Generalized Two Dimensional Fractional Sine Transform

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ABSTRACT-

As the sine transform ,cosine transform and Hartley transform are widely use in signal processing, the application of their fractional version in signal/image processing is very promising. In this paper distributional generalized two dimensional fractional sine transform is studied. Some properties are verified. Analyticity theorem of generalized two dimensional fractional sine transform is proved.

Keywords-

Fractional Fourier transform, fractional sine transform, fractional cosine transform, signal processing, image processing, speech processing.

1 Introduction

Before discussing the fractional sine transform (FRST) we first describe the concept of 'fractional operation'. Suppose there is an operation.

$$O(f(x)) = F(k) \quad (1.1)$$

Then its fractional operation (which is denoted by $O^{\alpha}(\ )$, where α is some real number) is the operation satisfying the following properties.

1. Boundary properties

$$O^{0}(f(x)) = f(x), O^{1}(f(x)) = F(k) \quad (1.2)$$

Aditivity properties-

$$O^{b}(O^{a} f(x) = O^{a}(O^{b}(g(x)) = O^{a+b}(g(x)) \quad (1.3)$$

From the additivity property, the inverse of the fractional operation is just $O^{-\alpha}(\ )$

$$O^{-\alpha}(F_{\alpha}(s)) = f(x)$$

Where $$F_{\alpha}(s) = O^{\alpha}(f(x))$$

Because it is free to choose the parameter a, using the fractional operation is more flexible then using the originaloperation, and some problems that cannot be solves by the fractional operation.

The concept of fractional order was only applied to the Fourier transform at first V. Namias constructed the fractional Fourier transform (FRFT)by using the Hermite polynomial in 1980,which put forward a definition of FRFT for the first time . In 1990 C.C.Shih redefined the FRFT based on the state function [1,2]. L.B. Almeida [3] H.M; 0zaktas [4]etc. found out that the FRFT of signal whose power is α equals $\frac{\alpha\pi}{2}$ angle rotation of the signal in its time frequency surface .FRFT is now a hotspot for research and has been widely used in domains like quantum mechanics optics and signal processing .

The success of FRFT in its application has of promoted the development of other kinds of fractional transform ,fractional Hartley transform ,fractional Hadamard transform and fractional cosine transform and fractional sine transform

(FRST) were put forward one by one .Pei Soo-Chang redefined the fractional cosine transform and fractional sine transform based on fractional Fourier transform in 2001 [5,6].

FRST is the extension of sine transform and it has been widely used in domain of digital signal processing [7].

The object we have studied in this paper is generalized two-dimensional Fractional sine transform in distributional sense .We have define Testing function space and Distributional generalized two-dimensional fractional sine transform , some properties of the kernel are verified. Also Analyticity of generalized two dimensional fractional sine transform is proved.

2 Two-dimensional generalized fractional sine transforms

2.1 Two dimensional fractional Sine transform with parameter α of *f(x, y)* denoted by $F_S^{\alpha}(x, y)$ performs a linear operation given by the integral transform.

$$F_S^{\alpha}\{f(x,y)\}(u,v) = \int_0^{\infty}\int_0^{\infty} f(x,y) k_{\alpha}(x,y,u,v)\,dxdy$$

(…2.1.)

Where the Kernel

$$k_s^{\alpha}(x,y,u,v) = \sqrt{\frac{1-i\cot\alpha}{2\pi}}\, e^{\frac{i}{2}(x^2+y^2+u^2+v^2)\cot\alpha}\, e^{i(\alpha-\frac{\pi}{2})}$$
$$\sin(\operatorname{cosec}\alpha.ux).\sin(\operatorname{cosec}\alpha.vy)$$

(...2.2)

2.2 The test function space *E*

An infinitely differentiable complex valued function ϕ on R^n belongs to $E(R^n)$ if for each compact set $I \subset S_{a,b}$,where

$$S_{a,b} = \{x, y : x.y \in R^{n,} |x| \le a, |y| \le b, a > 0, b > 0\}, I \in R^n$$

$$\gamma_{E_{p,q}}(\phi) = \sup_{x,y \in I} \left| D_{x,y}^{p,q} \phi(x,y) \right| < \infty$$

Where p,q=1,2,3……….

Thus E(R^n) will denote the space of all $\phi \in E(R^n)$ with support contained in $S_{a,b}$

Note that the space *E*is complete and therefore a Frechet space. Moreover, we say that *f* is a fractional sine transformable if it is a member of E^*, the dual space of *E*.

3 Distributional two-dimensional fractional Sine transform

The two-dimensional distributional fractional sine transform of $f(x,y) \in E^*(R^n)$ defined by

$$F_s^\alpha\{f(x,y)\} = F^\alpha(u,v) = < f(x,y), k_\alpha(x,y,u,v) >$$

(…3.1)

$$k_\alpha(x,y,u,v) = \sqrt{\frac{1-i\cot\alpha}{2\pi}} e^{\frac{i}{2}(x^2+y^2+u^2+v^2)\cot\alpha} e^{i(\theta-\frac{\pi}{2})}$$
$$\sin(\cos ec\alpha.ux).\sin(\cos ec\alpha.vy)$$

Where, (…3.2)

R.H.S. of eqn(3.1) has a meaning as the application of $f \in E^*$ to $K_\alpha(x,y,u,v) \in E$

It can be extended to the complex space as an entire function given by

$$F_s^\alpha\{f(x,y)\} = F^\alpha(g,h) = < f(x,y), k_\alpha(x,y,g,h) >$$

The right hand side is meaningful because for each $g,h \in C^n, k_\alpha(x,y,g,h) \in E$, as a function of x,y .

4 Properties of Kernel of Generalized two Dimensional fractional Sine transform

4.1 $k_{-\alpha}(x,y,u,v) = (-1)k_\alpha^*(x,y,u,v)$

$$k_{-\alpha}(x,y,u,v) = \sqrt{\frac{1-i\cot(-\alpha)}{2\pi}} e^{\frac{i}{2}(x^2+y^2+u^2+v^2)\cot(-\alpha)}$$
$$e^{i(-\alpha-\frac{\pi}{2})} \sin(\cos ec(-\alpha).ux).\sin(\cos ec(-\alpha).vy)$$

$$k_{-\alpha}(x,y,u,v) = \sqrt{\frac{1+i\cot\alpha}{2\pi}} e^{\frac{-i}{2}(x^2+y^2+u^2+v^2)\cot\alpha} e^{-i(\alpha+\frac{\pi}{2})}$$
$$\sin(\cos ec\alpha.ux).\sin(\cos ec\alpha.vy)$$

$$k_{-\alpha}(x,y,u,v) = \sqrt{\frac{1-(-i)\cot\alpha}{2\pi}} e^{\frac{-i}{2}(x^2+y^2+u^2+v^2)\cot\alpha}$$
$$(\cos(\alpha+\frac{\pi}{2}) - i\sin(\alpha+\frac{\pi}{2}))\sin(\cos ec\alpha.ux).\sin(\cos ec\alpha.vy)$$

$$k_{-\alpha}(x,y,u,v) = -\sqrt{\frac{1-(-i)\cot\alpha}{2\pi}} e^{\frac{-i}{2}(x^2+y^2+u^2+v^2)\cot\alpha}$$
$$(\cos(\alpha-\frac{\pi}{2}) - i\sin(\alpha-\frac{\pi}{2}))\sin(\cos ec\alpha.ux).\sin(\cos ec\alpha.vy)$$

$$k_{-\alpha}(x,y,u,v) = -\sqrt{\frac{1-(-i)\cot\alpha}{2\pi}} e^{\frac{-i}{2}(x^2+y^2+u^2+v^2)\cot\alpha} e^{-i(\alpha-\frac{\pi}{2})}$$
$$\sin(\cos ec\alpha.ux).\sin(\cos ec\alpha.vy)$$

$$k_{-\alpha}(x,y,u,v) = (-1)k_\alpha^*(x,y,u,v)$$

4.2 $K_\alpha(u,v,x,y) = K_\alpha(x,y,u,v)$

$$k_\alpha(u,v,x,y) = \sqrt{\frac{1-i\cot\alpha}{2\pi}} e^{\frac{i}{2}(u^2+v^2+x^2+y^2)\cot\alpha} e^{i(\alpha-\frac{\pi}{2})}$$
$$\sin(\cos ec\alpha.xu).\sin(\cos ec\alpha.yv)$$

$$k_\alpha(u,v.x,y) = \sqrt{\frac{1-i\cot\alpha}{2\pi}} e^{\frac{i}{2}(x^2+y^2+u^2+v^2)\cot\alpha} e^{i(\alpha-\frac{\pi}{2})}$$
$$\sin(\cos ec\alpha.ux).\sin(\cos ec\alpha.vy)$$

$$K_\alpha(u,v,x,y) = K_\alpha(x,y,u,v)$$

4:3 $K_\alpha(-x,-y,u,v) = K_\alpha(x,y,u,v)$

$$k_\alpha(-x,-y,u.v) = \sqrt{\frac{1-i\cot\alpha}{2\pi}} e^{\frac{i}{2}((-x)^2+(-y)^2+u^2+v^2)\cot\alpha} e^{i(\alpha-\frac{\pi}{2})}$$
$$\sin(\cos ec\alpha.u(-x)).\sin(\cos ec\alpha.v(-y))$$

$$k_\alpha(x,y,u,v) = \sqrt{\frac{1-i\cot\alpha}{2\pi}} e^{\frac{i}{2}(x^2+y^2+u^2+v^2)\cot\alpha} e^{i(\alpha-\frac{\pi}{2})}$$
$$\sin(\cos ec\alpha.ux).\sin(\cos ec\alpha.vy)$$

$$K_\alpha(-x,-y,u,v) = K_\alpha(x,y,u,v)$$

$$\int_{-\infty}^{\infty} k_\alpha(x,y,u,v) k_\alpha^*(x,y,u',v') dt = (-1)\delta(u-u',v-v')$$

$$= \int_{-\infty}^{\infty} k_\alpha(x,y,u,v) k_\alpha^*(x,y,u',v') dt$$

$$= (-1)\int_{-\infty}^{\infty} k_\alpha(x,y,u,v) k_{-\alpha}(x,y,u',v') dt$$

$$= (-1) k_0(u,v,u',v')$$

5. Analyticity of the generalized two-dimensional fractional Sine transform.

Theorem 1 Let f(x,y) $\in E^*(R^n)$ and let its fractional Sine transform be defined by (3.2). The $F_s^\alpha(g,h)$ is an analytic on C^n if the $\sup f \subset S_{a,b}$ where

$$S_{a,b} = \{x,y : x.y \in R^{n,} |x| \le a, |y| \le b, a>0, b>0\}$$

$$Moreover\ F_s^\alpha(g,h)\ is\ differentiable\ and$$

$$D_{g,h}^{p,q} F_s^\alpha(g,h) = \langle f(x,y)\rangle,\ D_{g,h}^{p,q} K_\alpha(x,y,g,h) \quad 5.1$$

Proof Let

$$g:(g_1, g_2 \ldots\ldots\ldots g_j \ldots\ldots g_n) \in C^n$$

and

$$h:(h_1, h_2, \ldots\ldots, h_j, \ldots\ldots\ldots h_n) \in C^n.$$

We first prove that,

$$\frac{\partial}{\partial q_j} F_S^\alpha(g,h) =< f(x,y), \frac{\partial}{\partial q_j} k_\alpha(x,y,g,h) >$$

For fined $g_j \neq 0$,chose two concentric circle c and c' with center at g_j and radii r and r1 respectively such that 0<r<r1< $|g_j|$.

Let Δg_j be a complex increment satisfying $0 < |\Delta g_j| < x.$
Consider,

$$\frac{F^\alpha(g_j + \Delta g_j) - F_S^\alpha(g_j)}{\Delta g_j} - < f(x,y), \frac{\partial}{\partial q_j} k_\alpha(x,y,g,h) >$$

$$= \langle f(x,y), \psi_{\Delta q_j}(x,y)\rangle$$

(…5.2)

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for any fixed x,y $\in R^n$ and fixed integer $p = (p_1, p_2, p_3, \ldots\ldots, p_n)$

$$D_X^P k_\alpha(x,y,g,h) = D_X^P [A e^{\frac{i}{2}(x^2+g^2)\cot\alpha} \sin(\operatorname{cosec}\alpha.gx).$$

$$. B(y)],$$

Where $B(y) = e^{\frac{i}{2}(y^2+h^2)\cot\alpha} \sin(\operatorname{cosec}\alpha.hy)$ and

$$A = \sqrt{\frac{1 - i\cot\alpha}{2\pi}} e^{i(\alpha - \frac{\pi}{2})}$$

$$D_X^p k_\alpha(x,y,g,h) = \sqrt{\frac{1-i\cot\alpha}{2\pi}} e^{i(\alpha-\frac{\pi}{2})} e^{\frac{i}{2}(x^2+y^2+g^2+h^2)\cot\alpha}$$

$$\sin(\operatorname{cosec}\alpha.hy) \sum_{n=0}^{p}\sum_{r=0}^{k} \binom{p}{n} \frac{n!}{(k-2r)!r!}$$

$$(i\cot\alpha)^{k-r}(2x)^{k-2r}(\operatorname{cosec}\alpha.gx)^{p-n}.$$

$$\sin(\operatorname{cosec}\alpha.gx + \frac{(p-n)\pi}{2})$$

(...5.3)

Since for any fixed $x, y \in R^n$ and fixed integer p and α is ranging from 0 to π/2.

$D_x^p k_\alpha(x,y,g,h)$ is analytic inside and on c', we have by Cauchy's integral formula

$$D_x^p \Psi_{\Delta g_j}(x,y) = \frac{1}{2\pi i} D_x^p \int_{c'} k_\alpha(x,y,\bar{g},h)[\frac{1}{\Delta g_j}(\frac{1}{z-g_j-\Delta g_j} - \frac{1}{z-g_j}) - \frac{1}{(z-g_j)^2}]dz$$

Where $\bar{g} = g_1, g_2, \ldots g_{j-1}, g_{j+1}, \ldots g_n$

$$D_x^p \Psi_{\Delta g_j}(x,y) = \frac{\Delta g_j}{2\pi i} \int_{c'} \frac{A(x,y,\bar{g},h)}{(z-g_j-\Delta g_j)(z-g_j)^2} dz$$

But for all $z \in c'$ and x restricted to a compact subset of R^n ,0<α<π/2,

$A(x,y,\bar{g},h) = D_x^p k_\alpha(x,y,\bar{g},h)$ is bounded by a constant Q

Moreover, $|z - g_j - \Delta g_j| > r_1 - r > 0$ and $|z - g_j| = r_{1.}$

Therefore we have ,

$$|D_x^p \Psi_{\Delta g_j}(x,y)| = \left| \frac{\Delta g_j}{2\pi i} \int_{c'} \frac{A(x,y,\bar{g},h)}{(z-g_j-\Delta g_j)(z-g_j)^2} dz \right|$$

$$\le \frac{|\Delta g_j| Q}{(r_1 - r) r_1}$$

Similarly, $\left|D_x^q \Psi_{\Delta h_j}(x, y)\right| = \dfrac{\left|\Delta h_j\right| p}{(r_1 - r) r_1}$,

Where $B(x, y, g, \overline{h}) = D_Y^q k_\alpha(x, y, g, \overline{h})$ is bounded by a constant P

Thus, as $\left|\Delta g_j\right| \to 0$, $D_x^p \Psi_{\Delta g_j}(x, y)$ tends to zero uniformly on the compact subsetof R^n,therefore it follows that $\Psi_{\Delta g_j}(x.y)$ converges in $E(R^n)$ to zero.

Since $f(x, y) \in E^*$ we conclude that equation (5.2) also tends to zero.

Therefore $F_\alpha(g, h)$ is differentiable with respect to g_j and h_j. But this is true for all j=1,2,3……n. Hence $F_\alpha(g, h)$ is analytic on C^n and

$$D_{x,y}^{p,q} F_\alpha(g, h) =< f(x, y).D_{g,h}^{p,q} k_\alpha(x, y, g, h)$$

6 CONCLUSIONS

We have extended two-dimensional fractional Sine transform in the distributional generalized sense, the testing function space and Distributional

Generalized two-dimensional fractional Sine transform is defined. Analyticity Theorem and some properties of kernel of generalized two-dimensional fractional Sine transform are proved.

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