Operation Transform Formulae on Generalized Fractional Fourier Transform

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ABSTRACT
Fractional Fourier transform (FrFT) is one of the most widely used tools in signal processing and optics. Several properties of FrFT have been studied recently and many are being investigated at present. The original purpose of FrFT is to solve the differential equation in quantum mechanics. In fact, most of the applications of FrFT now are application on optics. But there are still lots of unknowns to the signal processing community. Because of its simple and beautiful properties in Time-frequency plane we believe that many new applications are waiting to be proposed in signal processing.

In this paper FrFT is extended in the distributional generalized sense. Operation transform formulae for FrFT are discussed.

Keywords
Fractional Fourier Transform, Generalized Function, Quantum Mechanics, Optics, Signal Processing.

1. INTRODUCTION
Fourier analysis is one of the most frequently used tools in signal processing and many other scientific fields. Besides the Fourier transform (FT), time-frequency representation of signals, such as Wigner Distribution (WD), Short Time Fourier Transform (STFT), Wavelet Transform (WT) are also widely used in speech processing, image processing or quantum Physics [1].

The FrFT belongs to the class of time frequency representations that have been extensively used by the signal processing community. In all the time frequency representations, one normally uses a plane with two orthogonal axes corresponding to time and frequency.

If we consider a signal $x(t)$ to be represented along the time axis and its ordinary Fourier transform X(f) to be represented along the frequency axis, then the Fourier transform operator (denotes by F) can be visualized as a change in representation of the signal corresponding to a counter-clockwise rotation of the axis by an angle $\frac{\pi}{2}$. The FrFT is a linear operation that corresponds to the rotation of the signal through an angle which is not a multiple of $\frac{\pi}{2}$ i.e. it is the representation of the signal along the axis making on angle α with the time axis [2].

The FrFT was originally described by kober and was later introduced for signal processing by Namias as a Fourier transform of fractional order. A number of useful properties have been derived in [1]. In [5] Mc Bride and Kerr provide a rigorous mathematical formal work in which the formal work of Namias could be situated.

The FrFT has been found to have several applications in the areas of optics and signal processing. It also leads to generalization of motion of space (or time) and frequency domains which are central concepts of signal processing. It has many applications in solution of differential equations, optical beam propogation and spherical mirror resonators, optical diffraction theory, quantum mechanics, statistical optics, optical system design and optical signal processing, signal detectors, correlation and pattern recognition, space or time-variant filtering, multiplexing, signal and image recovery; Therefore, applications of the transform have been studied mostly in the areas of optics and wave propogation, and signal analysis and processing [6].

We Mansion very briefly some applications of FrFT and invite the reader to look up the details in the literature.

1.1 Filtering
If the components of a signal interact in the time and the frequency domain, then it may be difficult to separate them and filter out the noise. However using different rotations, it is possible to separate components from a signal as long as their [1].

1.2 Compression of Signal
The idea is that one or more fractional Fourier transforms are computed and filtered (e.g. by thresholding) to obtain simpler representation of the signal. It may well be that the FrFT components are much simpler in one d0main than in the other [3].

1.3 Image Encryption
A possible encryption technique for an image is to first multiply the image with a random phase (i.e. multiply it with a function of the form $e^{i\varphi(x,y)}$) then apply a 2-D FrFT of some order that may be different in the two-directions. Only the intensity of the result is stored. The some set of operations is applied a second time to the image but with a different phase and different orders. The original signal (intensity and phase) can be recovered from the two resulting images by a recursive Scheme [4].

1.4 Digital Watermarking
For copyright protection, a watermark can be embedded in a digital image. This watermark should be some signature embedded in the image without disturbing the original image visually. This embedding can be done by computing a 2-D FrFT of the image.

In the present work FrFT is extended in the distributional generalized sense. Some properties of the kernel and operation transform formulae is proved for the FrFT. Some operators as differential, scaling on testing function space are also discussed.

2. DISTRIBUTIONAL FRACTIONAL FOURIER TRANSFORMS:

2.1 Conventional fractional Fourier transform:

The Fractional Fourier transform with parameter α of $f(x)$ denoted by $FrFT\{f(x)\}$ performs a linear operation given by the integral transform.

$$\mathrm{FrFT}\{f(x)\} = F_\alpha(u)$$
$$= \int_{-\infty}^{\infty} f(x)\, k_\alpha(x,u) \quad \cdots\cdots\cdots (2.1),$$

Where

$$k_\alpha(x,u) = \sqrt{\frac{1-i\cot\alpha}{2\pi}}\; e^{\frac{i}{2\sin\alpha}[(x^2+u^2)\cos\alpha - 2xu]} \quad \cdots\cdots (2.2)$$

2.2 The testing function space E:

An infinitely differentiable complex valued smooth function φ on R^n belongs to $E(R^n)$ if for each compact set $I \subset S_a$, where $S_a = \{x; x \in R^n, |x| \le 0, a > 0\}, I \in R^n$.

$$\gamma_{E,p}(\varphi) = \sup_{x \in I} \left|D_x^p\right| \varphi(x)$$
$$< \infty, \quad \text{where p=1, 2, 3.....}$$

Thus $E(R^n)$ will denote the space of all $\varphi \in E(R^n)$ with suppose contained in S_a.

Note that the space E is complete and therefore a Frechet space. Moreover we say that f is a fractional Fourier transformable if it is a member of E^*, the dual space of E.

3. DISTRIBUTIONAL FRACTIONAL FOURIER TRANSFORM:

The Distributional fractional Fourier transform of $f(x,y) \in E(R^n)$ can be defined by

$$\mathrm{FrFT}\{f(x)\} = F_\alpha(u) = \langle f(x), K_\alpha(x,u)\rangle, \quad \cdots\cdots\cdots (3.1)$$

where

$$K_\alpha(x,u) = C_{1\alpha}\, e^{iC_{2\alpha}[(x^2+u^2)\cos\alpha - 2xu]} \quad \cdots\cdots\cdots (3.2)$$

$$\text{where } C_{1\alpha} = \sqrt{\frac{1-i\cot\alpha}{2\pi}}, \qquad C_{2\alpha} = \frac{1}{2\sin\alpha}$$

The right hand side of (3.1) has a meaning as the application of $f \in E^*$ to $k_\alpha(x,u) \in E$.

4. PROPOSITION:

4.1 Generalized fractional Fourier transform reduces to conventional Fourier transforms if $\theta = \frac{\pi}{2}$.

Proof: We know the generalized fractional Fourier transform is

$$\mathrm{FrFT}\{f(x)\} = F_\alpha(u)$$
$$= \int_0^\infty f(x)\, C_{1\alpha}\, e^{iC_{2\alpha}[(x^2+u^2)\cos\alpha - 2xu]}\, dx$$

putting $\alpha = \frac{\pi}{2}$

$$F_{\frac{\pi}{2}}(u) = \int_0^\infty f(x) \sqrt{\frac{1-i\cot\frac{\pi}{2}}{2\pi}}\; e^{\frac{i}{2\sin\frac{\pi}{2}}\left[(x^2+u^2)\cos\frac{\pi}{2} - 2xu\right]}\, dx$$
$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty f(x)\, e^{-ixu}\, dx$$
$$= F(u)$$

4.2 Proposition:

If f(x) is any signal then

$$F_\alpha(u) = e^{\frac{iu^2\cot\alpha}{2}}\, C_{1\alpha}\, F\{\tilde{f}(u)\}, \text{where } C_{1\alpha} = \sqrt{\frac{1-i\cot\alpha}{2\pi}}$$

Proof: Fractional Fourier transform is given as

$$F_\alpha(u) = \int_0^\infty f(x) \sqrt{\frac{1-i\cot\alpha}{2\pi}}\; e^{\frac{i}{2\sin\alpha}[(x^2+u^2)\cos\alpha - 2xu]}\, dx$$
$$F_\alpha(u) = e^{\frac{iu^2\cot\alpha}{2}} C_{1\alpha} \int_0^\infty f(x)\, e^{\frac{ix^2\cot\alpha}{2}}\, e^{\frac{-ixu}{\sin\alpha}}\, dx$$
$$= e^{\frac{iu^2\cot\alpha}{2}} C_{1\alpha} \int_0^\infty \tilde{f}(x)\, e^{-isx}\, dx,$$

Where

$$\tilde{f}(x) = f(x)\, e^{\frac{ix^2\cot\alpha}{2}}, \qquad s = \frac{u}{\sin\alpha}$$

$$\therefore F_\alpha(u) = e^{\frac{iu^2\cot\alpha}{2}}\, C_{1\alpha}\, F\{\tilde{f}(x)\}$$

5. PROPERTIES OF KERNEL OF FRFT:

5.1 To prove $\boldsymbol{k_{(-\alpha)}(x,u) = k_\alpha^*(x,u)}$

Consider

$$k_{(-\alpha)}(x,u) = \sqrt{\frac{1-i\cot(-\alpha)}{2\pi}}\; e^{\frac{i}{2\sin(-\alpha)}[(x^2+u^2)\cos(-\alpha) - 2xu]}$$

$= k_\alpha^*(x,u)$, where * denotes the conjugation.

5.2 To prove $\mathbf{k_\alpha(-x,u) = \frac{1}{e} k_\alpha(x,u)}$

Consider

$$k_\alpha(-x,u) = \sqrt{\frac{1-i\cot\alpha}{2\pi}}\; e^{\frac{i}{2\sin\alpha}[(x^2+u^2)\cos\alpha - 2(-x)u]}$$

$$= \frac{1}{e}\sqrt{\frac{1-i\cot\alpha}{2\pi}}\; e^{\frac{i}{2\sin\alpha}[(x^2+u^2)\cos\alpha - 2xu]}$$

$$= \frac{1}{e} k_\alpha(x,u)$$

5.3 $\mathbf{k_\alpha(x,0) = e^{\frac{-iu}{\sin\alpha}\left[\frac{u\cos\alpha}{2} - x\right]}\, k_\alpha(x,u)}$

Consider

$$k_\alpha(x,u) = \sqrt{\frac{1-i\cot\alpha}{2\pi}}\; e^{\frac{i}{2\sin\alpha}[x^2\cos\alpha]}$$

$$= e^{\frac{-iu}{\sin\alpha}\left[\frac{u\cos\alpha}{2}-x\right]} k_\alpha(x,u)$$

6. OPERATION TRANSFORM FORMULAE:

6.1 Linearity Property

If $FrFT\{f(x)\}$ is generalized fractional Fourier transform of $f(x)$ and $FrFT\{g(x)\}$ is generalized fractional Fourier transform of $g(x)$ then

$$FrFT\{C_1 f(x) + C_2 g(x)\}(u) = C_1 FrFT\{f(x)\}(u) + C_2 FrFT\{g(x)\}(u)$$

Proof: Consider

$$FrFT\{C_1 f(x) + C_2 g(x)\}(u) = \int_{\infty}^{-\infty} [C_1 f(x) + C_2 g(x)]\, k_\alpha(x,u)\, dx$$

$$= C_1 FrFT\{f(x)\}(u) + C_2 FrFT\{g(x)\}(u)$$

6.2 First Shifting Operator:

$$FrFT\{e^{-ax} f(x)\}(u) = \sqrt{\frac{1-icot\alpha}{2\pi}}\, \{A\, FrFT\big(e^{-ax}\, xf(x)\big)(u) + B\, FrFT\big(e^{-ax}\, f'(x)\big)(u)\},$$

Where, $A = \frac{2iC_{2\alpha}\cos\alpha}{2iC_{2\alpha}+a}$ and $B = \frac{1}{2iC_{2\alpha}+a}$

Proof:

$$FrFT\{e^{-ax} f(x)\}(u)$$

$$= \int_0^\infty e^{-ax} f(x)\, k_\alpha(x,u)\, dx$$

$$= e^{i\, C_{2\alpha}\, u^2 \cos\alpha}\, C_{1\alpha} \int_0^\infty f(x) e^{i\, C_{2\alpha}[x^2\cos\alpha - 2xu] - ax}\ dx$$

$$= e^{i\, C_{2\alpha}\, u^2 \cos\alpha}\, C_{1\alpha} \int_0^\infty \left[f(x) e^{i\, C_{2\alpha} x^2 \cos\alpha} \right] . e^{-2i\, C_{2\alpha} u - a}\ dx$$

$$= \sqrt{\frac{1-icot\alpha}{2\pi}} \left\{ \left(\frac{2iC_{2\alpha}\cos\alpha}{2iC_{2\alpha}+a}\right) FrFT\big(e^{-ax}\, xf(x)\big) f(u) + \left(\frac{1}{2iC_{2\alpha}+a}\right) FrFT\big(e^{-ax}\, f'(x)\big)(u) \right\}$$

$$= \sqrt{\frac{1-icot\alpha}{2\pi}}\, \{A\, FrFT\big(e^{-ax}\, xf(x)\big)(u) + B\, FrFT\big(e^{-ax}\, f'(x)\big)(u)\},$$

Where $A = \frac{2iC_{2\alpha}\cos\alpha}{2iC_{2\alpha}+a}$ and $B = \frac{1}{2iC_{2\alpha}+a}$

6.3 Differential property:

$$FrFT\{f'(x)\}(u) = (-i\cot\alpha) FrFT\{x\, f(x)\}(u) + (iu\ cosec\alpha) FrFT\{f(x)\}(u)$$

Proof:

$$FrFT\{f'(x)\}(u)$$

$$= \int_{-\infty}^{\infty} C_{1\alpha} e^{\frac{i}{2\sin\alpha}[(x^2+u^2)\cos\alpha - 2xu]} f'(x)\, dx$$

$$= C_{1\alpha} e^{\frac{iu^2\cot\alpha}{2}} \left\{ \left[e^{\frac{i}{2}x^2\cot\alpha - ixu\ cosec\alpha}\ f(x) \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} e^{\frac{i}{2}x^2\cot\alpha - ixu\ cosec\alpha} \left(\frac{i}{2}\cot\alpha\, 2x - iu\ cosec\alpha \right) f(x)\, dx \right\}$$

$$= (-i\cot\alpha) \left[C_{1\alpha} \int_{-\infty}^{\infty} e^{\frac{i}{2\sin\alpha}[(x^2+u^2)\cos\alpha - 2xu]} xf(x)\, dx \right] + (iu\ cosec\alpha) \left[C_{1\alpha} \int_{-\infty}^{\infty} e^{\frac{i}{2\sin\alpha}[(x^2+u^2)\cos\alpha - 2xu]} f(x)\, dx \right]$$

$$= (-i\cot\alpha)\ FrFT\{xf(x)\}(u) + (iu\ cosec\alpha) FrFT\{f(x)\}(u)$$

6.4 Shifting Property:

If $\{FrFT\ f(x)\}(u)$ is generalized fractional Fourier transform of Fourier transform of $f(x)$ then

$$\{FrFT\ f(x-x_0)\}(u) = e^{\frac{i}{2}{x_0}^2\cot\alpha - ix_0 u\ cosec\alpha}\ FrFT\{e^{ixx_0 cosec\alpha} f(x)\}$$

consider

$$\{FrFT\ f(x-x_0)\}(u)$$

$$= \langle (x-x_0), k_\alpha(x,u) \rangle$$

$$= \langle f(x), C_{1\alpha}\ e^{iC_{2\alpha}[x_0(2x+x_0)\cos\alpha - 2x_0 u]}\ e^{iC_{2\alpha}[(x^2+u^2)\cos\alpha - 2xu]} \rangle$$

$$= e^{\frac{i}{2\sin\alpha}[{x_0}^2\cos\alpha - 2x_0 u]} \langle f(x), e^{2iC_{2\alpha}\, xx_0}, k_\alpha(x,u) \rangle$$

$$= e^{\frac{i}{2}{x_0}^2\cot\alpha - i\, x_0\, u\ cosec\alpha}\ FrFT\{e^{ixx_0 cosec\alpha} f(x)\}$$

6.5 Multiplication by e^{iax}:

$$\{FrFT\ e^{iax}\ f(x)\}(u) = e^{\frac{a^2\sin 2\alpha}{2}}\ e^{ia(u - a\sin\alpha)\cos\alpha} [FrFT(f(x)](u - a\sin\alpha)$$

consider

$$\{FrFT\ e^{iax}\ f(x)\}(u)$$

$$= C_{1\alpha} \int_{-\infty}^{\infty} e^{\frac{i}{2\sin\alpha}[(x^2+u^2)\cos\alpha - 2xu]} e^{iax} f(x)\,dx$$

$$= C_{1\alpha} e^{\frac{i}{2}u^2\cos\alpha} \int_{-\infty}^{\infty} e^{\frac{i}{2}x^2\cot\alpha - ix[u\ cosec\alpha\ - a\sin\alpha\ cosec\alpha\ ]} f(x)\,dx$$

$$= C_{1\alpha} e^{\frac{i}{2}[(u - a\sin\alpha)^2 + (a\sin\alpha)^2 + 2(u - a\sin\alpha)(a\sin\alpha)\cot\alpha]}$$

$$\int_{-\infty}^{\infty} e^{\frac{i}{2}x^2\cot\alpha}\ e^{-ix\ cosec\alpha\ [u\ - a\sin\alpha\ ]} f(x)\,dx$$

$$= e^{\frac{a^2\sin 2\alpha}{2}}\ e^{ia[u\ - a\sin\alpha\ ]\cos\alpha}$$

$$\int_{-\infty}^{\infty} C_{1\alpha} e^{\frac{i}{2\sin\alpha}[[x^2+(u-a\sin\alpha)^2]\cos\alpha - 2x(u - a\sin\alpha)]}\ f(x)\,dx$$

$$= e^{\frac{a^2\sin 2\alpha}{2}}\ e^{ia(u - a\sin\alpha)\cos\alpha} \{FrFT(f(x)\}(u - a\sin\alpha)$$

7. CONCLUSION:

This paper presents Generalization of fractional Fourier transform in distributional sense. Some operators as differential, scaling on testing function space is proved. Also some operation transform formulae are also obtained, which will be useful when this transform will be used to solve differential equations.

8. REFERENCES:

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