# Stability Analysis of Fractional-order Systems

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## ABSTRACT
Fractional-order (FO) systems are a special subset of linear time-invariant (LTI) systems. The transfer functions (TFs) of these systems are rational functions with polynomials of rational powers of the Laplace variable 's'. FO systems are of interest for both controller design and modelling purpose. It has been shown that FOPID controller gives better response as compared to integer-order(IO) controllers. FO systems provide the accurate models for many real systems. The stability analysis of FO systems, which is quite different from that of integer-order(IO) systems analysis, is the main focus of this paper. Stability is defined using Riemann surface because of their multi- valued nature of the FO transfer functions (FOTFs). In this paper, various approaches viz., time domain analysis, frequency domain analysis, state space representation are discussed. Both the types of FO systems, with commensurate and incommensurate TFs, are discussed.

## Keywords
Fractional-order systems, fractional calculus, stability analysis.

## 1. INTRODUCTION
The mathematical modelling of FO systems and processes, based on the description of their properties in terms of fractional derivatives (FDs), leads to differential equations of involving FDs that must be analyzed. These are generally termed as Fractional Differential Equations (FDEs). The advantages of fractional calculus have been described and pointed out in the last few decades by many authors in [1], [2], [3], [8], [9], [24]. The latest and very exhaustive literature survey about the FC and FO systems is given in [17]. It has been shown that the FO models of real systems (especially distributed parameter type and memory type) are more adequate than the usually used IO models.

Fractional derivatives (FDs) provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. This is the main advantage models, which possess limited memory. The advantages of FDs become apparent in applications including modelling of damping behaviour of visco-elastic materials, cell diffusion processes [8], transmission of signals through strong magnetic fields, modelling mechanical and electrical properties of real materials, as well as in the description of rheological properties of rocks, and in many other fields [25].

In feedback control, by introducing proportional, integral and derivative control actions of the form $s^{\alpha}$, $1/s^{\alpha}$, $\alpha \in R^{+}$, we can achieve more satisfactory compromises between positive and negative effects, and combining the actions we could develop more powerful and flexible design methods to satisfy the controlled system specifications. Studies have shown that an FO controller can provide better performance than integer order (IO) controller and leads to more robust control in many practical applications.

The rest of the paper is organised as follows : Section 2 and 3 give special functions and definitions of fractional calculus theory. Section 4 describes the stability analysis of fractional-order systems, Section 5 explains the representations of fractional-order systems and in Section 6 analytical results of two examples with the conclusion in Section 7.

## 2. SPECIAL FUNCTIONS OF FRACTIONAL CALCULUS (FC)
Some special functions need to be used in FC.

### 2.1 Gamma Function
One of the most basic functions of FC is Euler's gamma function $\Gamma(z)$, which generalizes the factorial function z! and allows z to take also non-integer and even complex values. The gamma function ($\Gamma(z)$) is given by the following expression,

$$\Gamma(z) = \int_0^{\infty} e^{-u} u^{z-1} du \qquad (1)$$

Note that when $z \in \mathbb{Z}^{+}$ we have $\Gamma(z+1) = z!$

### 2.2 Mittag-Leffler Function
The exponential function $e^{z}$ plays a very important role in the theory of integer order differential equations. Its 1 parameter generalization function for a complex number z is given by,

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^{k}}{\Gamma(\alpha k + 1)}, \qquad (2)$$

The 2 parameter function of the ML function, which is also important in FC is defined as,

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^{k}}{\Gamma(\alpha k + \beta)}, \quad (\alpha > 0, \beta > 0)$$

(3)

This basic definition is very useful in deriving the response of an FO system to any forcing function, for example, step response, ramp response.

## 3. DEFINITIONS FOR FRACTIONAL-DIFFERINTEGRALS

The three equivalent definitions most frequently used for the general fractional derivatives (FD) are the Grunwald-Letnikov (GL) definition, the Riemann- Liouville and the Caputo definition [10]. In all the definitions below, the function $f(t)$ is assumed to be sufficiently smooth and locally integrable.

1) The Grunwald-Letnikov definition using Podlubny's limited memory principle [4] is given by

$$ {}_a D_t^{\alpha} f(t) = \lim_{h \to 0} h^{-\alpha} \sum_{j-0}^{[(t-a)/h]} (-1)^j \, {}^{\alpha}C_j \, f(t - jh), \quad (4)$$

where [.] means the integer part.

2) The Riemann-Liouville definition obtained using the Riemann-Liouville integral is given as,

$$ {}_a D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f^n(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \quad (5)$$

for $(n-1 < \alpha < n)$ and $\Gamma(.)$ is the Gamma function.

3) The Caputo definition can be written as,

$$ {}_a D_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha-n)} \int_a^t \frac{f^n(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \quad (6)$$

for $(n-1 < r < n)$, where $f^n(\tau)$ is the $n^{th}$-order derivative of the function $f(t)$. Since we deal with causal systems in control theory, the lower limit is fixed at $a = 0$. We see that the Caputo definition is more restrictive than the RL. Nevertheless, it is preferred by engineers and physicists because the FDEs with Caputo derivatives have the same initial conditions as that for the IODEs. Note that the FDs calculated using these 3 definitions coincide for an initially relaxed function (i.e. $f(t=0)=0$).

## 4. STABILITY OF FRACTIONAL-ORDER (FO) SYSTEMS

The stability analysis is important in control theory. Recently, there has been some advances in control theory of fractional differential systems for stability. In the FO systems the delay differential equation order is non-integer which makes it difficult to evaluate the stability by simply examining its characteristic equation or by finding its dominant roots or by using other algebraic methods. The stability of FO systems using polynomial criteria (e.g., Routh's or Jury's type) is not possible due to the fractional powers. A generalization of the Routh-Hurwitz criterion used for stability analysis for fractional-order systems is presented in [12]. However, this metho d is very complicated. Thus there remain only geometrical methods of complex analysis based on the argument principle (e.g; Nyquist type) which can be used for the stability check in the BIBO sense (bounded-input bounded-output). These are the techniques that inform about the number of singularities of the function within a rectifiable curve by observing the evolution of the function's argument through this curve. Root locus is another geometric method that can be used for analysis for FO systems. Also, for linear fractional differential systems of finite dimensions in state-space form, stability can be investigated. The stability of a linear fractional differential equation either by transforming the $s$-plane to the $F$-plane $(F = s^{\alpha})$ or to the $w$-plane $(w = s^{1/v})$, is explained in [13]. The robust stability analysis of a Fractional Order Interval Polynomial (FOIP) family is presented in [15] and [16].

### 4.1 Stability using Riemann surfaces

In a general way, the study of the stability of FO systems can be carried out by studying the solutions of the differential equations that characterize them. To carry out this study it is necessary to remember that a function of the type

$$ a_n s^{\alpha_n} + a_{n-1} s^{\alpha_{n-1}} + \ldots + a_0 s^{\alpha_0}, \quad (7)$$

with $\alpha_i \in R^+$, is a multivalued function of the complex variable $s$ whose domain can be seen as a Riemann surface of a number of sheets. The principal sheet is defined by $-\pi < \arg(s) < \pi$. In the case of $\alpha_i \in Q^+$, that is, $\alpha = 1/v$, $v$ being a positive integer, the $v$ sheets of the Riemann surface are determined by,

$$ s = |s| e^{j\phi}, \qquad (2k+1)\pi < \phi < (2k+3)\pi, \quad (8)$$
$$ k = -1, 0, \ldots, v-2 $$

Correspondingly, the case of $k = -1$ is the principal sheet. For the conformal mapping (transformation) $w = s^{\alpha}$, these sheets become the regions of the plane $w$ defined by :

$$ w = |w| e^{j\theta}, \alpha(2k+1)\pi < \theta < \alpha(2k+3)\pi, \quad (9)$$

Thus, an equation of the type (7) which in general is not a polynomial, will have an infinite number of roots, among which only a finite number of roots will be on the principal sheet of the Riemann surface. It can be said that the roots which are in the secondary sheets of the Riemann surface are related to solutions that are always monotonically decreasing functions (they go to zero without oscillations when $t \to \infty$) and only the roots that are in the principal sheet of the Rie- mann surface are responsible for a different dynamics: damped oscillation, oscillation of constant amplitude, oscillation of increasing amplitude. For the case of commensurate-order systems, whose characteristic equation is a polynomial of the complex variable $w = s^{\alpha}$ the stability condition is expressed as,

$$ |\arg(w_i)| > \frac{\alpha\pi}{2}, \quad (10)$$

where $w_i$ are the roots of the characteristic polynomial in $w$. For the particular case of $\alpha = 1$ the well known stability condition for linear time-invariant systems of integer-order is recovered:

$$\left|\arg(w_i)\right| > \frac{\pi}{2}. \tag{11}$$

## 4.2 Frequency Response - Bode Plot

In general, the frequency response has to be obtained by the direct evaluation of the irrational-order transfer function of the FO system along the imaginary axis for $s = j\omega$, $\omega \in (0,\infty)$ [6]. The frequency response can be obtained by the addition of the individual contributions of the terms of order $\alpha$ resulting,

$$G(s) = \frac{P(s)}{Q(s)} = \frac{\prod_{k=0}^{m}(s^\alpha + z_k)}{\prod_{k=0}^{n}(s^\alpha + \lambda_k)} \tag{12}$$

where $z_k$ and $\lambda_k$ are the zeros and poles respectively. For each of these term the magnitude plot will have a slope which starts at zero and tends to $\alpha 20$ dB/dec, and the phase plot will go from $0$ to $\alpha\pi/2$.

## 4.3 Root Locus

The closed loop stability can be determined using root locus. It can be used to obtain the values of gain, $k$ at which the closed loop system may be stable and become unstable for which values of gain, $k$. Thus stability is obtained as a function of gain. The locus of roots of the characteristic equation as $k$ varies is obtained. An algorithm for the calculation of the root locus of fractional linear systems is presented in [14].

# 5. REPRESENTATION OF FRACTIONAL-ORDER SYSTEMS

## 5.1 Laplace Transform

In system theory the analysis of dynamical behaviors is often made by means of transfer functions. With this in view, the introduction of the Laplace transform (LT) of non integer order derivatives is necessary for an optimal study. Fortunately, not very big differences can be found with respect to the classical case, confirming the utility of this mathematical tool even for fractional systems [10]. Inverse Laplace transformation (ILT) is also useful for time domain representation of systems for which only the frequency response is known. The most general formula is the following:

$$L\left\{\frac{d^m f(t)}{dt^m}\right\} = s^m L\{f(t)\}. \tag{13}$$

Eq.(13) is very useful in order to calculate the inverse Laplace transform of elementary transfer functions, such as non integer order integrators $1/s^m$.

## 5.2 State-space Representation

For linear fractional differential systems of finite dimensions in state-space form, stability is investigated [6]. Consider the commensurate-order TF defined as,

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\sum_{k=0}^{m} b_k (s^\alpha)^k}{\sum_{k=0}^{n} a_k (s^\alpha)^k} \tag{14}$$

where $a_n = 1$, $m > n$.

Associated with this TF, canonical state-space representa tions can be proposed, which are similar to the classical ones developed for IODE systems.

***Controllable Canonical Form :*** Defining the first state in terms of its Laplace transform as,

$$X_1(s) = \frac{1}{\sum_{k=0}^{n} a_k (s^\alpha)^k} U(s), \tag{15}$$

and the remaining elements of the state vector in a recursive away from this one as

$$x_{i+1} = D^\alpha x_i, \quad i = 1,2,..,n-1 \tag{16}$$

the state representation, expressed in the controllable canonical form, is given by the matrix equations

$$D^\alpha x = Ax + Bu, \tag{17}$$

where $D^\alpha x = \begin{bmatrix} D^\alpha x_1 \\ D^\alpha x_2 \\ \vdots \\ D^\alpha x_{n-1} \\ D^\alpha x_n \end{bmatrix}$,

$$A = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

$$y = \left[b_0 - b_n a_0 \cdots b_{n-1} - b_n a_{n-1}\right] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

where $b_i = 0$, for $m < i \le n$

$$C_0 = \left[B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B\right] \tag{18}$$

Controllability criterion is that the system is controllable if and only if matrix C defined by Eq.(18), which is called as controllable matrix is full-rank. Rearranging the above FO state equations, the observable canonical form can be obtained with the matrices A, B and C matrices. The observability condition is also same as for integer-order LTI systems.

# 6. ANALYTICAL RESULTS OF FRACTIONAL-ORDER SYSTEMS

Some FO systems are analyzed in this section. Their stability, step response, frequency response, and the SS representation is discussed. The analysis is done using MATLAB [20]. The standard commercially available simulation softwares cannot be used for evaluating the step, ramp, frequency response of the FO systems. Recently, in MATLAB two toolboxes dedicated to FO systems are available. They are CRONE [19] and NINTEGER toolbox [18].

## 6.1 Example 1

Consider the FO integrator system with TF of the form,

$$F(s) = \frac{1}{s^{\alpha}} . \tag{19}$$

***Initial Analysis of the System:*** For the FO integrator if $\alpha = 0.5$, then consider $w = s^{0.5}$, hence $\tilde{F}(w) = \frac{1}{w}$

The system with the above function has one open-loop pole at origin. The Riemann surface of the function $w = s^{1/v}$ has two Riemann sheets.

Now if $\alpha = 1.5$, and consider $w = s^{0.5}$, then

$$\tilde{F}(w) = \frac{1}{w^3}$$

The system with the above TF has three open-loop poles at origin.

***Step Response:*** The system transfer function is,

$$\frac{Y(s)}{U(s)} = \frac{1}{s^{\alpha}}, \tag{20}$$

Consider step input, $U(s) = 1/s$,

$$Y(s) = \frac{1}{ss^{\alpha}} = \frac{1}{s^{\alpha+1}} . \tag{21}$$

Taking inverse Laplace transform of the equation we get

$$y(t) = \frac{t^{\alpha}}{\Gamma(\alpha+1)} . \tag{22}$$

The fig.(1) shows the step response of the system for $\alpha = 0.1, 0.5, 0.8, 1$ and $1.5$.

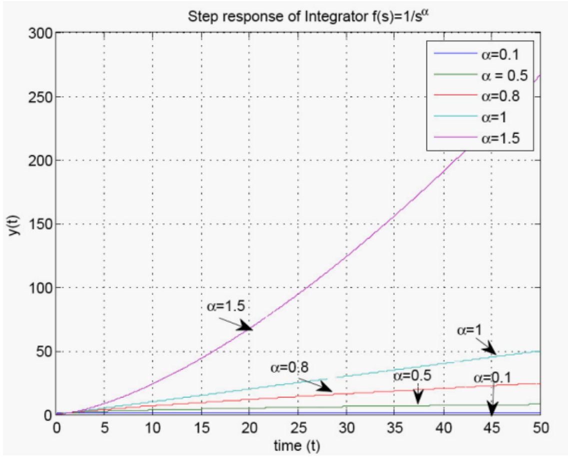


**Fig. 1. Step response of Example (1)**

***Frequency Response:*** Put $s = j\omega$ in the system function given by Eq.(19) we can plot the magnitude and phase plots. The magnitude and phase plot of the system for $\alpha =$ 0.1, 0.5, 0.8, and 1 is plotted as shown in the fig.(2). From the above response we can conclude that:

1) The magnitude has a constant slope of $-20\alpha$ dB/decade.

2) The phase plot is a horizontal line at $-\alpha\pi/2$.

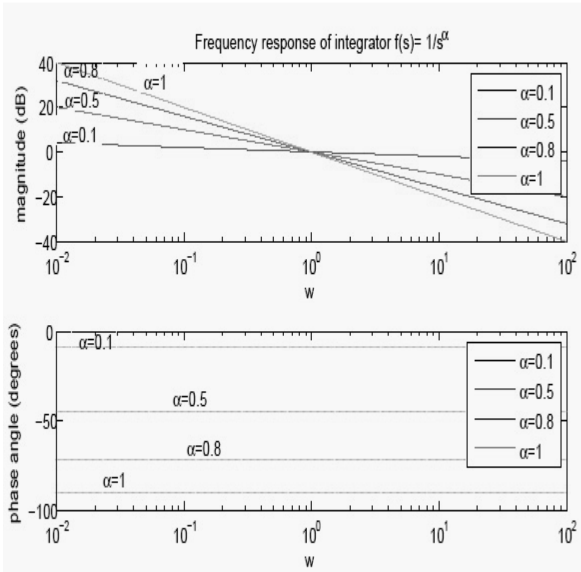


**Fig. 2. Frequency response of Example (1) for different values of α**

## 6.2 Example 2

Consider the incommensurate system given by the following transfer function [6] [7]

$$F(s)=\frac{1}{0.8s^{2..2}+0.5s^{0.9}+1}. \qquad (23)$$

***Initial Analysis of the System:*** The system given in the equation can be written as

$$F(s)=\frac{1}{0.8s^{(\frac{1}{10})22}+0.5s^{(\frac{1}{10})9}+1}. \qquad (24)$$

Consider $w = s^{1/10}$, the system has 10 Riemann sheets.

$$\widetilde{F}(w)=\frac{1}{0.8w^{22}+0.5w^{9}+1}. \qquad (25)$$

The open-loop poles and their appropriate arguments of the system are shown in table I. Physical significant roots are in the first Riemann sheet, which is expressed by relation $-\pi/v<\phi<\pi/v$, where $\phi=\arg(w)$.

In this example complex conjugate roots in first Riemann sheet are $w_{21,22}=1.0045\pm0.1684j,$

$|\arg(w_{21,22})|=0.1661$, which satisfy conditions $|\arg(w_{21,22})|>\pi/2v=\pi/20$ is as shown in Pole-zero plot shown in fig.(3).

**Table1: Open loop poles and corresponding arguments**

| Open-loop Poles | Arguments in radians |
|---|---|
| $w_{1,2} = -0.9970 \pm 0.1182j$ | $\|\arg(w_{1,2})\| = 3.023$ |
| $w_{3,4} = -0.9297 \pm 0.4414j$ | $\|\arg(w_{3,4})\| = 2.698$ |
| $w_{5,6} = -0.7465 \pm 0.6420j$ | $\|\arg(w_{5,6})\| = 2.431$ |
| $w_{7,8} = -0.5661 \pm 0.8633j$ | $\|\arg(w_{7,8})\| = 2.151$ |
| $w_{9,10} = -0.259 \pm 0.9625j$ | $\|\arg(w_{9,10})\| = 1.834$ |
| $w_{11,12} = -0.0254 \pm 1.0111j$ | $\|\arg(w_{11,12})\| = 1.595$ |
| $w_{13,14} = 0.3080 \pm 0.9772j$ | $\|\arg(w_{11,12})\| = 1.265$ |
| $w_{15,16} = 0.5243 \pm 0.8359j$ | $\|\arg(w_{15,16})\| = 1.010$ |
| $w_{17,18} = 0.7793 \pm 0.6795j$ | $\|\arg(w_{17,18})\| = 0.717$ |
| $w_{19,20} = 0.9084 \pm 0.3960j$ | $\|\arg(w_{19,20})\| = 0.411$ |
| $w_{21,22} = 1.0045 \pm 0.1684j$ | $\|\arg(w_{21,22})\| = 0.1661$ |

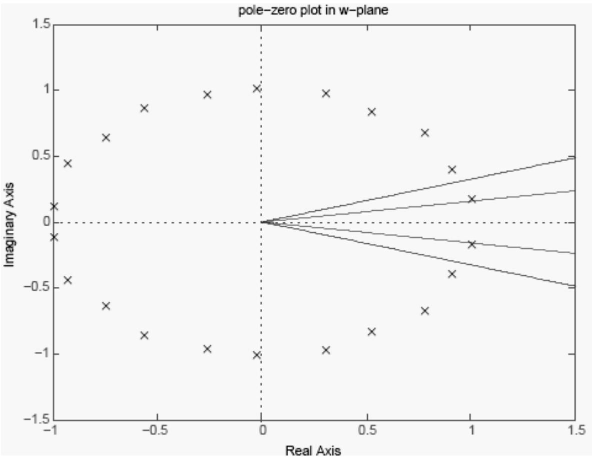


**Fig. 3. Pole-zero plot of Example (2)**

The roots in first Reimann sheet satisfy the stability criteria, hence the system is stable. Other roots of the system lie in region $|\phi|>\pi/v$ which are not physical (outside of the closed angular sector limited by the thick line in the fig.(3). The first Riemann sheet is transformed from $s$ plane to $w$-plane as follows:

$$-\pi/10<\arg(w)<\pi/10 \quad \text{and}$$
$$-\pi<10\arg(w)<\pi. \qquad (26)$$

Therefore from this consideration angle obtained is

$$|\arg(s)|=10|\arg(w)|. \qquad (27)$$

***Step Response:*** The system TF is,

$$\frac{Y(s)}{U(s)}=\frac{1}{0.8s^{2..2}+0.5s^{0.9}+1}. \qquad (28)$$

For step response of the system, $U(s)=1/s$.

Calculating the residues and poles by partial fractions are shown in table II.

**Table2: Open loop poles and corresponding arguments**

| Residues | Poles |
|---|---|
| −0.0264 ± 0.0209i | 0.7793 ± 0.6796i |
| 0.0147 ± 0.0313i | −0.5662 ± 0.8633i |
| 0.0355 ± 0.0079i | −0.9298 ± 0.4415i |
| −0.0006 ± 0.0391i | 0.3080 ± 0.9772i |
| −0.0422 ± 0.0068i | 1.0045 ± 0.1684i |
| −0.0142 ± 0.0447i | −0.0254 ± 1.0112i |
| 0.0467 ± 0.0210i | −0.9970 ± 0.1182i |
| 0.0271 ± 0.0477i | −0.2597 ± 0.9625i |
| −0.0476 ± 0.0323i | 0.9085 ± 0.3960i |
| −0.0369 ± 0.0464i | 0.5243 ± 0.8360i |
| 0.0441 ± 0.0409i | −0.7466 ± 0.6420i |

Using inverse Laplace transform

$$L^{-1}\left\{\sum_{i=1}^{n}\frac{r_i}{s(s^{\alpha}+p_i)}\right\}=\sum_{i=1}^{n} r_i t^{\alpha} E_{\alpha,\alpha+1}(-p_i t^{\alpha}),$$

(29)

where $E$ is the special function called as Mittag Leffler (ML) function, $r_i$ are the residues and $p_i$ are the corresponding poles for $i=1$ to $22$. To plot step response the use of ML function code [21] in MATLAB is done. It is concluded that the ML function calculation is time consuming and may not give proper results in all the cases. In such cases they can also be plotted u sing invlap.m subroutine (numerical ILT) [22], [23]. The step response plot is plotted as shown in fig.(4). The step response shows it gives bounded output for a bounded input.

***Frequency Response:*** Put $s = j\omega$ in the given system function. The magnitude plot and phase plot of the system using MATLAB is plotted as shown in the fig.(5).

***State-space Representation:*** The canonical form of the system is obtained as,

$$\frac{Y(s)}{X(s)}=\frac{1}{0.8s^{(0.1)22}+0.5s^{(0.1)9}+1}. \quad (30)$$

$$((s^{0.1})^{22}+0.625(s^{0.1})^{9}+1.25)Y(s)=1.25X(s).$$

(31)

Consider input u(t) and taking inverse Laplace transform we get,

$$D^{2.2}y(t)+0.625D^{0.9}y(t)+1.25y(t)=1.25u(t)$$

(32)

**Case 1**: Let $y(t)=x_1(t)$ and

$$D^{0.1}x_1(t)=x_2(t) \quad (33)$$

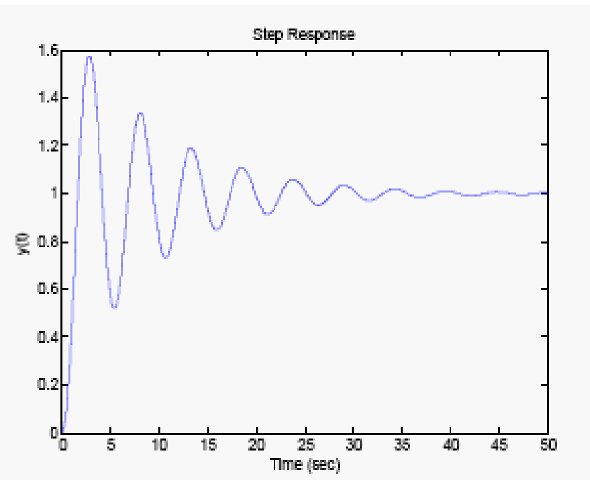


**Fig. 4. Step response of Example (2)**

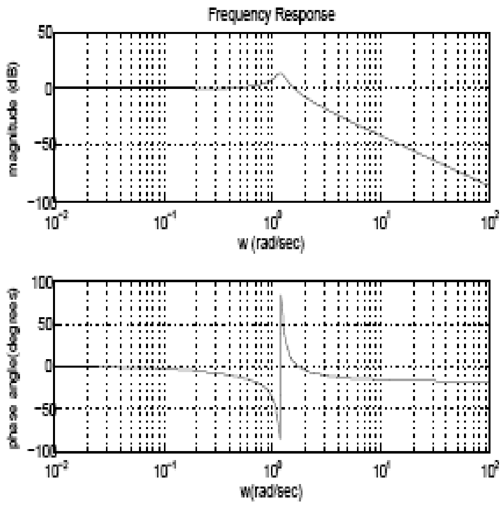


**Fig. 5. Frequency response of Example (2)**

$$D^{0.1}x_{22}(t)=-1.25x_1(t)-0.625x_{10}(t)+1.25u(t)$$

(34)

The controllable canonical form is therefore given by,

$$\begin{bmatrix} D^{0.1}x_1(t) \\ D^{0.1}x_2(t) \\ \vdots \\ D^{0.1}x_{22}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ -1.25 & 0 & \cdots & -0.625 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{22}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1.25 \end{bmatrix} u(t)$$

$$y(t)=\begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} u(t) \quad (35)$$

**Case 2:** Let $y(t)=x_1(t)$ and $D^{0.9}x_1(t)=x_2(t)$ (36)

The controllable canonical form is therefore given by,

$$\begin{bmatrix} D^{0.9}x_1(t) \\ D^{1.3}x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.125 & -0.625 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0.125 \end{bmatrix} u(t) \quad (37)$$

The controllable matrix of this system is full rank and hence the system is controllable. It is also shown that there can be no unique state space representation for a fractional-order system. In the analysis of this incommensurate FO system we conclude that the system is stable and its step response, states-space representation and frequency response have been obtained successfully.

## 7. CONCLUSION

The fractional order models of real systems are more adequate than the usually used integer order models. At the same time fractional integrals and derivatives are applied to the theory of control of dynamical systems, when the controlled system and/or the controller is described by fractional dif- ferential equations. The most important features of fractional systems are studied during the work. They are discussed using Bode diagrams, time response, state space representation usually adopted for integer order systems which allow an easy comparison among the two different behaviors. The multi- valued function is expressed as single-valued function by replacing $s^{1/v}$ by $w$ and analysis is done. Also it is shown that there can be no unique state space representation for a fractional-order system.

## 8. ACKNOWLEDGMENTS

Our thanks to all those who have directly or indirectly helped us in completion of this work. Special thanks to our family members for their support and encouragement.

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