Solution of Differential Equation using Fractional Hartley Transform

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ABSTRACT
This paper is concerned with the definition of generalized fractional Hartley transform. Fractional Hartley transform is extended to the distribution of compact support by using the kernel method and Fractional Hartley transform is used to solve some differential equations.

Key words
Fractional Fourier transforms, Fractional Hartley transform, Differential equation.

1.Introduction:
The fractional integral transforms play an important role in signal processing. Fourier analysis is one of the most frequently used tools in signal processing and many other scientific disciplines.

Namias [2] introduced the concept of Fourier transform of fractional order, which depends on a continuous parameter α. The fractional Fourier transform with $\alpha = 1$ corresponds to the classical Fourier transform and fractional Fourier transform with $\alpha = 0$ corresponds to the identity operator. The fractional Fourier transforms and its properties were discussed in Ozaktas [3]. Bhosale and Chaudhary [1] had extended it to the distribution of compact support.

Using the eigenvalue function, as used in fractional Fourier transform, different integral transform in Fourier class that is cosine transform, sine transform and Hartley transform, are generalized to fractional transform by Pei [5]. For the generalization of fractional Hartley transform, he had shown that for all non negative integer m, $e^{\frac{-t^2}{2}} H_m(t)$ is the eigen function of the Hartley transform and had given the formula for fractional Hartley transform as,

$$H^{\alpha}\{f(t)\}(s) = \int_{-\infty}^{\infty} f(t) K_{\alpha}(t,s)\,dt,$$

where

$$K_{\alpha}(t,s) = \sqrt{\frac{1 - i\cot\phi}{2\pi}}\, e^{i\frac{s^2}{2}\cot\phi}\, e^{i\frac{t^2}{2}\cot\phi}$$
$$\frac{1}{2}\left[(1 - ie^{i\phi}) cas(\csc\phi . st) + (1 + ie^{i\phi}) cas(-\csc\phi . st)\right].$$

In this paper first we have defined generalized fractional Hartley transform in section 2. Fractional Hartley transform is used to solve some differential equations in section 3.

2 GENERALIZED FRACTIONAL HARTLEY TRANSFORM

2.1 The test function space *E(R^n)*

An infinitely differentiable complex valued function ψ on R^n belongs to $E(R^n)$ if for each compact set $K \subset S_a$ where $S_a = \{t \in R^n, |t| \le a, a > 0\}$,

$$\gamma_{E,k}(\psi) = \sup_{t \in K} \left| D_t^k \psi(t) \right| < \infty,$$
$$k = 1,2,3.....$$

Note that the space *E* is complete and therefore a Frechet space.

2.2 The fractional Hartley transform on E'

It can be easily proved that function $K_{\alpha}(t,s)$ as a function of t, is a member of $E(R^n)$,

where

$$K_{\alpha}(t,s) = \sqrt{\frac{1 - i\cot\phi}{2\pi}}\, e^{i\frac{s^2}{2}\cot\phi} . e^{i\frac{t^2}{2}\cot\phi}$$
$$\frac{1}{2}\left[(1 - ie^{i\phi}) cas(\csc\phi . st) + (1 + ie^{i\phi}) cas(-\csc\phi . st)\right],$$

and $\phi = \frac{\alpha\pi}{2}$.

The generalized fractional Hartley transform of $f(t) \in E'(R^n)$, where $E'(R^n)$ is the dual of the testing function space, can be defined as,

$$H^{\alpha}\{f(t)\}(s) = \langle f(t), K_{\alpha}(t,s) \rangle. \qquad (2.2.1)$$

Another simple form of fractional Hartley transform as in Sontakke [6] is

$$H^{\alpha}\{f(t)\}(s) = \sqrt{\frac{1 - i\cot\phi}{2\pi}}\, e^{i\frac{s^2}{2}\cot\phi} \int_{-\infty}^{\infty} e^{i\frac{t^2}{2}\cot\phi}$$
$$\left[\cos(\csc\phi . st) - ie^{i\phi}\sin(\csc\phi . st)\right] f(t)\,dt$$

(2.2.2)

3 SOLUTIONS OF DIFFERENTIAL EQUATIONS

3.1 Solution of Differential Equation

$P(D)u = f$: Consider the differential equation

$$P(D)u = f \qquad (3.1.1)$$

where $f \in E'$ and $P(D) = \sum_{|\beta| \le m} a_\beta D^\beta$ is a linear differential operator of order m with constant coefficients.

Suppose that the equation (3.1.1) possesses a solution u.

Applying the fractional Hartley transform to (3.1.1) and using

$$D_t^n K_\alpha(t,s) = \sqrt{\frac{1 - i\cot\phi}{2\pi}}\, e^{i\frac{s^2}{2}\cot\phi} \sum_{J=0}^{n} \binom{n}{J}.$$

$$\sum_{r=1}^{J} \frac{J!}{(J-r)!(2r-J)!} (i\cot\phi)^r 2^{r-J} t^{2r-J} e^{\frac{i}{2}t^2\cot\phi}$$

$$(\csc\phi.s)^{n-J} \begin{bmatrix} \cos\left(\csc\phi.st + \frac{(n-J)\pi}{2}\right) \\ -ie^{i\phi}\sin\left(\csc\phi.st + \frac{(n-J)\pi}{2}\right) \end{bmatrix}.$$

(3.1.2)

$$H^\alpha\{P(D)u\} = H^\alpha f = f^\Lambda (say) \qquad (3.1.3)$$

For different values of n and J in (3.1.2) we can reform them to the fractional Hartley transform and hence we get

$$P(s,t).u^\Lambda = f^\Lambda$$

where $P(s,t)$ is polynomial in s and t,

$$u^\Lambda = H^\alpha\{u\}.$$

Under the assumption that the polynomial P is such that

$$P\left(D_t^u K_\alpha(t,s)\right) > \xi > 0 \qquad \text{for}$$

$$\xi = \xi_1, \xi_2, \ldots\ldots\xi_n \in IR^n \qquad (3.1.4)$$

Equation (3.1.3) gives

$$u^\Lambda = P\left[D_t^n K_\alpha(t,s)\right]^{-1} f^\Lambda \qquad (3.1.5)$$

Applying inversion of fractional Hartley transform to (3.1.5), we get

$$u = \left[H^\alpha\right]^{-1}\left[\frac{f^\Lambda}{P[D_t^n K_\alpha(t,s)]}\right]. \qquad (3.1.6)$$

Next we show that if $f \in E'$ then P satisfies (3.1.4) then equation (3.1.6) defines a tempered distribution which is the solution of equation (3.1.1) indeed since $f \in E'$ then for $0 < \alpha < 1$, $H^\alpha(f) \in E'$ and hence by assumption (3.1.4) and the definition that if $\theta \in \theta_M$ and $f \in E'$ then the product θf is defined by $\langle \theta f, \phi \rangle = \langle f, \theta\phi \rangle$ $\forall \phi \in E'$ we have

$$\frac{\left[H^\alpha(f)\right]}{P[D_t^n K_\alpha(t,s)]} \in E' \text{ and so } u \in E'.$$

To show that u satisfies (3.1.1) we apply H^α to both sides of (3.1.6) and (3.1.5). Since tempered distributions admit multiplication by polynomials, we hence obtain equality (3.1.3). Finally applying $\{H_\alpha\}^{-1}$ to (3.1.3), we get (3.1.1).

3.2 Solution of Differential Equation $P(\Lambda_t^*)u = f$

3.2.1 Operator Λ_t and Kernel of Fractional Hartley Transform:

Let us consider the operator

$$\Lambda_t = t^{-1}D + \frac{(\csc\phi.s)t^{-1}.[\sin(\csc\phi\, st) + ie^{i\phi}\cos(\csc\phi.st)]}{[\cos(\csc\phi.st) - ie^{i\phi}\sin(\csc\phi.st)]}$$

.

$$\Lambda_t(K_\alpha(t,s)) = \sqrt{\frac{1-i\cot\phi}{2\pi}}\, e^{i\frac{s^2}{2}\cot\phi}\left[(i\cot\phi)e^{i\frac{t^2}{2}\cot\phi}[\cos(\csc\phi.st) - ie^{i\phi}\sin(\csc\phi.st)]\right]$$

$$= (i\cot\phi)K_\alpha(t,s). \qquad (3.2.1)$$

$\Lambda_t^2(K_\alpha(t,s)) = \Lambda_t\left[i\cot\phi K_\alpha(t,s)\right]$ by equation (3.2.1)

$$= (i\cot\phi)^2 K_\alpha(t,s).$$

Similarly if we operate the operator Λ_t again on $(i\cot\phi)^2 K_\alpha(t,s)$, we get

$$\Lambda_t^3\,(K_\alpha(t,s)) = (i\cot\phi)^3\,K_\alpha(t,s) \text{ and so on}$$

$$\Lambda_t^k\,(K_\alpha(t,s)) = (i\cot\phi)^k\,K_\alpha(t,s)\,.$$

Since the operator $\Lambda_t^k(K_\alpha(t,s)) = (i\cot\phi)^k K_\alpha(t,s)$ is obviously linear and continuous for $0<\alpha\le 1$

$$H^\alpha\left[\Lambda_t^k f(t)\right] = \left\langle \Lambda_t^k f(t), K_\alpha(t,s)\right\rangle$$

for all $f\in E'$

$$= \left\langle f(t), \Lambda_t^k K_\alpha(t,s)\right\rangle$$

$$= \left\langle f(t), (i\cot\phi)^k K_\alpha(t,s)\right\rangle.$$

We therefore have

$$H^\alpha\left[\Lambda_t^k f(t)\right] = (i\cot\phi)^k H^\alpha\{f(t)\}$$ for all $f\in E'$

and for $0<\alpha\le 1$ we define an operator $\Lambda_t^k : E' \to E'$ by means of the relation

$$\left\langle \Lambda_t^k f(t), \phi(t)\right\rangle = \left\langle f(t), \Lambda_t^k \phi(t)\right\rangle$$

where

$$\Lambda_t = t^{-1}D + \frac{(\csc\phi.s)t^{-1}.\left[\sin(\csc\phi\, st) + ie^{i\phi}\cos(\csc\phi.st)\right]}{\left[\cos(\csc\phi.st) - ie^{i\phi}\sin(\csc\phi.st)\right]}$$

$$,\quad D = \frac{d}{dt}$$

for all $f\in E'$ and $\phi\in E$. The operator Λ_t^* is called the adjoint operator of Λ_t. For each $k = 1,2,3,.........$ we easily get

$$\left\langle (\Lambda_t^*)^k f(t),\, \phi(t)\right\rangle = \left\langle f(t),\, (\Lambda_t^*)^k \phi(t)\right\rangle.$$

It can be readily shown that if f is a regular distribution is generated by an element in $D(I)$ then $\Lambda_t^* f = \Lambda_t f$.

For each $k = 1,2,3,.........$ and for $0<\alpha\le 1$ we have

$$\left\langle (\Lambda_t^*)^k f(t),\, K_\alpha(t,s)\right\rangle = \left\langle f(t),\, (\Lambda_t^*)^k K_\alpha(t,s)\right\rangle$$

$$= \left\langle f(t),\, (i\cot\phi)^k K_\alpha(t,s)\right\rangle$$

$$. \qquad = (i\cot\phi)^k \left\langle f(t),\, K_\alpha(t,s)\right\rangle.$$

Thus we arrive at the important results, for each $k = 1,2,3,.........$ and for $0<\alpha\le 1$,

$$H^\alpha\left[(\Lambda_t^*)^k f(t)\right] = (i\cot\phi)^k H^\alpha\{f(t)\} \quad (3.2.2)$$

for all $f\in E'$.

3.2.2 Solution of $P(\Lambda_t^*)u = f$:

Consider the differential equation

$$P(\Lambda_t^*)u = f\,. \qquad (3.2.3)$$

Where $f\in E'$ and P is any polynomial degree m. Suppose that the equation (3.2.3) possesses a solution u.

Applying the fractional Hartley transform to (3.2.3) and using the formula (3.2.2), we get

$$H^\alpha\left[P(\Lambda_t^*)u(t)\right] = f^\Lambda$$

$$P(i\cot\phi)H^\alpha\left[u(t)\right] = f^\Lambda \qquad (3.2.4)$$

Let $\quad u^\Lambda = H^\alpha\{u\},\quad f^\Lambda = H^\alpha\{f\}.$

If we further assume that the polynomial P is such that

$$\left|P(i\cot\phi)\right| < \in\, > 0 \text{ for } 0<\alpha\le 1. \qquad (3.2.5)$$

Then under this assumption (3.2.4) gives

$$u^\Lambda = \left[P(i\cot\phi)\right]^{-1} f^\Lambda. \qquad (3.2.6)$$

Applying inversion of fractional Hartley transform to (3.2.6), we get

$$u = \left[H^\alpha\right]^{-1}\left(\frac{f^\Lambda}{P(i\cot\phi)}\right). \qquad (3.2.7)$$

Next we show that if $f\in E'$ and P satisfy (3.2.5) then equation (3.2.7) defines a tempered distribution which is the solution of equation (3.2.3).

Indeed, since $f\in E'$ then for $0<\alpha\le 1$ $H^\alpha\{f(t)\}\in E'$ and hence by assumption (3.2.5) and $f\in E'$, product θf is defined by

$$\langle \theta f\ \phi\rangle = \langle f\ \theta\phi\rangle \quad \forall \phi\in E'$$

We have $\dfrac{\left[H^\alpha(f)\right]}{P\left[i\cot\phi\right]}\in E'$ and so $u\in E'$.

To show that u satisfies (3.2.3) we apply H^α to both sides of (3.2.7) and (3.2.6). Since tempered distributions admit multiplication by polynomials, we hence obtain equality (3.2.4). Finally applying $\{H_\alpha\}^{-1}$ to (3.2.4), we get (3.2.3).

$$\{H_\alpha\}^{-1}\left(P(i\cot\phi)H^\alpha[u(t)]\right)=\{H_\alpha\}^{-1} f^\wedge$$

$$\left(P(i\cot\phi)u\right)=\{H_\alpha\}^{-1} H^\alpha\left\{f^\wedge\right\}$$

$$\left(P(i\cot\phi)u\right)= f^\wedge .$$

CONCLUSION

In this paper we have defined generalized fractional Hartley transform. Fractional Hartley transform is extended to the distribution of compact support by using the kernel method and solution of differential equations is found by using fractional Hartley transform.

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