g*s-I Closed Sets in Topological Spaces

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ABSTRACT
G.B.Navalagi [2] introduced a new class of set called g*s-closed set in topological space. In this paper , we introduce and study the properties of g*s-I closed and open sets in ideal topological spaces. Also by using , g*s-I closed sets we introduce g*s-I continuous functions, g*s-I closed and open maps.

Keywords
g*s-I closed sets, g*s-I open sets, g*s-I continuous, g*s-I closed and open maps.

1. INTRODUCTION AND PRELIMINARIES
An ideal I on a topological space (X,τ) is a non-empty collection of subsets of X which satisfies the following properties. (1) $A \in I$ and $B \subseteq A$ implies $B \in I$, (2) $A \in I$ and $B \in I$ implies $A \cup B \in I$. An ideal topological space is a topological space (X, τ) with an ideal I on X and is denoted by (X, τ, I). For a subset $A \subseteq X$, $A^*(I,\tau)=\{x \in X : A \cap U \notin I$ for every $U \in \tau\ (X,x)\}$ is called the local function of A with respect to I and τ [4]. We simply write A* in case there is no chance for confusion.A kuratowski closure operator cl*(.) for a topology $\tau^*(I, \tau)$ called theτ *- topology , finer than τ is defined by $cl^*(A) = A \cup A^*$ [9]. If $A \subseteq X$, then cl(A) and int(A) will respectively, denote the closure and interior of A in (X, τ).

Definition 1.1. A subset A of a topological space (X, τ) is called
1)Generalised closed (briefly g-closed) [1],if cl (A) $\subseteq$ U whenever $A \subseteq U$ and U is open in (X, τ)
2) Generalised semiclosed (briefly gs-closed) [11], if scl(A) $\subseteq$ U whenever $A \subseteq U$ and U is open in (X, τ)
3) Semi-Generalised closed (briefly sg-closed) [9], if scl(A) $\subseteq$ U whenever $A \subseteq U$ and U is Semi-open in (X, τ).Every Semi-closed set is sg-closed.
4)Weakly Closed(briefly w-closed) [10], if cl(A) $\subseteq$ U whenever $A \subseteq U$ and U is Semi-open in (X, τ).
5) Weakly Generalised Closed(briefly wg-closed)[7] if cl(int A) $\subseteq$ U whenever $A \subseteq U$ and U is open in (X, τ)
6)α-generalised closed (briefly αg-closed) [4], if α-cl (A) $\subseteq$ U whenever $A \subseteq U$ and U is open in (X, τ)
7)Generalized-α closed (briefly gα-closed) [3], if α-cl(A) $\subseteq$ U whenever $A \subseteq U$ and U is α-open in (X, τ)
8)regular w-closed(briefly rw-closed) [12], if α-cl (A) $\subseteq$ U whenever $A \subseteq U$ and U is regular semi- open in (X, τ)
9) Strongly g-closed [1], if cl (A) $\subseteq$ U whenever $A \subseteq U$ and U is g-open in(X, τ)

The complements of the above mentioned closed sets are called their respective open sets.

Definition 1.2. [1] A subset A of an Ideal topological space (X, τ, I) is said to be αI-open if $A \subseteq int(Cl^*(int(A)))$. The complement of the αI-open set is called αI-closed.

Definition 1.3. A subset A of an ideal topological space (X, τ, I) is said to be a
1) gα - I – closed [16], if α I cl(A) $\subseteq$ U whenever $A \subseteq U$ and U is α - open in X
2) αg - I – closed [16], if α I cl(A) $\subseteq$ U whenever $A \subseteq U$ and U is open in X.
3) wgα-I-closed [14], if αIcl (int(A)) $\subseteq$ U whenever $A \subseteq U$ and U is α-open in (X, τ).
4) wαg-I-closed [14], if αIcl (int(A)) $\subseteq$ U whenever $A \subseteq U$ and U is open in (X, τ).
5) Ig-closed [15] , if $A^* \subseteq U$ whenever $A \subseteq U$ and U is open in X.

The complements of the above mentioned closed sets are called their respective open sets.

Definition 1.4. For a subset A of an ideal topological space (X, τ, I),
1) $sI\text{-}cl(A) = A \cup int^*(cl(A))$
2) $sI\text{-}cl(A) \subseteq cl(A)$
3) $sI\text{-}cl(A) \subseteq scl(A)$

2. g*S-I CLOSED SETS IN TOPOLOGICAL SPACES
Definition 2.1:A subset A of (X,τ,I) is called a g*s-I closed set, if $sI\text{-}cl(A) \subseteq U$, whenever $A \subseteq U$ and U is gs-open in (X, τ, I).The class of all g*s-I closed sets in a topological space (X,τ,I) is denoted by g*s-I-C(X, τ,I).

Remark 2.2: The complement of g*s-I closed set is g*s-I open set.

Theorem 2.3: Every closed set in (X,τ,I) is g*s-I closed set.
Proof: Let A be a closed set in X. Let U be a gs-open set such that $A \subseteq U$. Since A is closed , cl(A) = A , $cl(A) \subseteq U$. But $sI\text{-}cl(A) \subseteq cl(A) \subseteq U$. Therefore $sI\text{-}cl(A) \subseteq U$. Hence A is g*s-I closed set in X.

Theorem2.4: Union of two closed sets is g*s-I closed.
Proof: Let A and B be two closed and hence g*s-I closed sets in X. Let U be a gs-open set in X such that $A \cup B \subseteq U$. Then $A \subseteq U$ and $B \subseteq U$. Since A and B are g*s-I closed sets , we have $sI\text{-}cl(A) \subseteq U$ and $sI\text{-}cl(B) \subseteq U$. Hence $sI\text{-}cl(A \cup B) = sI\text{-}cl(A) \cup sI\text{-}cl(B) \subseteq U$. Therefore $A \cup B$ is g*s-I closed.

Theorem 2.5: Every g*s-I closed set is gs closed.

Proof:Let A be g*s-I closed set in X. Let U be a open set such that A ⊆ U. Since every open set is gs-open and A is g*s-I closed , we have s-cl(A) ⊆ U . Therefore A is gs closed in X.

Theorem 2.6: Every g*s-I closed set in X is sg closed .
Proof:Let A be g*s-I closed set in X. Let U be a Semi-open set in X such that A ⊆ U. Since every Semi-open set is gs-open and A is g*s-I closed , we have s-cl(A) ⊆ U . Therefore A is sg closed in X.

Theorem 2.7: Every g*s-I closed set in X is g*s-closed.
Proof:Let A be g*s-I closed set in X. Let U be a gs-open set in X such that A ⊆ U. Since A is g*s-I closed , we have sI-cl(A) ⊆ U and also s-cl(A) ⊆ U. Therefore A is g*s-closed in X.

Theorem 2.8: Every g*s-I closed set in X is gs-I closed.
Proof:Let A be g*s-I closed set in X. Let U be an open set in X such that A ⊆ U. Since every open set is gs-open and A is g*s-I closed, we have sI-cl(A) ⊆ U. Therefore A is gs-I closed in X.

Theorem 2.9: Every g*s-I closed set in X is sg-I closed.
Proof:Let A be g*s-I closed set in X. Let U be an Semi-open set in X such that A ⊆ U. Since every Semi-open set is gs-open and A is g*s-I closed, we have sI-cl(A) ⊆ U. Therefore A is sg-I closed in X.

Theorem 2.10: Every α-I closed set in X is g*s-I closed.
Proof:Let A be α-I closed set in X. Let U be an open set such that A ⊆ U. Since every open set is gs-open and cl(int*(cl(A))) ⊆ A, we have sI-cl(A) ⊆ A ⊆ U. Therefore A is g*s-I closed in X.
The converse of the above theorems need not be true as seen from the following examples.

Example2.11: Consider the topological space X= {a,b,c} with topology τ={∅, {a}, X } and I={∅, {b}}. The sets {b} and {c} are g*s-I closed but not closed.

Example 2.12: Consider the topological space X= {a,b,c} with topology τ={∅, {a},{a,c}, X } and I={∅, {a}}. The sets {a,b} and {c} are gs closed but not g*s-I closed.

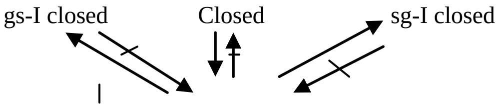
Example 2.13: Consider the topological space X= {a,b,c} with topology τ={∅, {a,b}, X } and I={∅, {a}}. The sets {a,c} and {b,c} are sg closed sets but not g*s-I closed.

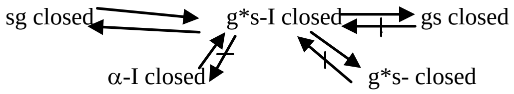
Example 2.14: Consider the topological space X= {a,b,c} with topology τ={∅, {a},{a,c}, X } and I={∅, {a}}. The set { c} is g*s-closed set but not g*s-I closed.

Example 2.15: Consider the topological space X= {a,b,c} with topology τ={∅, {a},{a,c}, X } and I={∅, {a}}. The set { a,b} is gs-I closed set but not g*s-I closed.

Example 2.16: Consider the topological space X= {a,b,c} with topology τ={∅, {a,b}, X } and I={∅, {a}}. The set { a,c} and { b,c} are sg-I closed sets but not g*s-I closed.

Example 2.17: Consider the topological space X= {a,b,c} with topology τ={∅, {a},{c},{a,c}, X } and I={∅, {a}}. The set { a} is g*s-I closed but not α-I closed.
The following diagram is depicted based on the above Theorems:





Remark 2.18: The concept of g*s-I closed Set is independent of the following classes of sets ,namely g-closed set, g* - closed Set, w- closed set, pre- closed set, gα– closed set, αg- closed set, Strongly generalized closed sets, α-g-I closed sets and g-α-I closed set , as seen from the following examples.

Example 2.19: Consider the topological space X= {a,b,c} with topology τ={∅, {a},{a,c}, X } and I={∅, {a}}. In this space , the set { a,b} is g -closed but not g*s-I closed. In topology τ={∅, {a},{c},{a,c}, X } and I={∅, {a}}, the set { a} is g*s-I closed but not g closed.

Example 2.20: Consider the topological space X= {a,b,c} with topology τ={∅, {a,b}, X } and I={∅, {a}}. In this space , the sets { a,c}and{ b,c} are g*-closed but not g*s-I closed. In topology τ={∅, {a},{c},{a,c}, X } and I={∅, {a}}, the set { a} is g*s-I closed but not g* closed.

Example 2.21: Consider the topological space X= {a,b,c} with topology τ={∅, {a,b}, X } and I={∅, {a}}. In this space , the sets { a,c}and{ b,c} are w-closed but not g*s-I closed. In topology τ={∅, {a}, X } and I={∅, {b}}, the sets { b} and{c} are g*s-I closed but not w closed.

Example 2.22: Consider the topological space X= {a,b,c} with topology τ={∅, {a},{a,c}, X } and I={∅, {a}}. In this space , the set { c} is pre-closed but not g*s-I closed. In topology τ={∅, {a}, {c},{a,c},X } and I={∅, {a}}, the set { a} is g*s-I closed but not pre closed.

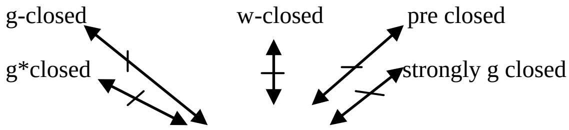
Example 2.23: Consider the topological space X= {a,b,c} with topology τ={∅, {a},{c},{a,c}, X } and I={∅, {a}}. In this space , the set {a} is g*s-I closed but not strongly g closed. In topology τ={∅, {a,b},X } and I={∅, {a}}, the sets { a,c} and {b,c} are strongly g closed but not g*s-I closed.

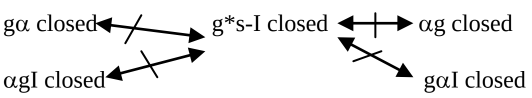
Example 2.24: Consider the topological space X= {a,b,c} with topology τ={∅, {a,b}, X } and I={∅, {a}}. In this space , the sets { a,c}and{ b,c} are gα closed but not g*s-I closed. In topology τ={∅, {a}, {c},{a,c},X } and I={∅, {a}}, the set { a} is g*s-I closed but not gα closed.

Example 2.25: Consider the topological space X= {a,b,c} with topology τ={∅, {a,b}, X } and I={∅, {a}}. In this space , the sets { a,c}and{ b,c} are αg closed but not g*s-I closed. In topology τ={∅, {a}, {c},{a,c},X } and I={∅, {a}}, the set { a} is g*s-I closed but not αg closed.

Example 2.26: Consider the topological space X= {a,b,c} with topology τ={∅, {a,b}, X } and I={∅, {a}}. In this space , the sets { a,c}and{ b,c} are αgI closed but not g*s-I closed. In topology τ={∅, {a}, {c},{a,c},X } and I={∅, {a}}, the set { a} is g*s-I closed but not αgI closed.

Example 2.27: Consider the topological space X= {a,b,c} with topology τ={∅, {a},{c},{a,c}, X } and I={∅, {a}}. In this space , the set {a} is g*s-I closed but not g αI closed. In topology τ={∅, {a,b},X } and I={∅, {a}}, the sets { a,c} and {b,c} are gαI closed but not g*s-I closed.
From the above Examples we get the following diagram:





Definition 2.28: A subset A of (X, τ,I) is called g*s-I open set if A^c is g*s-I closed. The class of all g*s-I open sets is denoted as g*s-I O(X, τ, I).

Definition 2.29: A collection $\{u_i\}$ of g*s-I open sets are called a g*s-I covering for an ideal topological space (X, τ,I), if $X \subseteq \cup u_i$

Definition 2.30: An ideal topological space (X,τ,I) is called a g*s-I compact space if every g*s-I covering has a finite subcover.

Definition 2.31: Let (X, τ, I) be a g*s-I compact space.If A is g*s-I closed subset of X, then A is g*s-I compact.

Theorem 2.32: Every closed subset of a g*s-I compact space is g*s-I compact

Proof: Let Y be a closed set in a g*s-I compact space (X,τ,I). Therefore, Y is g*s-I compact, since every closed set is g*s-I closed.Given, a covering A of Y by g*s-I open sets in X, we can form an open covering *B* of X by adjoining to A the single g*s-I open set X-Y. ie, *B* = A∪(X-Y).Since X is g*s-I compact , some finite subcollection of *B* covers X. If this subcollection contains the set X-Y, discard X-Y, otherwise leave the subcollection alone. The resulting collection is a finite subcollection A that covers Y. Hence Y is g*s-I compact.

3. g*S-I-CONTINUOUS FUNCTIONS IN TOPOLOGICAL SPACES

Levine[5] introduced semi continuous functions using semi open sets. The study on the properties of semi continuous functions is further carried out by Noiri[8], Cossely and Hilbrand and many others .Sundaram introduced the concept of generalized continuous functions which includes the class of continuous functions and studies several properties related to it.

In this section,we introduce the concepts of g*s-I Continuous functions , g*s-I closed maps , g*s-I open maps in Ideal Topological Spaces and study their properties.

Definition 3.1: A map f : (X, τ, I) → (Y, σ) is called g*s-I Continuous if the inverse image of every closed set in Y is g*s-I closed in X.

Theorem 3.2: If a map f : (X, τ, I) → (Y, σ) is continuous , then it is g*s-I Continuous but not conversely.

Proof: Let f : (X, τ, I) → (Y, σ,) be continuous. Let F be any closed set in Y. Then the inverse image $f^{-1}(F)$ is closed in Y. Since every closed set is g*s-I closed, $f^{-1}(F)$ is also g*s-I closed in X. Therefore f is g*s-I Continuous.

The converse of the above theorem need not be true as seen from the following example.

Example 3.3: Consider the topological space X=Y= {a,b,c} with topology τ={∅, {a}, X }, σ = {ϕ, {a,b}, Y} and I={∅, {a}}. Let f : (X, τ, I) → (Y, σ) be an identity map. Then f is not continuous, since for the closed set {c} in Y , $f^{-1}(\{c\})=\{c\}$ is not closed in X. But f is g*s-I Continuous.

Theorem 3.4: If a map f : (X, τ, I) → (Y, σ) is g*s-I continuous , then it is gs- Continuous but not conversely.

Proof: Let f : (X, τ, I) → (Y, σ) be g*s-I continuous. Let F be any closed set in Y. Then F is g*s-I closed in Y. Then the inverse image $f^{-1}(F)$ is also g*s-I closed in X. Since every g*s-I closed set is gs-closed , $f^{-1}(F)$ is also gs-closed in X. Therefore f is gs-Continuous.

The converse of the above theorem need not be true as seen from the following example.

Example 3.5: Consider the topological space X=Y= {a,b,c} with topology τ={∅, {a}, X }, σ = {ϕ, {b}, Y} and I={∅, {a}}. Let f : (X, τ, I) → (Y, σ) be an identity map. Then f is gs-Continuous but not g*s-I continuous, since for the closed set {a,c} in Y , $f^{-1}(\{a,c\})=\{a,c\}$ is not g*s-I closed in X.

Theorem 3.6: Let X=A∪B, where A and B are closed in X. Let f: A→Y and g:B→Y be continuous. If f(x)=g(x) , for every x∈A∩B ,then f and g give a g*s-I continuous function h:X→Y ,defined as h(x)=f(x) , if x∈A and h(x)=g(x), if x∈B

Proof: By Pasting Lemma, there exists a continuous function h:X→Y ,such that h(x)=f(x) , if x∈A and h(x)=g(x), if x∈B. Since every continuous function is g*s-I continuous, h is g*s-I continuous.

Theorem 3.7: If f : (X, τ, I) → (Y,σ,J) and g: (Y,σ,J) → (Z,ν) are g*s-I Continuous then g∘f: (X, τ, I) → (Z,ν) is also g*s-I continuous.

4. g*S-I-CLOSED AND OPEN MAPS IN TOPOLOGICAL SPACES

In this section, we introduce the concepts of g*s-I –Closed maps and g*s-I-Open maps in Ideal Topological Spaces.

Definition 4.1: A map f : (X, τ, I) → (Y, σ, J) is called g*s-I Closed map, if for each closed set F in X , f(F) is g*s-I closed in Y.

Theorem 4.2: If a map f: (X, τ, I) → (Y, σ, J) is a Closed map, then it is g*s-I Closed but not conversely.

Proof: Since every closed set is g*s-I Closed, the result follows.The converse of the above theorem need not be true as seen from the following example.

Example 4.3: Consider the topological space X=Y= {a,b,c} with topology τ={∅, {a,b}, X }, σ = {ϕ, {a}, Y} and I=J={∅, {a}}. Let f: (X, τ, I) → (Y, σ, J) be an identity map. Then f is not a closed map, since for the closed set { c} in X , f({c})={c} is not closed in Y. But f is g*s-I Closed map.

Definition 4.4: A map f : (X, τ, I) → (Y, σ, J) is called g*s-I Open map, if for each open set U in X , f(U) is g*s-I open in Y.

Theorem 4.5: If a map f : (X, τ, I) → (Y, σ, J) is a Open map, then it is g*s-I Open but not conversely.

Proof: Let f : (X, τ, I) → (Y, σ, J) be an open map. Let U be any open set in X. Then f(U) is an open set in Y. Then f(U) is g*s-I open in Y , since every open set is g*s-I open. Therefore f is g*s-I open.The converse of the above theorem need not be true as seen from the following example.

Example 4.6: Consider the topological space X=Y= {a,b,c} with topology τ={∅, {a,b}, X }, σ = {ϕ, {a}, Y} and I=J={∅, {a}}. Let f : (X, τ, I) → (Y, σ, J) be an identity map. Then f is not an open map , since the open set { a,b} in X , is not open in Y. But{ a,b} in X, is g*s-I open map in Y.

Theorem 4.7: If f : (X, τ, I) → (Y, σ, J) and g: (Y,σ,J) → (Z,ν,K) are g*s-I Closed maps ,then g∘f: (X, τ, I) → (Z,ν,K) is also g*s-I closed.

5. REFERENCES

[1] A.Pushpalatha, Studies on Generalizations of Mappings in Topological spaces, Ph.D Thesis, Bharathiar University , Coimbatore , 2000..

[2] Definition Bank in General Topology by G.B.Navalagi.

[3] H.Maki, R.Devi and K.Balachandran ,Generalized α-closed sets in Topology ,Bull.Fukuoka Univ.Ed.Part-III 42 (1993).

[4] H.Maki, R.Devi and K.Balachandran , Associated Topologies of Generalized α-closed sets and α-Generalized closed sets, Mem.Sci.Kochi Univ.Ser.A.Math.15(1994),51-63.

[5] Levine N.Strong Continuity in Topological spaces ,Amer.Math.Monthly 67 (1960) 269

[6] N. Levine , Generalized closed sets in Topology, Rend.Circ.Mat.Palermo 19(1970) ,89-96.

[7] N.Nagaveni , Studies on Generalizations of Homeomorphisms in Topological spaces, Ph.D Thesis, Bharathiar University , Coimbatore , 1999.

[8] Noiri, T.Strong form of Continuity in Topological spaces, Rend .Circ.Mat.Palermo (1986) 107-113

[9] P.Bhattacharya and B.K.Lahari , Semi- Generalized closed sets in Topology,Indian J.Math,29(3),pp.375-382(1987).

[10] P.Sundaram and M.Sheik John , On w-closed sets in Topology,Acta ciencia Indica 4 (2000),389-392.

[11] S.P.Arya and T.Nour ,Characterizations of S-normal Spaces , Indian J.pure Appl.Math.,21,pp.717-719(1990).

[12] S.S.Benchali and R.S.Wali, On Rw- closed sets in Topological spaces, Bull.Malays.Math.Sci.Soc.(2) 30(2)(2007),99-110.

[13] P.Sundaram, Studies on Generalizations of continuous maps in Topological spaces, Ph.D Thesis, Bharathiar University , Coimbatore , 1991

[14] K.Indirani, V.Rajendran and P.Sathishmohan, On wgα-I-continuity and wαg-I-continuity,

[15] M. Navaneethakrishnan and J. Paulraj Joseph, g-closed sets in ideal topological spaces, Acta Math. Hunger, 119(4) (2008), 365 – 371.

[16] M.Rajamani and V.Rajendran, A study on gα-closed sets in ideal topological spaces, M.Phil., Thesis (2009).

[17] Pushpalatha.A and K.Anitha , g*s-Closed sets in topological spaces, Int.J.Contemp.Math. Sciences, Vol.6,2011,no.19, 917-929.