On Generlized Mittag-Leffler Function and Fractional Operator

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ABSTRACT
The paper is devoted to study properties of a generalized function of Mittag-Leffler type, including various fractional integral operators like Riemann – Liouville operator, Hilfer operator etc. Certain unified integral formulas including this function are established. Image of this function under Saigo operator is also obtained.

Keywords
Fractional integral operators; fractional differential operators; generalized Mittag-Leffler function; Fox-Wright ${}_p\psi_q$-function.

1. INTRODUCTION
Importance of Mittag-Leffler function has been realized during the last two decades due to its involvement in the problems of applied sciences such as physics, chemistry, biology and engineering (Hilfer [2]). Mittag-Leffler function occurs naturally in the solution of fractional order differential or integral equations (Saxena et al. [8, 9]).

Prabhakar [6] introduced a generalized Mittag-Leffler function function $E_{\alpha,\beta}^{\gamma}(z)$ in the form

$$E_{\alpha,\beta}^{\gamma}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)n!}z^n \qquad (1.1)$$

where$z,\alpha,\beta,\gamma\in C, Re(\alpha)>0,\ Re(\beta)>0,\ Re(\gamma)>0$ and $(\gamma)_n$ is the Pochhammer symbol (Rainville[6]), $(\gamma)_0=1, (\gamma)_n=\gamma(\gamma+1)(\gamma+2)\ldots(\gamma+n-1)$.

Shukla and Prajapati [13] introduced the function $E_{\alpha,\beta}^{\gamma,q}(z)$ which is defined for $z,\alpha,\beta,\gamma\in C, min\{Re(\alpha)>0, Re(\beta)>0, Re(\gamma)\}>0$ and $q\in(0,1)\cup N$ as:

$$E_{\alpha,\beta}^{\gamma,q}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\alpha n+\beta)n!}z^n \qquad (1.2)$$

In this paper, a further generalization of the Mittag-Leffler function is investigated with various fractional operators and defined as

$$E_{\alpha,\beta}^{\gamma,\delta,q}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\alpha n+\beta)(\delta)_{qn}}z^n \qquad (1.3)$$

where$z,\alpha,\beta,\gamma,\delta\in C$, $min\{Re(\alpha)>0, Re(\beta)>0, Re(\gamma), Re(\delta)\}>0,\quad q\in(0,1)\cup N$ and $(\gamma)_{qn}=\frac{\Gamma(\gamma+qn)}{\Gamma(\gamma)}$ denotes the generalized Pochhammer symbol (Rainville [7]) which in particular reduces to

$$q^{qn}\prod_{r=1}^{q}\left(\frac{\gamma+r-1}{q}\right)_n \text{ if q} \in \text{N}.$$

$E_{\alpha,\beta}^{\gamma,\delta,q}(z)$ contains the aforementioned Mittag-Leffler function. Note that $E_{\alpha,\beta}^{\gamma,1,1}(z)=E_{\alpha,\beta}^{\gamma}(z)$,

$E_{\alpha,\beta}^{1,1,1}(z)=E_{\alpha,\beta}(z)andE_{\alpha,1}^{1,1,1}(z)=E_{\alpha}(z)$.

The following well known facts are needed throughout the present investigation

- The Riemann – Liouville operator:

The right sided Riemann – Liouville fractional integral operator I_{a+}^{v} and the right sided Riemann – Liouville fractional derivative operator D_{a+}^{v} are defined by (Samko et al. [12])for $Re(v)>0$

$$(I_{a+}^{v}f)(x)=\frac{1}{\Gamma(v)}\int_a^x(x-t)^{v-1}f(t)\,dt \qquad (1.4)$$

and

$$(D_{a+}^{v}f)(x)=\left(\frac{d}{dx}\right)^n(I_{a+}^{n-v}f)(x)(n=[Re(v)]+1) \qquad (1.5)$$

where $[x]$ denotes the greatest integer in the real number x.

Hilfer [2] generalized the Riemann – Liouville fractional derivative operator D_{a+}^{v} in (1.7) by introducing a right sided fractional derivative operator of order and type with respect to x as follows:

$$(D_{a+}^{v,\mu}f)(x)=I_{a+}^{\mu(1-v)}\frac{d}{dx}(I_{a+}^{(1-\mu)(1-v)}f)(x) \qquad (1.6)$$

Miller and Rose [3] defined the fractional differential operator of order μ defined as

$$D^{\mu}f(t)=D^n\{I^{k-\mu}f(t)\} \qquad (1.7)$$

where $Re(\mu)>0$ and if k is the smallest integer with the property that $k\geq Re(\mu)$.

- Saigo fractional operator (Saigo [11]):

Saigo, introduced the following hypergeometric operator of fractional integration for $Re(\alpha)>0$ and real numbers β and η

$$\mathcal{I}_{0,x}^{\alpha,\beta,\eta}f(x)=\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}\times{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{t}{x}\right)f(t)dt \qquad (1.8)$$

- Generalized hypergeometric function (Rainville [7]):

The generalized hypergeometric function is defined as

$$ {}_pF_q(\alpha_1,\dots,\alpha_p;\beta_1,\dots,\beta_q;z)=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(\alpha_i)_n}{\prod_{j=1}^{q}(\beta_j)_n}\frac{z^n}{n!} \qquad (1.9)$$

where the infinite series converges for all $z\in C$ when $p\le q$.

- Wright generalized hypergeometric function is defined as (Srivastava and Manocha[15]):

$$ {}_p\psi_q\left[\begin{matrix}(a_1,A_1),\dots,(a_p,A_p);\\(b_1,B_1),\dots,(b_q,B_q);\end{matrix}z\right]=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+A_in)}{\prod_{j=1}^{q}\Gamma(b_j+B_jn)} \qquad (1.10)$$

2. FRACTIONAL CALCULUS OF $E_{\alpha,\beta}^{\gamma,\delta,q}(z)$

In this section we studied certain differential and integral properties of the function $E_{\alpha,\beta}^{\gamma,\delta,q}(z)$ along with fractional integral and differential operators. Certain new unified integrals involving generalized Mittag-Leffler function are established in last two theorems. Results are given in the form of following four theorems.

Theorem 2.1.

Let $z,\alpha,\beta,\gamma,\delta\in C,\min\{\mathrm{Re}(\alpha)>0,Re(\beta)>0,Re(\gamma),\mathrm{Re}(\delta)\}>0$, $q\in N$ then for $k\in N$

$$\left(\frac{d}{dz}\right)^k E_{\alpha,\beta}^{\gamma,\delta,q}(z)=\frac{(\gamma)_{qk}}{(\delta)_{qk}}\sum_{n=0}^{\infty}\frac{(\gamma+qk)_{qn}(n+1)_k}{\Gamma(\alpha n+\beta+qk)(\delta+qk)_{qn}}z^n \qquad (2.1)$$

and

$$\left(\frac{d}{dz}\right)^k[z^{\beta-1}E_{\alpha,\beta}^{\gamma,\delta,q}(\omega z^\alpha)]=z^{\beta-k-1}E_{\alpha,\beta-k}^{\gamma,\delta,q}(\omega z^\alpha) \qquad (2.2)$$

Proof: From (1.3)

$$\left(\frac{d}{dz}\right)^k\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\alpha n+\beta)(\delta)_{qn}}z^n =\sum_{n=m}^{\infty}\frac{(\gamma)_{qn}\ n(n-1)\dots(n-k+1)}{\Gamma(\alpha n+\beta)(\delta)_{qn}}z^{n-k}$$

$$=\frac{(\gamma)_{qk}}{(\delta)_{qk}}\sum_{n=0}^{\infty}\frac{(\gamma+qk)_{qn}(n+1)_k}{\Gamma(\alpha n+\beta+qk)(\delta+qk)_{qn}}z^n$$

which is the proof of (2.1).

Again using (1.3) and term-by-term differentiation under the sign of summation, we get

$$\left(\frac{d}{dz}\right)^k[z^{\beta-1}E_{\alpha,\beta}^{\gamma,\delta,q}(\omega z^\alpha)] =\left(\frac{d}{dz}\right)^k\sum_{n=0}^{\infty}\left[\frac{(\gamma)_{qn}}{\Gamma(\alpha n+\beta)(\delta)_{qn}}\omega^n z^{\alpha n+\beta-1}\right]$$

$$=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\alpha n+\beta-k)(\delta)_{qn}}\omega^n z^{\alpha n+\beta-k-1}$$

$$=z^{\beta-k-1}E_{\alpha,\beta-k}^{\gamma,\delta,q}(\omega z^\alpha).$$

which proves (2.2).

Lemma 2.1. Following result holds true for the fractional derivative operator $D_{a+}^{\upsilon,\mu}f$ defined by (1.6)

$$\left(D_{0+}^{\upsilon,\mu}[t^{\lambda-1}]\right)(x)=\frac{\Gamma\lambda}{\Gamma(\lambda-\upsilon)}x^{\lambda-\upsilon-1} \qquad (2.3)$$

where $x>0;0<\upsilon<1;0\le\mu\le1;Re(\lambda)>0$.

Proof: We observe from the definition (1.4) that

$$\left(I_{0+}^{(1-\mu)(1-\upsilon)}[t^{\lambda-1}]\right)(x)=\frac{\Gamma\lambda}{\Gamma[(1-\mu)(1-\upsilon)+\lambda]}x^{(1-\mu)(1-\upsilon)+\lambda-1}$$

and

$$\frac{d}{dx}\left(I_{0+}^{(1-\mu)(1-\upsilon)}[t^{\lambda-1}]\right)(x)=\frac{[(1-\mu)(1-\upsilon)+\lambda-1]\Gamma\lambda}{\Gamma[(1-\mu)(1-\upsilon)+\lambda]}x^{(1-\mu)(1-\upsilon)+\lambda-2}$$

which in light of the definition (1.8), yield

$$\left(D_{0+}^{\upsilon,\mu}[t^{\lambda-1}]\right)(x)=\frac{\Gamma\lambda}{\Gamma[(1-\mu)(1-\upsilon)+\lambda-1]}\times\left(I_{a+}^{\mu(1-\upsilon)}[t^{(1-\mu)(1-\upsilon)+\lambda-2}]\right)(x)$$

$$=\frac{\Gamma\lambda}{\Gamma(\lambda-\upsilon)}x^{\lambda-\upsilon-1}$$

which proves lemma 2.1.

Theorem 2.2. Let $x>a;\ (a\in[0,\infty))$, and $\lambda,\alpha,\beta,\gamma,\delta\in C$, $\min\{\mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(\lambda),\mathrm{Re}(\delta)\}>0$, $q\in N$ then

$$\left(I_{a+}^{\upsilon}\left[(t-a)^{\beta-1}E_{\alpha,\beta}^{\gamma,\delta,q}(\omega(t-a)^\alpha)\right]\right)(x)=(x-a)^{\beta+\upsilon-1}E_{\alpha,\beta+\upsilon}^{\gamma,\delta,q}(\omega(x-a)^\alpha) \qquad (2.4)$$

$$\left(D_{a+}^{\upsilon}\left[(t-a)^{\beta-1}E_{\alpha,\beta}^{\gamma,\delta,q}(\omega(t-a)^\alpha)\right]\right)(x)=(x-a)^{\beta-\upsilon-1}E_{\alpha,\beta-\upsilon}^{\gamma,\delta,q}(\omega(x-a)^\alpha) \qquad (2.5)$$

$$\left(D_{0+}^{\upsilon,\mu}\left[(t-a)^{\beta-1}E_{\alpha,\beta}^{\gamma,\delta,q}(\omega(t-a)^\alpha)\right]\right)(x)=(x-a)^{\beta-\upsilon-1}E_{\alpha,\beta-\upsilon}^{\gamma,\delta,q}(\omega(x-a)^\alpha) \qquad (2.6)$$

Proof: The formula (1.3) and (1.4), the term by term fractional integration and the application of the relation [11, Eq.(2.44)]

$$\left(I_{a+}^{\upsilon}[(t-a)^{\beta-1}]\right)(x)=\frac{\Gamma\beta}{\Gamma(\lambda+\beta)}(x-a)^{\upsilon+\beta-1} \qquad (2.7)$$

where $\upsilon,\beta\in C,\mathrm{Re}(\upsilon),\mathrm{Re}(\beta)>0$, yield for $x>a$

$$\left(I_{a+}^{\upsilon}\left[(t-a)^{\beta-1}E_{\alpha,\beta}^{\gamma,\delta,q}(\omega(t-a)^{\alpha})\right]\right)(x) = \sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\alpha n+\beta)(\delta)_{qn}}\frac{\omega^n}{n!} \times \left(I_{a+}^{\upsilon}\left[(t-a)^{\alpha n+\beta-1}\right]\right)(x)$$

$$= (x-a)^{\beta+\upsilon-1} \times \sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\alpha n+\beta+\upsilon-1)(\delta)_{qn}}\frac{[\omega(x-a)^{\alpha}]^n}{n!}$$

$$= (x-a)^{\beta+\upsilon-1}E_{\alpha,\beta+\upsilon}^{\gamma,\delta,q}(\omega(x-a)^{\alpha}).$$

which proves (2.4).

Again by use of (1.5) and (2.4), we have

$$\left(D_{a+}^{\upsilon}\left[(t-a)^{\beta-1}E_{\alpha,\beta}^{\gamma,\delta,q}(\omega(t-a)^{\alpha})\right]\right)(x) = \left(\frac{d}{dx}\right)^k\left(I_{a+}^{k-\upsilon}\left[(t-a)^{\beta-1} \times E_{\alpha,\beta}^{\gamma,\delta,q}(\omega(t-a)^{\alpha})\right]\right)(x)$$

$$= \left(\frac{d}{dx}\right)^k\left[(x-a)^{\beta+k-\upsilon-1}E_{\alpha,\beta+k-\upsilon}^{\gamma,\delta,q}(\omega(x-a)^{\alpha})\right]$$

Now applying (2.2), we get

$$= (x-a)^{\beta-\upsilon-1}E_{\alpha,\beta-\upsilon}^{\gamma,\delta,q}(\omega(x-a)^{\alpha}).$$

which proves (2.5).

Similarly, proof of (2.6) follows by applying Hilfer operator (1.6) to (1.3) and using lemma 2.1.

Theorem 2.3. If $z,\alpha,\beta,\gamma,\delta,\upsilon \in C, \min\{Re(\alpha), Re(\beta), Re(\gamma), Re(\delta), Re(\upsilon), Re(s)\}$, $q \in (0,1)\cup N$ then

$$\frac{1}{\Gamma(\upsilon)}\int_t^x (x-u)^{\upsilon-1}(u-t)^{\beta-1}E_{\alpha,\beta}^{\gamma,\delta,q}(z(u-t)^{\alpha})du = (x-t)^{\beta+\upsilon-1}E_{\alpha,\beta+\upsilon}^{\gamma,\delta,q}(z(x-t)^{\alpha}) \tag{2.8}$$

$$\frac{\Gamma(\gamma)}{\Gamma(\delta)}\int_0^{\infty} e^{-su}(u-t)^{\beta-1}E_{\alpha,\beta}^{\gamma,\delta,q}(z(u-t)^{\alpha})du = s^{-\beta}e^{-st}\,{}_2\psi_1\left[\begin{matrix}(\gamma,q),(1,1); \\ (\delta,q);\end{matrix}\; s^{-\alpha}z\right] \tag{2.9}$$

Proof: Putting $\frac{u-t}{x-t}=\theta$ and using (1.3), then L.H.S. of equation (2.8) reduces to

$$= \frac{1}{\Gamma(\upsilon)}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{\Gamma(\alpha n+\beta)(\delta)_{qn}}z^n(x-t)^{\alpha n+\beta+\upsilon-1} \times \int_0^1 \theta^{\alpha n+\beta-1}(1-\theta)^{\upsilon-1}d\theta$$

$$= \frac{1}{\Gamma(\upsilon)}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}z^n(x-t)^{\alpha n+\beta+\upsilon-1}}{\Gamma(\alpha n+\beta)(\delta)_{qn}}\frac{\Gamma(\alpha n+\beta)\Gamma(\upsilon)}{\Gamma(\alpha n+\beta+\upsilon)}$$

finally using (1.3), we obtained R.H.S. of equation (2.8).

Again putting $u-t=\theta$ and using (1.3), then L.H.S. of equation (2.9) reduces to

$$= \frac{\Gamma(\gamma)}{\Gamma(\delta)}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}z^n}{\Gamma(\alpha n+\beta)(\delta)_{qn}}e^{-st}\int_0^{\infty} e^{-s\theta}\theta^{\alpha n+\beta-1}du$$

$$= e^{-st}\frac{\Gamma(\gamma)}{\Gamma(\delta)}\sum_{n=0}^{\infty}\frac{\Gamma(\gamma+qn)\Gamma(\delta)z^n}{\Gamma(\alpha n+\beta)\Gamma(qn+\delta)\Gamma(\gamma)} \times \frac{\Gamma(\alpha n+\beta)}{s^{\alpha n+\beta}}$$

finally using (1.10), we obtained R.H.S. of equation (2.9).

Remark 1. In particular, setting $t = 0$ and $x = 1$ in (2.8), we get

$$\frac{1}{\Gamma(\upsilon)}\int_0^1 u^{\beta-1}(1-u)^{\upsilon-1}E_{\alpha,\beta}^{\gamma,\delta,q}(zt^{\alpha})du = E_{\alpha,\beta+\upsilon}^{\gamma,\delta,q}(z). \tag{2.10}$$

In the next theorem the results are given in terms of Wright generalized hypergeometric function ${}_p\psi_q$ and are obtained with the help of following interesting integral [5]

$$\int_0^{\infty} x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda}dx = 2\lambda a^{-\lambda}\left(\frac{a}{2}\right)^{\mu}\frac{\Gamma(2\mu)\Gamma(\lambda-\mu)}{\Gamma(1+\lambda+\mu)} \tag{2.11}$$

where $0 < Re(\mu) < Re(\lambda)$.

Theorem 2.4. The following integral formulas holds true for $z,\alpha,\beta,\gamma,\delta \in C, 0 < Re(\mu) < Re(\lambda), q \in (0,1)\cup N, x > 0$

$$\frac{\Gamma(\gamma)}{\Gamma(\delta)}\int_0^{\infty} x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda} \times E_{\alpha,\beta}^{\gamma,\delta,q}(xz)dx = \frac{\lambda a^{\mu-\lambda}}{2^{\mu-1}}\,{}_4\psi_3\left[\begin{matrix}(\gamma,q),(2\mu,2),(\lambda-\mu,1),(1,1); \\ (\delta,q),(\beta,\alpha),(1+\lambda+\mu,1);\end{matrix}\;\frac{az}{2}\right] \tag{2.12}$$

and

$$\frac{\Gamma(\gamma)}{\Gamma(\delta)}\int_0^{\infty} x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda} \times E_{\alpha,\beta}^{\gamma,\delta,q}\left(\frac{z}{x+a+\sqrt{x^2+2ax}}\right)dx = \frac{a^{\mu-\lambda}\Gamma(2\mu)}{2^{\mu-1}} \times {}_4\psi_4\left[\begin{matrix}(\gamma,q),(1+\lambda,1),(\lambda-\mu,1),(1,1); \\ (\delta,q),(\beta,\alpha),(\lambda,1),(1+\lambda+\mu,1);\end{matrix}\;\frac{z}{a}\right] \tag{2.13}$$

Proof: Using equation (1.3) and applying (2.11) in L.H.S. of (2.12), under the conditions $0 < Re(\mu) < Re(\lambda)$, we obtained

$$\frac{\Gamma(\gamma)}{\Gamma(\delta)}\int_0^{\infty} x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda} \times E_{\alpha,\beta}^{\gamma,\delta,q}(xz)dx$$
$$= \frac{\lambda a^{\mu-\lambda}}{2^{\mu-1}}\frac{\Gamma(\gamma)}{\Gamma(\delta)} \times \sum_{n=0}^{\infty}\frac{(\gamma)_{qn}}{(\delta)_{qn}\Gamma(\alpha n+\beta)}\frac{\Gamma(2\mu+2n)\Gamma(\lambda-\mu+n)}{\Gamma(1+\lambda+\mu+n)}\left(\frac{az}{2}\right)^n$$

finally using (1.10), we obtained R.H.S. of equation (2.12).

Again using equation (1.3) and applying (2.11) in L.H.S. of (2.13), under the conditions $0 < Re(\mu) < Re(\lambda) \leq Re(\lambda + n), n \in N_0$, we obtained

$$\frac{\Gamma(\gamma)}{\Gamma(\delta)}\int_0^\infty x^{\mu-1}\left(x + a + \sqrt{x^2 + 2ax}\right)^{-\lambda}$$
$$\times E_{\alpha,\beta}^{\gamma,\delta,q}\left(\frac{z}{x + a + \sqrt{x^2 + 2ax}}\right) dx$$
$$= \frac{a^{\mu-\lambda}\Gamma(2\mu)}{2^{\mu-1}}\frac{\Gamma(\gamma)}{\Gamma(\delta)}$$
$$\times \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{(\delta)_{qn}\Gamma(\alpha n + \beta)} \frac{\Gamma(1 + \lambda + n)\Gamma(\lambda - \mu + n)}{\Gamma(\lambda + n)\Gamma(1 + \lambda + \mu + n)}\left(\frac{z}{a}\right)^n$$

Finally using (1.10), we obtained R.H.S. of equation (2.12).

3. SAIGO OPERATOR AND GENERALIZED MITTAG-LEFFLER FUNCTION

We conclude with the following theorem, which gives the image of generalized Mittag-Leffler function $E_{\upsilon,\rho}^{\sigma,\delta,q}(\omega z^\upsilon)$ under Saigo's operator (1.8).

Theorem 3.1 Let $\mathcal{J}_{0,t}^{\alpha,\beta,\gamma}[.]$ be the Saigo's left-sided fractional integral operator (1.8), then there holds the formula

$$\mathcal{J}_{0,t}^{\alpha,\beta,\gamma}\left[z^{\rho-1}E_{\upsilon,\rho}^{\sigma,\delta,q}(\omega z^\upsilon)\right]$$
$$= z^{\rho-\beta-1}\frac{\Gamma(\delta)}{\Gamma(\sigma)}$$
$$\times\ {}_3\psi_3\left[\begin{matrix}(\sigma,q),(\rho+\gamma-\beta,\upsilon),(1,1); \\ (\delta,q),(\rho-\beta,\upsilon),(\rho+\gamma+\alpha,\upsilon);\end{matrix}\ \omega z^\upsilon\right] \quad (3.1)$$

The conditions for the validity of (3.1) are

(i) $\alpha, \beta, \gamma\ (Re(\alpha) > 0)$ and ω are any complex numbers.

(ii) ρ and υ are arbitrary such that $Re(\rho + \gamma - \beta + k\upsilon > 0$.

Proof : Using the definition (1.8) in the left hand side of (3.1), writing the function in the forms given by (1.3) and (1.9), interchanging the order of integration and summation and evaluating the integral as beta integral, finally applying Gauss summation formula and by virtue of (1.10), we easily arrive at the result (3.1) under the valid conditions.

Corollary 3.1. Let $\beta = -\alpha$ in (3.1) and using definition (1.3), we have

$$I_{0+}^{\alpha}\left[z^{\rho-1}E_{\upsilon,\rho}^{\sigma,\delta,q}(\omega z^\upsilon)\right] = z^{\alpha+\rho-1}E_{\upsilon,\rho+\alpha}^{\sigma,\delta,q}(\omega z^\upsilon). \quad (3.2)$$

Remark 2. For $\delta = 1$ and $q = 1$ we arrive at the result [10, p.145, Eq.14] given by Saxena and Saigo.

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