# Method of Initial Functions for Composite Laminated Beams

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## ABSTRACT
In this paper method of initial functions is used for the study of composite laminated beams. The distribution of bending and shear stresses in composite laminated beams are different from beams of small thickness. The equations of two dimensional elasticity have been used for deriving governing equations. The order of the derived equations depends on the stage at which the series representing the stresses and displacements are truncated. No assumptions regarding physical behavior of beams are made. The beam theories which are based on assumptions are of a practical utility in the case of beams of moderate thickness. However in the case of thick or laminated beams it becomes difficult to obtain useful results using these theories.

## 1. INTRODUCTION

Composite laminated beams are widely used in many structures, because this concept is very suitable for the development of lightweight structures. Beams that are built of more than one material are called composite beams. Examples are bimetallic beams, sandwich beams, laminated beams and reinforced concrete beams. It is difficult to analyze the laminated beams by the same bending theory we used for ordinary beams. In the present paper equations governing the flexure of composite laminated beams are derived without making any assumption regarding the physical behavior of beams. The method of initial functions (MIF) has been used for the purpose of deriving the equations. The method of initial function (MIF) is an analytical method of elasticity theory. The method makes it possible to obtain exact solutions of different types of problems, i.e., solutions without the use of hypotheses about the character of stress and strain.

According to this method, the basic desired functions are the displacements and stresses.

Method of initial functions is used for two dimensional elasto dynamic problems for plain stress and plain strain conditions [2]. And it is used for the analysis of thick circular plates. The governing equations are derived from the three-dimensional elasticity equations in cylindrical polar coordinates using Maclaurin series [6]. MIF has been applied for deriving higher order theories for laminated composite thick rectangular plates [5]. The governing differential equations of plate with arbitrary thickness and basic equations of three-dimensional theory of elasto-dynamics are formulated using MIF [8]. They have used MIF for the static analysis of simply supported, orthotropic, and laminated circular cylindrical shell of revolution subjected to axisymmetric load.By using the continuity conditions of displacements and stresses on each interface between adjacent layers, the state equation for the laminate is obtained [1].

In the case of laminated beams it is quite difficult to assume a distribution of stresses and deflection with a fair amount of accuracy. It requires to developed simple models to explain this behavior as a function of material, geometry and loading parameters [7]. Developed governing equations for composite laminated deep beams by using method of initial functions. The beam theory developed can be used for beam sections of any depth [3].Applied method of initial functions (MIF) for the analysis of orthotropic deep beams and compared the results with the available theory based on assumptions [4].

## 2. PROBLEM FORMULATION

The laminated composite beam consists of $N$ number of layers, serially numbered beginning from the bottom most layer. The thickness of any $i^{th}$ layer ($i = 1,\ldots\ldots\ldots N$) is equal to $h_i$ and its elastic constants are $E_{x,i}$ , $E_{y,i,}$ , $G_i$, $\mu_{xy,i}$ , and $\mu_{yx,i.}$

Each layer has its own local coordinates system $x_i$ and $y_i$ ($i = 1,\ldots\ldots.N$) which is parallel to the Cartesian co-ordinate system $x$ and $y$ for the overall beam. The principal axes of orthotropy of each layer are parallel to the coordinate axes.

The constitutive relation for the material in the $i^{th}$ layer ($i = 1, \ldots\ldots N$) are :

$$\sigma_x = C_{11,i}\varepsilon_x + C_{12,i}\varepsilon_y$$

$$\sigma_y = C_{12,i}\varepsilon_x + C_{22,i}\varepsilon_y \quad (1)$$

$$\tau_{xy} = C_{33,i}\gamma_{xy}$$

Where $\sigma_x$, $\sigma_y$ and $\tau_{xy}$ are bending, normal and shear stresses respectively.

And $\varepsilon_x$ and $\varepsilon_y$ are strains in $x$ and $y$ directions respectively.

The constants $C_{11,i}$ to $C_{33,i}$ expressed in terms of the elastic moduli of the material, for orthotropic material.

The equations of equilibrium for solids ignoring the body forces for two dimensional cases are:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \qquad (2)$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \qquad (3)$$

For small displacements, the strain-displacement relations are:

$$\varepsilon_x = \frac{\partial u}{\partial x}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} \qquad (4)$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

Where $u$ and $v$ are displacements in $x$ and $y$ directions.

Local coordinate's $x_i$ and $y_i$ for each layer lie in the same plane. Hence, the operators $\alpha$ and $\beta$ and the coordinates $x$ and $y$ need not have suffix $i$. stresses, strain and displacements can be expressed as functions of the global coordinates and need not have subscripts.

Eliminating $\sigma_x$ from above equations the following equations are obtained, which can be written in matrix form as

$$\frac{\partial}{\partial y}\begin{bmatrix} u \\ v \\ Y \\ X \end{bmatrix} = \begin{bmatrix} 0 & -\alpha & 0 & 1/G \\ C_{1,i}\alpha & 0 & C_{2,i} & 0 \\ 0 & 0 & 0 & -\alpha \\ C_{3,i}\alpha^2 G & 0 & C_{1,i}\alpha & 0 \end{bmatrix}\begin{bmatrix} u \\ v \\ Y \\ X \end{bmatrix} \qquad (5)$$

Where,

$$X = \tau_{xy} \quad , Y = \sigma_y = C'_{12}\varepsilon_x + C'_{22}\varepsilon_y$$

$$C_{1,i} = \frac{-C_{11,i}}{C_{22,i}}; C_{2,i} = \frac{1}{C_{22,i}}; C_{3,i} = \frac{C_{12,i}}{C_{22,i}} - C_{11,i}$$

Expressions for the constants $C_{11}$ to $C_{33}$ are given in Appendix.

The equation (5) is written as:

$$\frac{\partial}{\partial y}\{S\} = [D_i]\{S\} \qquad (6)$$

The solution of equation (6) is

$$\{S\} = \left[e^{[D_i]y_i}\right]\{S_i\} \qquad (7)$$

Where $\{S_i\}$, is the vector of initial functions, as the value of the state vector $\{S\}$, at the bottom of the i-th layer i.e. at $y_i=0$, *(i = 1,…….N).*

Hence, $\{S_i\} = [u_i, \quad v_i, \quad Y_i, \quad X_i]^T$ (8)

If $u_i$, $vi$, $Y_i$ and $X_i$ are values of $u$, $v$, $Y$ and $X$ respectively, at the bottom plane of the $i^{th}$ layer

*(i =1,…… N)*

The solution of equation (7) can be written as:

$$\{S\} = [L_i]\{S_i\} \qquad (9)$$

The equation (9) represents the general solution of two-dimensional problem for orthotropic materials.

Where $[L_i] = e^{[D_i]y_i}$ (10)

The transfer matrix $[L_i]$ relates the stresses and displacements at the bottom plane of the $i^{th}$ layer to the same at any other parallel plane within the same layer *(i =1,……, N).*

It is a square matrix of the form

$$[L_i] = \begin{bmatrix} L_{uu} & L_{uv} & L_{uY} & L_{uX} \\ L_{vu} & L_{vv} & L_{vY} & L_{vX} \\ L_{Yu} & L_{Yv} & L_{YY} & L_{YX} \\ L_{Xu} & L_{Xv} & L_{XY} & L_{XX} \end{bmatrix}, \quad (11)$$

Expending (10) in the form of a series

$$[L_i] = [I] + y_i[D_i] + \frac{y_i^2}{2!}[D_i]^2 + ....... \quad (12)$$

Where, [I] is a unit matrix.

The truncation of series (12) depend on the order of the beam theory desired.

## 3. ANALYSIS OF THE COMPOSITE LAMINATED BEAMS

In the case of a layered composite beam loaded at the top surface, the state of stresses and displacements at the free bottom surface of the beam is given by:

$$\{S_1\} = [u_1, \quad v_1, \quad 0, \quad 0]^T \quad (13)$$

Let

$$\{S_T\} = [u_T, \quad v_T, \quad Y_T, \quad X_T]^T \quad (14)$$

Where $u_T$, $v_T$, $Y_T$ and $X_T$ are the values of stresses and displacements at the top surface of the
layered beam.

Relating the stresses and displacements at the top surface of the layer to those at the bottom surface by successive application of the transfer matrix [$L_i$] across each layer, one obtains:

$$\{S_T\} = [A][S_1] \quad (15)$$

Where,

$$[A] = [L_N]_{y_N=h_N} ..................[L_2]_{y_2=h_2} .[L_1]_{y_1=h_1}$$
(16)

The terms of the matrix [A] are evaluated after expanding the exponential in the form of a series.

The matrix has a form:

$$[A] = \begin{bmatrix} A_{uu} & A_{uv} & A_{uY} & A_{uX} \\ A_{vu} & A_{vv} & A_{vY} & A_{vX} \\ A_{Yu} & A_{Yv} & A_{YY} & A_{YX} \\ A_{Xu} & A_{Xv} & A_{XY} & A_{XX} \end{bmatrix} \quad (17)$$

The equation (15) relates the boundary conditions at the top surface to those at the bottom surface and is useful for deriving governing differential equations for a layered beam having a particular number of layers.

The method adopted for analyzing layered beams involves the determination of initial functions at the bottom surface of the beam by relating them through the matrix [A] to the stresses at the top surface.

## 4. APPLICATION TO THE PROBLEM OF COMPOSITE LAMINATED BEAM HAVING TWO LAYERS

A composite beam consists of the two layers. Therefore the matrix [A] becomes

$$[A] = [L_2]_{y_2=h_2} .[L_1]_{y_1=h_1} \quad (18)$$

Where $h_1$ and $h_2$ are the thickness of two layers.

The conditions at top are given by:

$$\{S_T\} = [u_T, \quad v_T, \quad -p, \quad 0]^T \quad (19)$$

Substituting the expressions (13) and (17) in the equations (15) we get:

$$A_{Xu}u_1 + A_{Xv}v_1 = 0 \quad (20)$$

$$A_{Yu}u_1 + A_{Yv}v_1 = -p \quad (21)$$

These equations are exactly satisfied by

$$u_1 = A_{Xv}\phi, \quad (22)$$

$$v_1 = A_{Xu}\phi \quad (23)$$

Where φ, is an unknown auxiliary function substituting the value of $u_1$ and $v_1$ from the equations (22) and (23) in the equation (21), the differential equation governing the problem of a normally loaded composite beam is obtained:

$$(A_{Yu}.A_{Xv} - A_{Yv}.A_{Xu})\phi = -p \quad (24)$$

The order of the governing differential equation (28) depends on the order of the terms in the matrix [A].

The auxiliary function $\varphi$ is chosen such that it satisfies the governing differential equation (24), as well as the boundary conditions at the edges of the beam. Initial functions are obtained from equations (22) and (23). By operating on the initial functions by the transfer matrix $[L_i]$ successively across each layer, we can determine the stresses and displacements, within the entire beam.

## 5. CONCLUSION

The stresses evaluated at top surface should be quite close in value to the intensities of the corresponding applied loads. MIF have advantage over other theories because no assumptions regarding physical behavior of beams are made. MIF have capabilities to converge to an exact linear elasticity solution and so provide a governing equation of desired order according to the requirements of a beam problem of any specific thickness and material orthotropy. MIF gives accurate results in case of small thickness, large thickness and layered members.

## 6. NOTATION

$l$ - Span of beam
$E$ - Young's modulus of Elasticity
$G$ - Shear modulus of Elasticity
$\mu$ - Poisson's ratio

## 7. APPENDIX

$$C'_{11} = \frac{E_x}{1-\mu_{xy}\mu_{yx}},$$

$$C'_{12} = \frac{E_y\mu_{xy}}{1-\mu_{xy}\mu_{yx}},$$

$$C'_{22} = \frac{E_y}{1-\mu_{xy}\mu_{yx}},$$

$$C'_{33} = G_{xy},$$

$$\mu_{xy} = \mu_{yx}\frac{E_x}{E_y}$$

## 8. REFERENCES

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