Inversion Formula for Generalized Two-Dimensional Offset Fractional Fourier Transform

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ABSTRACT

The Fractional Fourier Transform is very useful tool in digital signal processing. The Offset Fractional Fourier transform, however, is generalization of the Fractional Fourier transform. The Offset Fractional Fourier Transform is more flexible than the original Fractional Fourier Transform and can solve some problem that cannot be solved well by the original fractional Fourier transform. In this paper, we present generalization of Two Dimensional offset fractional Fourier transform (2D offset FRFT) in distributional sense. Inversion formula for 2D offset FRFT is also proved.

Keywords

Fractional Fourier Transform, Two Dimensional Offset Fractional Fourier Transform, Digital Signal processing, image processing, speech Recognition.

1. INTRODUCTION

The FRFT was originally describe by KOBER and was introduce for signal processing by Namias [2].This was done, starting from fractional power of the Eigen value of the Fourier transform and their corresponding Eigen function. With this formation on integral representation of this operator was derived in heuristic manner [1].

The FRFT can be interpreted as a rotation of the time frequency plane and has been proved to relate to other time frequency representation [4].

The FRFT process signal in a unified time frequency domain and time domain and frequency domain are the special cases of the fractional Fourier domain .Due to additional angle parameter, FRFT is flexible and suitable for processing non-stationary signal, especially the chirp signal [3].

FRFT has many applications in spectrum analysis, in magnetic resonance (MRI) for image reconstruction, moving target detection at sea based on fractional characters in FRFT domain. It have been extensively use in optics ,signal analysis and processing a especially for wave and beam propagation, wave field reconstruction ,phase retrieval and phase space. Tomography, study of time and space frequency distribution.[5].

In signal processing application it is basically use for filtering ,signal recovery .signal synthesis ,signal detector ,core later, beam forming restoration, and enhancement, pattern recognition, optical winner filtering and match filtering.

The properties of FRFT are useful not only deriving direct and inverse transform of many time varying function but also in several valuable result in signal processing.[6].

Many application of the original FRFT are also the potential application of the offset fractional Fourier transform [7].

In the present work Generalization of Two Dimensional Offset Fractional Fourier Transform is presented in distributional sense. Inversion formula for 2D offset FRFT is proved.

2. TESTING FUNCTION SPACE E

An infinitely differentiable complex valued smooth function on $\phi\ (R^n)$, belongs to $E\ (R^n)$, if for each compact $I \subset S_{a,b}$

Where

$$S_{a,b}\{t, x: t, x \,\varepsilon\, R^n\ , |t| \leq a, |x| \leq b, a > o, b > o\}\ , I \,\varepsilon\, R^n$$

$$\gamma_{I,l,q}(\phi) = \sup_{t,\, x\varepsilon I} \left| D_{t,x}^{l,q} \phi(t,x) \right| \quad \ldots \ldots (2.1)$$

$$< \infty\ , l, q = 0,1,2 \ldots \ldots \ldots$$

Thus $E\ (R^n)$ will denote the space of all $\phi\varepsilon\ E\ (R^n)$ with support contained in $S_{a,b}$.

II. THE DEFINITION OF GENERALIZED DISTRIBUTIONAL TWO-DIMENSIONAL OFFSET FRACTIONAL FOURIER TRANSFORM ON E.

The Two-Dimensional Offset Fractional Fourier Transform $\left[F_\alpha^{\tau,\eta,\xi,\gamma} f(t,x)\right](s,u)$ of generalization function $f(t,x)$ through an angle α is defined as

$$\left[F_\alpha^{\tau,\eta,\xi,\gamma} f(t,x)\right](s,u) = [\mathrm{FRFT}\ f(t,x)]_\alpha^{\tau,\eta,\xi,\gamma}(s,u)$$

$$= \langle f(t,x), K_\alpha(s-\eta, t, u-\gamma, x)\rangle \qquad \ldots \ldots \ldots (3.1)$$

Where

$$K_\alpha(t, s-\eta, x, u-\gamma) = \sqrt{\frac{1-i\cot\alpha}{2\pi}}\ e^{i(s\tau+u\xi)}$$

$$e^{\frac{i}{2\sin\alpha}\left[\left((s-\eta)^2+t^2+(u-\gamma)^2+x^2\right)\cos\alpha - 2\left((s-\eta)t+(u-\gamma)x\right)\right]}$$

$$K_\alpha(t, s-\eta, x, u-\gamma) = C_{1\alpha} e^{i(s\tau+u\xi)} e^{ic_{2\alpha}\left[\left((s-\eta)^2+t^2+(u-\gamma)^2+x^2\right)\cos\alpha - 2\left((s-\eta)t+(u-\gamma)x\right)\right]}$$

Where $C_{1\alpha} = \sqrt{\frac{1-i\cot\alpha}{2\pi}}$ and $C_{2\alpha} = \frac{1}{2\sin\alpha}$

3. INVERSION FORMULA FOR TWO DIMENSIONAL OFFSET FRACTIONAL FOURIER TRANSFORMS

If it possible to recover the function f by means of the inversion formula

$$\left[F_\alpha^{\tau,\eta,\xi,\gamma} f(t,x)\right](s,u) = [\mathrm{FRFT}]_\alpha^{\tau,\eta,\xi,\gamma}(s,u) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,x) K_\alpha(t, s-\eta, x, u-\gamma)\, dt\, dx$$

where

$$K_\alpha(t, s-\eta, x, u-\gamma) = \sqrt{\frac{1-i\cot\alpha}{2\pi}}\; e^{i(s\tau+u\xi)} e^{\frac{i}{2\sin\alpha}\left[((s-\eta)^2+t^2+(u-\gamma)^2+x^2)\cos\alpha - 2((s-\eta)t+(u-\gamma)x)\right]}$$

$$= C_{1\alpha} e^{i(s\tau+u\xi)} e^{ic_{2\alpha}\left[((s-\eta)^2+t^2+(u-\gamma)^2+x^2)\cos\alpha - 2((s-\eta)t+(u-\gamma)x)\right]}$$

Where $C_{1\alpha} = \sqrt{\frac{1-i\cot\alpha}{2\pi}}$ and $C_{2\alpha} = \frac{1}{2\sin\alpha}$

As obtain below.
Recall the two dimensional offset fractional Fourier transform

$$\left[F_\alpha^{\tau,\eta,\xi,\gamma} f(t,x)\right](s,u) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,x) K_\alpha(t, s-\eta, x, u-\gamma)\, dt\, dx \quad \ldots (4.1)$$

Where

$$K_\alpha(t, s-\eta, x, u-\gamma) = \sqrt{\frac{1-i\cot\alpha}{2\pi}}\; e^{i(s\tau+u\xi)} e^{\frac{i}{2\sin\alpha}\left[((s-\eta)^2+t^2+(u-\gamma)^2+x^2)\cos\alpha - 2((s-\eta)t+(u-\gamma)x)\right]}$$

Can be written as

$$K_\alpha(t, s-\eta, x, u-\gamma) = C_{1\alpha} e^{i(s\tau+u\xi)} e^{ic_{2\alpha}\left[((s-\eta)^2+t^2+(u-\gamma)^2+x^2)\cos\alpha - 2((s-\eta)t+(u-\gamma)x)\right]}$$

Where $C_{1\alpha} = \sqrt{\frac{1-i\cot\alpha}{2\pi}}$ and $C_{2\alpha} = \frac{1}{2\sin\alpha}$

Now

$$K_\alpha(t, s-\eta, x, u-\gamma) = C_{1\alpha} e^{i(s\tau+u\xi)} e^{[ic_{2\alpha} t^2\cos\alpha]} e^{[ic_{2\alpha}(s-\eta)^2\cos\alpha]} e^{[ic_{2\alpha} x^2\cos\alpha]} e^{[ic_{2\alpha}(u-\gamma)^2\cos\alpha]} e^{[-2ic_2 t(s-\eta)]} e^{[-2ic_{2\alpha} x(u-\gamma)]}$$

Equation (4.1)

$$\left[F_\alpha^{\tau,\eta,\xi,\gamma} f(t,x)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,x)\, C_{1\alpha} e^{i(s\tau+u\xi)} e^{[ic_{2\alpha} t^2\cos\alpha]} e^{[ic_{2\alpha}(s-\eta)^2\cos\alpha]} e^{[ic_{2\alpha} x^2\cos\alpha]} e^{[ic_{2\alpha}(u-\gamma)^2\cos\alpha]} e^{[-2ic_2 t(s-\eta)]} e^{[-2ic_{2\alpha} x(u-\gamma)]} dt\, dx$$

$$\left[F_\alpha^{\tau,\eta,\xi,\gamma} f(t,x)\right](s,u)\; e^{-ic_{2\alpha}[(s-\eta)^2+(u-\gamma)^2]\cos\alpha} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(t,x)\, e^{-2i\, C_{2\alpha}[t(s-\eta)+x(u-\gamma)]} dt\, dx$$

where $g(t,x) = C_{1\alpha} f(t,x) e^{i(s\tau+u\xi)} e^{ic_{2\alpha}(t^2+x^2)\cos\alpha}$

$$\left[F_\alpha^{\tau,\eta,\xi,\gamma} f(t,x)\right](s,u)\; e^{-ic_{2\alpha}[(s-\eta)^2+(u-\gamma)^2]\cos\alpha}$$

$$= F[g(t,x)][2c_{2\alpha}(s-\eta)(u-\gamma)]$$

The Fourier transform of $g(t,x)$ with the argument $2c_{2\alpha}(s-\eta) = \sigma$ And $2c_{2\alpha}(u-\gamma) = \rho$

$$\left[F_\alpha^{\tau,\eta,\xi,\gamma} f(t,x)\right]\left(\eta + \frac{\sigma}{2c_{2\alpha}}, \gamma + \frac{\rho}{2c_{2\alpha}}\right) e^{-ic_{2\alpha}\left[\left(\frac{\sigma}{2c_{2\alpha}}\right)^2+\left(\frac{\rho}{2c_{2\alpha}}\right)^2\right]\cos\alpha} = F[g(t,x)](\sigma,\rho)$$

Invoking the Fourier inversion we can write

$$g(t,x) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\sigma,\rho) e^{ixt(\sigma,\rho)}\, d\sigma\, d\rho$$

Therefore

$$f(t,x) = \frac{1}{4\pi^2\sin^2\alpha}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[F_\alpha^{\tau,\eta,\xi,\gamma} f(t,x)\right](s,u)\; e^{-i(s\tau+u\xi)\left(\sqrt{\frac{1-i\cot\alpha}{2\pi}}\right)^{-1}} e^{-ic_{2\alpha}[((s-\eta)^2+t^2+(u-\gamma)^2+x^2)\cos\alpha]} e^{2ic_{2\alpha}[t(s-\eta)+x(u-\gamma)]} ds\, du$$

$$f(t,x) = \frac{1}{(2\pi)^{3/2}}\frac{1}{\sin^2\alpha}\frac{1}{\sqrt{1-i\cot\alpha}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[F_\alpha^{\tau,\eta,\xi,\gamma} f(t,x)\right](s,u) e^{-i(s\tau+u\xi)} e^{-i/2\sin\alpha[((s-\eta)^2+t^2+(u-\gamma)^2+x^2)\cos\alpha - 2t(s-\eta)-2x(u-\gamma)]}\, ds\, du$$

$$f(t,x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[F_\alpha^{\tau,\eta,\xi,\gamma} f(t,x)\right](s,u)\frac{1}{(2\pi)^{3/2}}\frac{1}{\sin^2\alpha}\frac{1}{\sqrt{1-i\cot\alpha}}\, e^{-i(s\tau+u\xi)} e^{-i/2\sin\alpha[((s-\eta)^2+t^2+(u-\gamma)^2+x^2)\cos\alpha - 2t(s-\eta)-2x(u-\gamma)]}\, ds\, du$$

where

$$K_\alpha(t, s-\eta, x, u-\gamma) = \frac{1}{(2\pi)^{3/2}}\frac{1}{\sin^2\alpha}\frac{1}{\sqrt{1-i\cot\alpha}}\, e^{-i(s\tau+u\xi)} e^{-i/2\sin\alpha[((s-\eta)^2+t^2+(u-\gamma)^2+x^2)\cos\alpha - 2t(s-\eta)-2x(u-\gamma)]}$$

Hence proved

4. CONCLUSION

The present paper focuses on the generalization of two dimensional Offset-Fractional Fourier transform in distributional sense. Inversion formula for 2D Offset FRFT is proved.

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