.1)$$

where$\tilde{f}_k$ is the piecewise linear interpolator for f whose nodes are chosen at $t_j = jh, j = 0,1\dots,n$ and $h = \frac{b-a}{n}$, is considered. Odibat [24, 25] uses a modified trapezoidal rule to approximate the fractional integral ${}_0J_x^\alpha f(x)$and the Caputo fractional derivative ${}_a^cD_x^\alpha f(x)$of order $\alpha > 0$. The other approximations for fractional operators can be referred to [30-32].

Theorem 3.1 (See [24, 25]).Let $b > 0, \alpha > 0$, and suppose that the interval [0, b] is subdivided into ksubintervals $[x_j,x_{j+1}]$j=0,…,k-1, of equal distances $h = \frac{b}{k}$ by using the node s$x_j = jh$, $j = 0,\dots,k$. Then the modified trapezoidal rule

$$ T(f,h,\alpha) = \frac{h^a}{\Gamma(\alpha+2)}\Big(((k-1)^{\alpha+1} - (k-\alpha-1)k^\alpha)f(0) + f(b)\Big) + \frac{h^a}{\Gamma(\alpha+2)} \times \sum_{j=1}^{k-1}\Big((k-j+1)^{\alpha+1} - 2(k-j)^{\alpha+1} + (k-j-1)^{\alpha+1}\Big)f(x_j) \quad (3.2)$$

is an approximation to the fractional integral ${}_0J_x^\alpha f\,|_{x=b}$:

$$ {}_0J_x^\alpha f\,|_{x=b} = T(f,h,\alpha) - E_T(f,h,\alpha). $$

Furthermore, if$f \in C^2[0,b]$, then

$$ |E_T(f,h,\alpha)| \le c'_\alpha \|f''\|_\infty b^\alpha h^2 = O(h^2), $$

wherec'_α is a constant depending only on α.

The following theorem gives an algorithm to approximate the Caputo fractional derivative of an arbitrary order $\alpha > 0$.

Theorem 3.2 (See [24, 25]).Let $b > 0, \alpha > 0$ with $n-1 < \alpha \le n, n \in \mathbb{N}$, and suppose that the interval [0,b] is subdivided into k subintervals $[x_j,x_{j+1}]$, $j = 0,\dots,k-1$,of equal distances $h = \frac{b}{k}$ by using the nodes $x_j = jh, j = 0,1,\dots,k$.

Then the modified trapezoidal rule

$$ C(f,h,\alpha) = \frac{h^{n-\alpha}}{\Gamma(n+2-\alpha)} \times \Bigg[\Bigg(((k-1)^{n-\alpha+1} - (k-n+\alpha-1)k^{n-\alpha})f^{(n)}(0) + f^{(n)}(b) + \sum_{j=1}^{k-1}\Big((k-j+1)^{n-\alpha+1} - 2(k-j)^{n-\alpha+1} + (k-j-1)^{n-\alpha+1}\Big)f^{(n)}(x_j)\Bigg)\Bigg] \quad (3.3)$$

is an approximation to the Caputo fractional derivative ${}_0^cD_x^\alpha f(b)$.

Furthermore, if $f \in C^{n+2}[0,b]$, then

$$ |E_C(f,h,\alpha)| \le c\grave{}_{n-\alpha}\|f^{(n+2)}\|_\infty b^{n-\alpha}h^2 = O(h^2), $$

where$c\grave{}_{n-\alpha}$ is a constant depending only on α .

Theorem 3.3 The following Abel integral equation of first kind due to Jahanshahi [21] is given as

$$f(x) = \int_0^x \frac{g(t)}{(x-t)^\alpha} dt , 0 < \alpha < 1, 0 \le x \le b \quad (3.4)$$

where $f \in C^1[a,b]$ is a given function satisfying $f(0) = 0$ and g is the unknown function we are looking for.

The solution to problem (3.4) is given as

$$g(x) = \frac{\sin(\alpha\pi)}{\pi} \int_0^x \frac{f'(t)}{(x-t)^{1-\alpha}} dt \quad (3.5)$$

Due to Mathai and Saxena [23] the derivative formula of H-function for m ≥1 , $\gamma > 0$ is given as follows:

$$\frac{d}{dz}\left[z^{-\gamma\frac{b_1}{B_1}} H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_i,A_i)_{1,P}\\(b_j,B_j)_{1,Q}\end{matrix}\right.\right]\right] = \left(\frac{-1}{B_1}\right) z^{-\gamma\frac{b_1}{B_1}} H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_i,A_i)_{1,P}\\(1+b_1,B_1)(b_j,B_j)_{2,q}\end{matrix}\right.\right] \quad (3.6)$$

Lemma 1. Let $\alpha > 0$ with $n-1 < \alpha \le n,\ n \in \mathbb{N}and0 < x < b$. Riemann–Liouville fractional integrals of power function can be easily obtained as

$$_0J_x^\alpha[(t)^{\beta-1}](x) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} x^{\alpha+\beta-1} \quad (3.7)$$

4. MAIN RESULTS

Theorem 4.1. With all conditions mentioned above on parameters and for $m \ge 1, \gamma > 0$, the solution of following class of abel integral equation

$$x^{-\gamma\frac{b_1}{B_1}} H_{p,q}^{m,n}\left[x\left|\begin{matrix}(a_i,A_i)_{1,P}\\(b_j,B_j)_{1,Q}\end{matrix}\right.\right] = \int_0^x \frac{g(t)}{(x-t)^\alpha} dt \quad (4.1)$$

is given as

$$g(x) = \left(\frac{-\Gamma(\alpha)}{B_1}\right)\frac{\sin\alpha\pi}{2\pi^2 i} x^{\alpha-\gamma\frac{b_1}{B_1}} \times H_{p+1,q+1}^{m,n+1}\left[x\left|\begin{matrix}\left(\frac{\gamma b_1}{B_1},1\right)(a_i,A_i)_{1,P}\\(-\alpha+\frac{\gamma b_1}{B_1},1)(b_1+1,B_1)(b_j,B_j)_{2,q}\end{matrix}\right.\right] \quad (4.2)$$

Proof: Let$f(x) = x^{-\gamma\frac{b_1}{B_1}} H_{p,q}^{m,n}\left[x\left|\begin{matrix}(a_i,A_i)_{1,P}\\(b_j,B_j)_{1,Q}\end{matrix}\right.\right]$ be taken in equation (3.4), and with the help of equation (3.5) and (3.6) we get

$$g(x) = \frac{\sin(\alpha\pi)}{\pi}\int_0^x \frac{1}{(x-t)^{1-\alpha}}\left(\frac{-1}{B_1}\right) t^{-\gamma\frac{b_1}{B_1}} \times H_{p,q}^{m,n}\left[t\left|\begin{matrix}(a_i,A_i)_{1,p}\\(1+b_1,B_1)(b_j,B_j)_{2,q}\end{matrix}\right.\right] dt$$

After changing the order of integration and using definition of H-function (2.16), (2.17), we get by the virtue of (3.7)

$$= \left(\frac{-\Gamma(\alpha)}{B_1}\right)\frac{\sin\alpha\pi}{2\pi i\ \pi}\int_c \mathcal{H}_{p,q}^{m,n}\left[s\left|\begin{matrix}(a_i,A_i)_{1,p}\\(1+b_1,B_1)(b_j,B_j)_{2,q}\end{matrix}\right.\right] \times {}_0J_x^\alpha\left(t^{-\gamma\frac{b_1}{B_1}-s}\right) ds$$

$$= \left(\frac{-\Gamma(\alpha)}{B_1}\right)\frac{\sin\alpha\pi}{2\pi^2 i}\int_c \mathcal{H}_{p,q}^{m,n}[s]\frac{\Gamma\left(1-\frac{\gamma b_1}{B_1}-s\right)}{\Gamma\left(1+\alpha-\frac{\gamma b_1}{B_1}-s\right)} x^{\alpha-\frac{\gamma b_1}{B_1}-s} ds$$

$$= \left(\frac{-\Gamma(\alpha)}{B_1}\right)\frac{\sin\alpha\pi}{2\pi^2 i} x^{\alpha-\frac{\gamma b_1}{B_1}}\int_c \mathcal{H}_{p,q}^{m,n} \mid \left[\begin{matrix}(a_i,A_i)_{1,p}\\(1+b_1,B_1)(b_j,B_j)_{2,q}\end{matrix}\right| s\Bigg] \times \frac{\Gamma\left(1-\frac{\gamma b_1}{B_1}-s\right)}{\Gamma\left(1+\alpha-\frac{\gamma b_1}{B_1}-s\right)} x^{-s} ds$$

finally by the virtue of H-function definition (2.17) we get (4.2).

Corollary 4.1. On putting $m = 1, n = 1, p = 1, q = 2, \gamma = 1,$ $a_1 = 1-\eta, A_1 = 1, 1+b_1 = 0, B_1 = 1,\ b_2 = 1-\beta, B_2 = \alpha$in main result (4.1), we get solution of the following problem

$$x^\gamma\Gamma(\eta)E_{\alpha,\beta}^\eta(-x) = \int_0^x \frac{g(t)}{(x-t)^\alpha} dt \quad (4.3)$$

is given as

$$g(x) = \frac{-\Gamma(\alpha)\sin(\alpha\pi)}{2\pi^2 i} x^{\alpha+\gamma} H_{2,3}^{1,2}\left[x\left|\begin{matrix}(-\gamma,1)(1-\eta,1)\\(-\alpha-\gamma,1)(0,1)(1-\beta,\alpha)\end{matrix}\right.\right]$$

Corollary 4.2. On putting $m = 1, n = 0, p = 0, q = 1, \gamma = 0$, in (4.1) we get the solution of

$$B^{-1}x^{\frac{b}{B}}\exp\left(-x^{\frac{1}{B}}\right) = \int_0^x \frac{g(t)}{(x-t)^\alpha} dt \quad (4.4)$$

as follows

$$g(x) = \frac{-\Gamma(\alpha)\sin(\alpha\pi)}{2\pi^2 iB} x^\alpha H_{1,2}^{1,1}\left[x\left|\begin{matrix}(0,\ 1)\\(-\alpha,1)\ (b+1,B)\end{matrix}\right.\right] \quad (4.5)$$

Further specializing parameter in above result as putting B=1, b=0,$\alpha = 1/2$ we obtain result due to Jahanshahi [21].

Theorem 4.2. Let $0 < x <$ b and suppose that the interval [0, x] is subdivided into k subintervals $[t_j, t_{j+1}],\ j = 0, \ldots, k-1$, of equal distances $h = \frac{x}{k}$ by using the nodes $t_j = jh, j = 0,1,\ldots,k$.An approximate solution $\tilde{g}$ to the solution g of the abel integral equation (4.1) is given as ($f'(0) = 0$)

$$\tilde{g}(x) = \frac{h^\alpha}{\Gamma(1-\alpha)\Gamma(2+\alpha)} \times \left(\frac{-1}{B_1}\right) x^{-\gamma\frac{b_1}{B_1}} H_{p,q}^{m,n}\left[x\left|\begin{matrix}(a_i,A_i)_{1,P}\\(1+b_1,B_1)(b_j,B_j)_{2,q}\end{matrix}\right.\right] + \sum_{j=1}^{k-1}((k-j+1)^{\alpha+1} - 2(k-j)^{\alpha+1} + (k-j-1)^{\alpha+1})\left(\frac{-1}{B_1}\right) t_j^{-\gamma\frac{b_1}{B_1}} H_{p,q}^{m,n}\left[t_j\left|\begin{matrix}(a_i,A_i)_{1,P}\\(1+b_1,B_1)(b_j,B_j)_{2,q}\end{matrix}\right.\right] \quad (4.6)$$

5. CONCLUSION

In the present paper the analytical and approximated solutions of the abel integral equation involving Fox –H function have obtained .The analytical solutions have given in closed forms in terms of H function and due to general nature of the H function, these solutions have vast importance in various physical and engineering problems. We also derived some interesting particular cases involving Mittag Leffler and exponential functions in corollary 4.1 and 4.2. In theorem 4.2, an approximated solution of abel integral equation is given, using numerical trapezoidal rule. The method is based on approximating of fractional derivatives and integrals.

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