H_∞ norm Computation for Fractional Order Control System

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ABSTRACT

In this paper, we propose two tools for fractional systems H_∞ norm computation; one is based on an extension of widelyused tools designed for integer systems that is by manipulating properties of singular values. The other is the computing H_∞ norm graphically as peak value of bode magnitude plot.

General Terms

H_∞ norm, Fractional systems

Keywords

Fractional order system (FOS),

1. INTRODUCTION

In recent years there has been an increasing attention to fractional-order systems. These systems are of interest for both modelling and control purposes. In general, Fractional differentiation is now a well known tool for controller synthesis. Several presentations and applications of the fractional PID controller [1], [2], [3] [4] and of CRONE control [5] demonstrate their efficiency. Fractional differentiation also permits a simple representation of some high order complex integer systems [6]. Consequently, basic properties of fractional systems have been investigated these last ten years and criteria and theorems are now available in the literature concerning stability [7], observability, and controllability [8] of fractional systems. Thus, several studies have been made on FOS stability and the most well known stability criteria is Matignon'scriteria which enables to test systems stability through thelocation of the state matrix Eigen values in the complex plane.

The H_∞ norm reflects how much a dynamic system amplifies or attenuates its input at the frequency at which the amplification is maximal.This control technique may be applied to both SISO (single-input, single-output) and MIMO (multiple-input, multiple-output) plants, and its results achieve a remarkable robustness (Lublin *et al.*, 1996; Doyle *et al.*, 1989)[11].By computing H_∞ norm we continue to explore the use of frequency domain techniques for design of feedback systems. H_∞ control is strongly linked to the weighted sensitivity functions. Performance specification is then of great importance in H_∞ control approach as means of loop shaping[9].

In this paper, we propose two tools for fractional systems H_∞ norm computation; one is based on an extension of widelyused tools designed for integer systemsthat is by manipulating properties of singular values[13]. The other is the computing H_∞ norm graphically as peak value of bode magnitude plot.

2. NOTATIONS AND DEFINATIONS

2.1 Fractional Calculus

Riemann-Liouville [10] fractional differentiation is used and the fractional integral of a function *f(t)* is defined by

$$D^{-\alpha}f(t)=D^{m}\,J^{m-\alpha}f(t)=\frac{d^m}{dt^m}\left[\frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{(\alpha-m-1)}f(\tau)d\tau\right]$$

where

$$\Gamma(x)=\int_0^\infty t^{x-1}\,exp^{-t}\,dt$$

is the Gamma function which is an important special function in fractional calculus. It is an extension of the factorial function, that is, if n is a positive integer.

$$\Gamma(n)\ =(n-1)!$$

2.2 LTI COMMENSURATE FRACTIONAL ORDER SYSTEMS

In case of commensurate-order system [9], the transfer function is given by

$$G(s)=\frac{\Sigma_{k=0}^{m} b_k (s^\alpha)^k}{\Sigma_{k=0}^{n} a_k (s^\alpha)^k}$$

which can be considered as pseudorational function, $H(\lambda)$, of the variable $\lambda = s^\alpha$,as in equation

$$\lambda = s^\alpha = \frac{\Sigma_{k=0}^{m} b_k \lambda^k}{\Sigma_{k=0}^{n} a_k \lambda^k}$$

The pseudo state space representation is of the form

$$D^\alpha = Ax(t)+Bu(t) \quad (1)$$
$$y(t)=Cx(t)+Du(t)$$

where $x(t)\epsilon R^n$is the pseudo state vector, $u(t)\ \epsilon\ R^m$is the input vector, $y(t)\epsilon R^p$is theoutput vector, α is the fractional order of the system and A, B, C and D are constantmatrices. $D^\alpha$is the fractional differentiation operator of order α and the transfer functionrepresentation for fractional order is given as in equation

$$G(s) = C(s^{\alpha} I - A)^{-1} B + D$$

2.3 Stability

Stability analysis for fractional-order systems is difficult by simply examining the characteristic equation, either by finding its dominant roots or by using algebraic methods. LTI IOS stability can be checked via the location of the state matrixAEigenvalues in the complex plane. This result was extended to LTI commensurate FOS of order 0 < ν <1 byD. Matignon.

Theorem 1 (Matignon, 1996 [7]):System, with minimal triplet (A,B,C) and 0 < ν <1, is BIBO stable if

$$|arg(eig(A))| > \alpha \frac{\pi}{2}$$

This result remains valid when 1 < ν <2.Stability domain is thus defined as follow

$$D_s = \left\{ z\epsilon C \left| arg\left(eig(A)\right)\right| > \alpha \frac{\pi}{2} \right\}.$$

3. H∞ NORM FOR FRACTIONAL ORDER SYSTEMS (FOS)

As for LTI IOS, the H∞ norm of stable FOS from its transfer function G(s) as follows:

Defination1:H∞norm of stable FOS system (1) is:

$$\|G(s)\|_{\infty} \triangleq \max_{\omega\epsilon R} \bar{\sigma}\left(G(j\omega)\right),$$

where$\bar{\sigma}\left(G(j\omega)\right)$ is the largest singular value $\left(\mathrm{G(j\omega)}\right)$at frequency ω:

$$\bar{\sigma}\left(G(j\omega)\right) = \max_{i=\{1\ldots\min(m,p)\}} \sigma_i\left(G(j\omega)\right)$$

$$= \max_{i=\{1\ldots\min(m,p)\}} \sqrt{\lambda_i\left(G(j\omega)\right)^{*}\left(G(j\omega)\right)} \quad (2)$$

Steady state response of FOS (1) to sinusoidal input u$(j\omega)$is y$(j\omega)$=G$(j\omega)$u$(j\omega)$. At frequencyω, the gain $\frac{\|y(t)\|_2}{\|u(t)\|_2}$, depending on vector u $(j\omega)$ is:

$$\bar{\sigma}\left(G(j\omega)\right) = \max_{u(j\omega)\neq 0} \frac{\|y(j\omega)\|_2}{\|u(j\omega)\|_2}$$

Worst case frequency gain is thus given by H∞ norm of FOS:

$$\|G(s)\|_{\infty} = \max_{\omega\epsilon R} \max_{u(j\omega)\neq 0} \frac{\|y(j\omega)\|_2}{\|u(j\omega)\|_2}$$

In time domain, above equation can be written as:

$$\|G(s)\|_{\infty} = \max_{u(t)\neq 0} \frac{\|y(t)\|_2}{\|u(t)\|_2} = \max_{\|u(t)\|_{2=1}} \|y(t)\|_2$$

Therefore, H∞ norm can be interpreted in time domainas the largest energy among output signals resulting from all inputs of unit energy. Consequently, H∞ norm physical interpretation, in frequency and time domains, is the same for FOS as for IOS.

4. COMPUTATION OF FRACTIONAL SYSTEM H∞ NORM BASED ON SINGULAR VALUES

Definition 1 and relation (2) imply that H∞norm of FOS is less than γ if:

$$\forall\omega\epsilon R,\ \max_{i=\{1\ldots\min(m,p)\}} \sqrt{\lambda_i\left(G(j\omega)\right)^{*}\left(G(j\omega)\right)} < \gamma \quad (3)$$

whereγdenotes a real positive number satisfying

$$\gamma > \sigma_{max}\,(D).$$

H∞norm of the FOS described in (1) is bounded by γ and equation (3) can be written as

$$\forall\omega\epsilon R,\ \lambda_i\ \left(\left(G(j\omega)\right)^{*}\left(G(j\omega)\right)\right) < \gamma^2$$

Due to Eigenvalue properties, above relation can be rewritten as:

$$\forall\omega\epsilon R,\ \lambda_i\left(\left(\gamma^2 I - G(j\omega)\right)^{*}\left(G(j\omega)\right)\right) > 0$$

which is equivalent to the Linear Matrix Inequality (LMI):

$$\forall\omega\epsilon R,\ \lambda_i\left(\left(\gamma^2 I - \left(G(j\omega)\right)^{*}\left(G(j\omega)\right)\right)\right) > 0$$

Above equation is satisfied if and only if

$\forall\omega\epsilon R$,$\left(\gamma^2 I - \left(G(j\omega)\right)^{*}\left(G(j\omega)\right)\right)$isnon singular, that is if and only if $\left(\gamma^2 I - \left(G(-s)\right)^{T}\left(G(s)\right)\right)$ has no zero on imaginary axis.

Hence, the H∞ norm is bounded by γif and only if system whose transfer matrix is $G_\gamma(s) = \left(\gamma^2 I - \left(G(-s)\right)^{T}\left(G(s)\right)\right)^{-1}$ is asymptotically stable.

5. COMPUTATION OF FRACTIONAL SYSTEM H∞ NORM BASED ON PEAK MAGNITUDE

A control engineering interpretation of the infinity norm of a scalar transfer functionis the distance in the complex plane from the origin to the farthest point on the Nyquistplot. As in equation, H∞ norm can also be evaluate as the peak value on the Bode magnitude plot a transfer function

$$\|G(s)\|_{\infty} = \sup(|G(j\omega)| : \omega\epsilon R)$$

5.1 Procedure

H∞ norm of fractional systemcan be computed graphically based on following steps,

Step1: Check that transfer function given is *proper* (order of denominator is greater thannumerator).

Step2: Prove that transfer function given is stable.

Step3: If it satisfies step1 and step2, i.e., it belongs to *RH∞*.

Step4: Locate the peak magnitude from Bode magnitude plot.

5.2 Examples

In this section, H_∞ norm of two fractional systemsare calculated by using the above mentioned procedure.

5.2.1 Example 1

Podlubny Transfer Function:

$$F_1(s) = \frac{1}{0.8s^{2.2} + 0.5s^{0.9} + 1}$$

Step1:$F_1(s)$ is proper.
Step2 : $F_1(s)$ is stable as poles in higher reimann sheet are always stable, thus poles inprincipal reimann sheet is to verify whether they fulfil the following condition forstability,

$$\frac{\pi}{m} < |arg(\omega)| < \frac{\pi}{2m}$$

Step2.1 Mapping, $\omega = s^{1/10}$

$$F_1(s) = \frac{1}{0.8\omega^{22} + 0.5\omega^{9} + 1}$$

Step2.2 solve

$$0.8\omega^{22} + 0.5\omega^{9} + 1 = 0$$

The poles that are on the principal Riemann sheet areω=1.0045±0.1684jas shown in Fig (1)

Step2.3: Take absolute value of the poles that are on the principal Riemann sheetwhich is 1.661.

Step2.4: Check the condition for stability

$$\frac{\pi}{10} < 1.661 < \frac{\pi}{20}$$

Hence, $F_1(s)$ is stable.

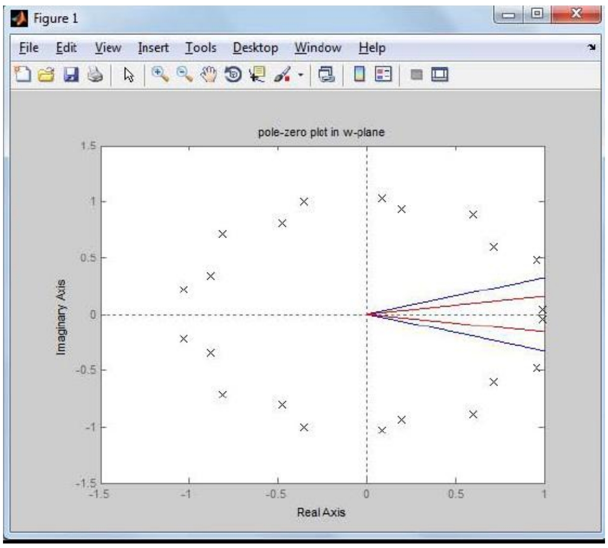

Fig 1:ω plane for $F_1(s)$

Step3: Hence, $F_1(s) \epsilon RH_\infty$.

Step4: The peak magnitude is 13.2875db as shown in figure (2) is the H_∞ for $F_1(s)$.

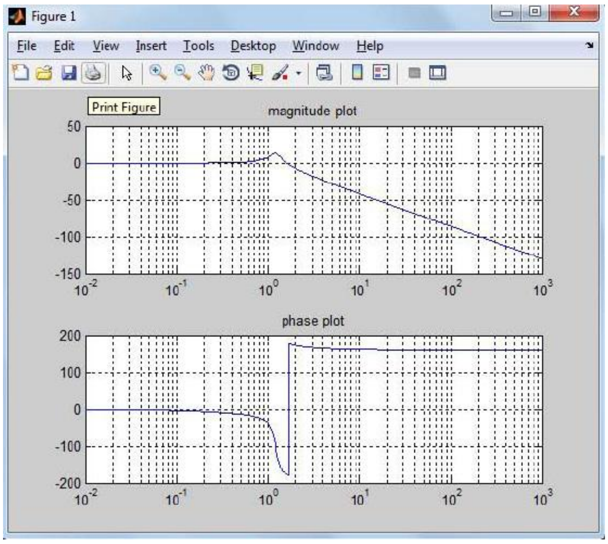

Fig 2: Bode plot for $F_1(s)$

5.2.2 Example 2

$$F_2(s) = \frac{1}{s^{2.3} + 3.2s^{1.4} + 2.4s^{0.9} + 1}$$

Step1:$F_2(s)$ is proper.
Step2 : $F_2(s)$ is stable as poles in higher reimann sheet are always stable, thus poles inprincipal reimann sheet is to verify whether they fulfil the following condition forstability,

$$\frac{\pi}{m} < |arg(\omega)| < \frac{\pi}{2m}$$

Step2.1 Mapping, $\omega = s^{1/10}$

$$F_2(s) = \frac{1}{\omega^{23} + 3.2\omega^{14} + 2.4\omega^{9} + 1}$$

Step2.2 solve

$$\omega^{23} + 3.2\omega^{14} + 2.4\omega^{9} + 1 = 0$$

The poles that are on the principal Riemann sheet areω=0.0149±0.1906j as shown in Fig (3)

Step2.3: Take absolute value of the poles that are on the principal Riemann sheetwhich is 0.1912.

Step2.4: Check the condition for stability

$$\frac{\pi}{10} < 0.1912 < \frac{\pi}{20}$$

Hence, $F_2(s)$ is stable.

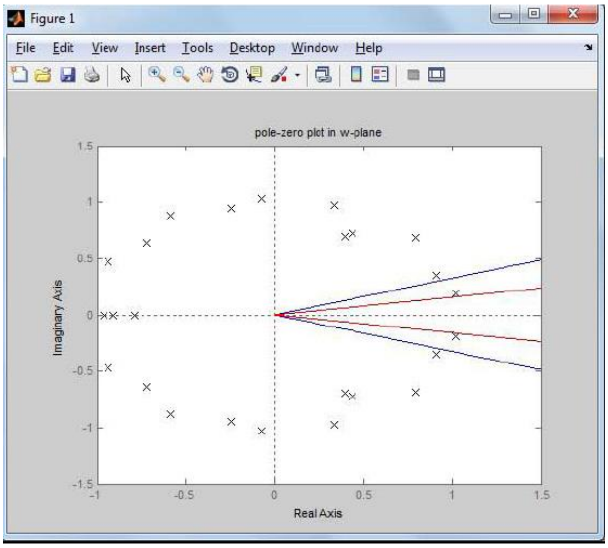

Fig 3:ω plane for $F_2(s)$

Step3: Hence, $F_2(s) \epsilon RH_\infty$.

Step4: The peak magnitude is 94.4323dbas shown in figure (4) is the H_∞ for $F_2(s)$.

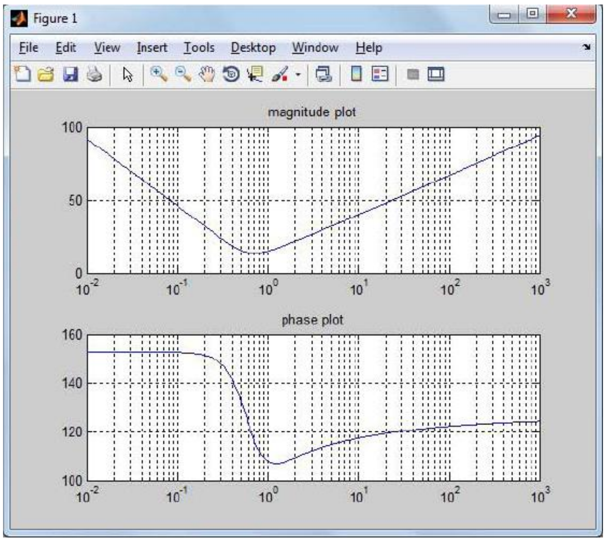

Fig 4: Bode plot for $F_2(s)$

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7. CONCLUSION

Fractional PID regulator and CRONE [8] robust regulatorsare now well known in the field of fractional differentiation application in control theory. H_∞ norm plays a vital role in designing an efficient controller and to handle systems with uncertainties and disturbances and with high performance. In order to develop control methods for more complex fractional systems than the linear one, this paper proposes two tools for the computation of a fractional system H_∞ norm. The first one is based singular value properties of an integer order system and the other is based on peak magnitude plot of bode plot. Computing the H_∞ norm with these methods includes stability analysis of a fractional system and can be easily implemented. Our goal is now to design a H_∞ controller for fractional order system.

8. REFERENCES

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