Sampling Theorems for Fractional Laplace Transform of Functions of Compact Support

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ABSTRACT

Linear canonical transform is an integral transform with four parameters and has been proved to be powerful tool for optics, radar system analysis, filter design etc. Fractional Fourier transform and Fresnel transform can be seen as a special case of linear canonical transform with real parameters. Further generalization of linear canonical transform with complex parameters is also developed and Fractional Laplace transform is one of the special case of linear canonical transform with complex entities. Here we have studied the fractional Laplace transform of a periodic function of compact support. New sampling formulae for reconstruction of the functions that are of compact support in fractional Laplace transform domain have been proposed. More specifically it is shown that only (2k+1) coefficients are sufficient to construct any fractional Laplace transform domain of periodic function with compact support, where k is the order of positive highest nonzero harmonic component in the $\propto^{\text{th}}$ domain.

Keywords: *Periodic function, Fractional Laplace transform, Sampling theorem*.

1. INTRODUCTION

The linear canonical transform is an integral transform with four parameters *a, b, c, d.* It was first introduced in 1970 and has found to be useful in many applications in [4, 5]. Fractional Fourier transform, fractional Laplace transform, scaling operations are all special cases of linear canonical transform. The properties of the fractional Laplace transform have been studied in [2]. The convolution theorem in two versions of fractional Laplace transform has also been studied in [3].

Sampling theory can be used in any discipline where function need to be reconstructed from sampled data. The fundamental result in this theory is 'sampling theorem'. Sampling theorem is the basic result for converting the continuous functions to discrete functions. Sampling theorems in the context of fractional Fourier transform have been investigated in [9, 10]. The Sampling theorem for the linear canonical transform of function of compact support in time domain has been derived in [7]. Sampling theorem in fractional Fourier transform of band limited periodic functions have been derived in [6].

After a brief introduction, the review of the fractional Laplace transform and preliminaries are given in terms of some basic definitions related with fractional Laplace transform in section 2. The expressions for the function which is 0-periodic and of $\frac{\pi}{2}-$ compact support is given in section 3. In section 4, two sampling relations in fractional Laplace transform domain of periodic function with compact support are presented. Here we have considered the function of $\left(\propto-\frac{\pi}{2}\right)$ period with $\propto-$ compact support and the function of 0- period with $\propto$-compact support, $\propto\neq\frac{\pi}{2}$. It is also shown that (2k+1) samples are required for reconstruction of fractional Laplace transform. Finally the paper is concluded in section 5.

2. PRELIMINARIES

We now give some basic definitions involving fractional Laplace transform of a function $f(t)$ which are useful in this correspondence.

2.1 Fractional Laplace transform: In full analogy with fractional Fourier transform, Torre [8] introduced fractional Laplace transform.

Fractional Laplace transform of a function $f(t)$, denoted by $L^{\propto}$ (u) , given by

$L^{\propto}(u)=\int_{-\infty}^{\infty} f(t)K_{\propto}(t,u)dt,$ where $f(t)$ is any square integrable function and

$$K_{\propto}(t,u)=\sqrt{\frac{1-icot\propto}{2\pi i}}\, e^{\frac{t^2}{2}cot\propto+\frac{u^2}{2}cot\propto-tucosec\propto}, \quad \propto \text{ is not multiple of } \pi$$
$$=\delta(t-u), \quad \propto \text{ is multiple of } \pi \qquad (1)$$

Inversely

$$f(t)=\int_{-\infty}^{\infty} L^{\propto}(u)\, K_{\propto}{}^{*}(t,u)du$$

$\propto=\frac{p\pi}{2}$, denotes the rotation angle of the transformed function for fractional Laplace transform and " $*$ " is the complex conjugate operator. The fractional Laplace transform is reduced to conventional Laplace transform when $\propto=\frac{\pi}{2}$.

2.2 Periodic function: A function $f(t)$ is said to be $\propto$-periodic in fractional Laplace transform domain with period $u_{\propto}$ if its fractional Laplace transform with angle $\propto$ satisfies,

$L^{\propto}(u)=L^{\propto}(u+A_{\propto})$, for all u, for $-\pi<\propto<\pi$.

2.3 Definition: A function $f(t)$ is said to be of $\propto-$ compact support in fractional Laplace domain if its fractional Laplace transform with angle $\propto$**,**

$L^{\propto}(u)=0$ for $|u|>B_{\propto}$, where $-\pi<\propto<\pi$.

2.4 Definition: The fractional Laplace transform of exponential function as given by [2] is

$$L^{\propto}\left(e^{iat}\right)=\sqrt{1+itan\propto}\quad e^{-\left(\frac{u^2-a^2}{2}\right)tan\propto}\{cos(ausec\propto)+isin(ausec\propto)\} \qquad (2)$$

3. Fractional Laplace transform of 0-periodic function with $\frac{\pi}{2}$- compact support in fractional Laplace transform domain:

If a function $f(t)$ is 0- periodic and of highest frequency $\frac{k}{T}$ satisfying the Dirichlet conditions , where T is the period then $f(t)$ can be reconstructed from its (2k+1) samples as [1].

$$f(t) = \sum_{m=0}^{2k} f(m\tau) g_m(t) \quad (3)$$

$$= f(0)g_0(t) + f(\tau)g_1(t) + \ ... + f(2k\tau)g_{2k}(t) \ ,$$

$g_m(t)$ is periodic in T, As shown in [Brown] it is to be expanded in terms of Fourier series.

$$g_m(t) = \sum_{k=-\infty}^{\infty} \sin c \,.\frac{1}{\tau}(t - m\tau - kT)$$

$$= \sum_{n=-k}^{k} (2k+1)^{-1} e^{i\left(\frac{2\pi}{T}\right)n(t-m\tau)} \quad (4)$$

$$= \sum_{n=-k}^{k} (2k+1)^{-1} e^{i\left(\frac{2\pi nt}{T}\right)} e^{-i\left(\frac{2\pi n}{T}\right)m.\left(\frac{T}{2k+1}\right)}$$

$$= \sum_{n=-k}^{k} \left\{(2k+1)^{-1} e^{-in\left(\frac{2\pi m}{2k+1}\right)}\right\} e^{i\left(\frac{2\pi}{T}\right)nt}$$

$$= \sum_{n=-k}^{k} C_n \, e^{i\left(\frac{2\pi}{T}\right)nt} \quad (5)$$

Where $C_n = \begin{cases} (2k+1)^{-1} e^{-in\left(\frac{2\pi m}{2k+1}\right)} & ,n \le k \\ 0 & otherwise \end{cases}$

Use of equation (5) in (3) and taking fractional Laplace transform on both side,

$$L^{\propto}\{f(t)\}(\mathrm{u}) = \sum_{m=0}^{2k} f(m\tau) \left[\sum_{n=-k}^{k} C_n \, L^{\propto}\left\{e^{i\left(\frac{2\pi nt}{T}\right)}\right\}(u)\right]$$

(6)

$$= \sum_{m=0}^{2k} f(m\tau) \sum_{n=-k}^{k} C_n \sqrt{1 + itan \propto} \; e^{-\left(\frac{u^2 - \left(\frac{2\pi n}{T}\right)^2}{2}\right)tan\propto}$$

$$\left\{cos\left(\frac{2\pi n}{T} usec \propto\right) + isin\left(\frac{2\pi n}{T} usec \propto\right)\right\}$$

$$= \sqrt{1 + itan \propto} \; \mathrm{e}^{-\frac{\mathrm{u}^2}{2}\tan\propto}$$

$$\sum_{m=0}^{2k} f(m\tau) \left[\sum_{n=-k}^{k} C_n \, e^{\frac{2\pi^2 n^2}{T^2} tan\propto} e^{i\frac{2\pi n}{T} u.sec\alpha}\right] \quad (7)$$

Let

$$\varphi(u, \propto, T) = \sum_{m=0}^{2k} f(m\tau) \left[\sum_{n=-k}^{k} C_n \, e^{\frac{2\pi n}{T}\left(\frac{\pi n tan\alpha}{T} + iu.sec\alpha\right)}\right]$$

(8)

$\varphi(u, \propto, T)$ contains the main features of the transformed function . For the periodic function $f_1(T)$, when T=1 and $\tau = \frac{T}{2k+1}$ equation (7) and (8) becomes

$$L^{\propto}\{f(t)\}(\mathrm{u}) = \sqrt{1 + itan \propto} \; \mathrm{e}^{-\frac{\mathrm{u}^2}{2}\tan\propto}$$

$$\sum_{m=0}^{2k} f\left(m\frac{T}{2k+1}\right) \left[\sum_{n=-k}^{k} C_n \, e^{\frac{2\pi^2 n^2}{T^2} tan\propto} e^{i\frac{2\pi n}{T} u.sec\alpha}\right]$$

(9)

Let $\varphi(u, \propto, T) =$

$$\sum_{m=0}^{2k} f\left(m\frac{T}{2k+1}\right) \left[\sum_{n=-k}^{k} C_n \, e^{\frac{2\pi n}{T}\left(\frac{\pi n tan\alpha}{T} + iu.sec\alpha\right)}\right]$$

(10)

Sampling theorem in fractional Laplace transform of a function with 0- periodic and $\frac{\pi}{2}-$compact support can be represented by equation (9) and $f(t)$ can be calculate from (2k+1) samples of it, in the time domain.

When $\propto = \pi\left(n + \frac{1}{2}\right)$ then $\tan\propto = \infty$, $sec \propto = \infty$.

$\varphi(u, \propto, T)$ is periodic for $\propto \neq \pi\left(n + \frac{1}{2}\right)$ with period equal to $T|cos \propto|$.

4. Sampling theorem for $\left(\propto - \frac{\pi}{2}\right)$ periodic function with compact support $\propto$ in fractional Laplace transform domain:

We have discussed the sampling theorem in fractional Laplace transform of 0 -periodic function with $\frac{\pi}{2}-$compact support and it is to be used to reconstruct the function or its fractional Laplace transform from the samples of the function. Here we consider the sampling relation for the compact support $\propto$ and the function of $\left(\propto - \frac{\pi}{2}\right)$ period and compact support$\propto$.

4.1 Theorem: Any function $f(t)$ of $\left(\propto - \frac{\pi}{2}\right)$ period with compact support $\propto$ with fundamental frequency $u_{\propto_0}$, can be reconstructed uniquely with (2k+1) samples of its fractional Laplace transform of order $\propto$,that is by

$$f(t) = e^{\frac{t^2}{2}cot\propto'} e^{\frac{i\pi}{4}} \sqrt{\frac{1 - cot\alpha'}{cot\alpha'}} \sum_{n=-k}^{k} C_n e^{\frac{\left(itcsc\propto' + nu_{\propto_0}\right)^2}{2cot\propto'}}$$

where $\propto' = \frac{\pi}{2} - \propto$ and $u_{\propto - \frac{\pi}{2}}$ is the periodic time in the $(\propto - \frac{\pi}{2})$ domain. ‘ $C_n{}'$ are the coefficients of the periodic function $L^{\propto - \frac{\pi}{2}}(u)$. K is the order of the highest nonzero harmonic component in the $\propto^{th}$ domain.

Proof: Using inversion formula in fractional Laplace transform

$$f(t) = \sqrt{\frac{1 + icot\left(\propto - \frac{\pi}{2}\right)}{2\pi i}}$$

$$\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}cot\left(\propto - \frac{\pi}{2}\right) - \frac{u^2}{2}cot\left(\propto - \frac{\pi}{2}\right) + tucsc\left(\propto - \frac{\pi}{2}\right)} L^{\propto - \frac{\pi}{2}}(\mathrm{u}).\,\mathrm{du}$$

$$= \sqrt{\frac{1 - itan\propto}{2\pi i}} \int_{-\infty}^{\infty} L^{\propto - \frac{\pi}{2}}(\mathrm{u}) e^{\frac{t^2}{2}tan\propto + \frac{u^2}{2}tan\propto - tusec\propto} \,\mathrm{du}$$

$$= \sqrt{\frac{1 - icot\left(\frac{\pi}{2} - \propto\right)}{2\pi i}}$$

$$\int_{-\infty}^{\infty} L^{\propto - \frac{\pi}{2}}(\mathrm{u}) e^{\frac{t^2}{2}cot\left(\frac{\pi}{2} - \propto\right) + \frac{u^2}{2}cot\left(\frac{\pi}{2} - \propto\right) - tucsc\left(\frac{\pi}{2} - \propto\right)} .\,\mathrm{du}$$

$$= e^{\frac{t^2}{2}cot\propto'} \sqrt{\frac{1 - icot \propto'}{2\pi i}} \int_{-\infty}^{\infty} L^{\propto - \frac{\pi}{2}}(\mathrm{u}) e^{\frac{u^2}{2}cot\propto' - tucsc\propto'} du.$$

(11)

As $L^{\propto - \frac{\pi}{2}}(\mathrm{u})$ is periodic in $\left(\propto - \frac{\pi}{2}\right)^{th}$domain. We can express it using the conventional Fourier series as

$$L^{\propto - \frac{\pi}{2}}(\mathrm{u}) = \sum_{n=-k}^{k} C_n e^{inu_{\propto_0}u} \quad (12)$$

where C_n are the Fourier series coefficients and k is the order of the highest nonzero harmonic components in the $\propto^{th}$ domain.

Now, from equation (11),

$$f(t) = e^{\frac{t^2}{2}cot\propto'} \sqrt{\frac{1 - icot\propto'}{2\pi i}}$$

$$\int_{-\infty}^{\infty}\left[\sum_{n=-k}^{k} C_n e^{inu_{\propto_0}u}\right] e^{\frac{u^2}{2}cot\propto' - tucsc\propto'}\,.\mathrm{du}$$

Interchanging summation and integration signs as in [5]

$$,f(t) = \quad e^{\frac{t^2}{2}cot\propto'}\sqrt{\frac{1-icot\propto'}{2\pi i}}\sum_{n=-k}^{k} C_n$$

$$\int_{-\infty}^{\infty} e^{\frac{u^2}{2}cot\propto' + i(itcsc\propto' + nu_{\propto_0})u}\,.\mathrm{du}$$

$$= e^{\frac{t^2}{2}cot\propto'}\sqrt{\frac{1-icot\propto'}{2\pi i}}\sum_{n=-k}^{k} C_n$$

$$\int_{-\infty}^{\infty} e^{i\pi\left[\frac{cot\alpha'}{2\pi i}u^2 + 2\left(\frac{i}{2\pi}tcsc\propto' + \frac{1}{2\pi}nu_{\propto_0}\right)u\right]}\,.\mathrm{du}$$

$$= e^{\frac{t^2}{2}cot\propto'}\sqrt{\frac{1-icot\propto'}{2\pi i}}\sum_{n=-k}^{k} C_n$$

$$\int_{-\infty}^{\infty} e^{i\pi[\chi u^2 + 2\xi u]}\,.\mathrm{du} \quad \text{(say)}$$

where $\chi = \frac{cot\alpha'}{2\pi i}$ and $\boldsymbol{\xi} = \frac{i}{2\pi}tcsc\propto' + \frac{1}{2\pi}nu_{\propto_0}$

$$= e^{\frac{t^2}{2}cot\propto'}\sqrt{\frac{1-icot\propto'}{2\pi i}}\sum_{n=-k}^{k} C_n\left\{\frac{1}{\sqrt{\chi}}e^{\frac{i\pi}{4}}e^{-\frac{i\pi\xi^2}{\chi}}\right\} =$$

$$e^{\frac{t^2}{2}cot\propto'}\sqrt{\frac{1-icot\propto'}{2\pi i}}\sum_{n=-k}^{k} C_n$$

$$\left\{\frac{1}{\sqrt{\frac{cot\alpha'}{2\pi i}}}e^{\frac{i\pi}{4}}e^{-\frac{i\pi(itcsc\propto' + nu_{\propto_0})^2}{4\pi^2\left(\frac{cot\alpha'}{2\pi i}\right)}}\right\} =$$

$$e^{\frac{t^2}{2}cot\propto'}\sqrt{\frac{1-icot\propto'}{2\pi i}}\sqrt{\frac{2\pi i}{cot\alpha'}}e^{\frac{i\pi}{4}}\sum_{n=-k}^{k} C_n$$

$$\left\{e^{\frac{(itcsc\propto' + nu_{\propto_0})^2}{2cot\propto'}}\right\} \qquad (13)$$

It is clear from (13) that we can reconstruct the time domain function $f(t)$ which is of compact support $\propto$ and of $\left(\propto - \frac{\pi}{2}\right)$ periodic ,from (2k+1) Fourier series coefficient C_n of $L^{\propto - \frac{\pi}{2}}(\mathrm{u})$ in $(\propto - \frac{\pi}{2})$ domain.

4.2 Sampling theorem for a function of 0-period with compact support - $\propto$.

Any function $f(t)$ of $0-$period with compact support $\propto$, can be reconstructed uniquely from the (2k+1) samples of the function in time domain using the relation [6] as

$$f(t) = e^{-\frac{t^2}{2}cot\propto}\sum_{v=0}^{N-1} f(v\Delta_\propto)G(t,\propto,v,N) \qquad (14)$$

where

$$G(t,\propto,v,N) = \sum_{S=-\infty}^{\infty} e^{i\{(v+sN)\Delta_\propto\}^2\frac{cot\propto}{2}}\frac{sin[\Omega_\alpha csc\propto\{t-(v+sN)\Delta_\propto\}]}{\Omega_\alpha csc\propto\{t-(v+sN)\Delta_\propto\}}$$

N=2k+1, $\Omega_\alpha = ku_{\propto_0}$, $u_{\propto_0}$ is fundamental frequency $,\Delta_\propto = \frac{\pi sin\alpha}{\Omega_\alpha}$

Proof: By using inverse fractional Laplace transform of the function having compact support to $\mathrm{k}u_{\propto_0}$ is given by [4]:

$$f(t) = \sqrt{\frac{1+icot\propto}{2\pi i}}\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}cot\propto - \frac{u^2}{2}cot\propto + tucsc\propto} L^\propto(\mathrm{u}).\mathrm{du}$$

$$= e^{-\frac{t^2}{2}cot\propto}\sqrt{\frac{1+icot\propto}{2\pi i}}\int_{-ku_{\propto_0}}^{ku_{\propto_0}} L^\propto(\mathrm{u})e^{-\frac{u^2}{2}cot\propto + tucsc\propto}\,.\mathrm{du}$$

(15)

Let $F(t) = \int_{-ku_{\propto_0}}^{ku_{\propto_0}} L^\propto(\mathrm{u})e^{-\frac{u^2}{2}cot\propto + tucsc\propto}\,.\mathrm{du}$, $F(t)$ is of compact support to $ku_{\propto_0}csc\propto$ in the conventional form.

By using Shannon sampling theorem to $F(t)$ [9]

$$F(t) = \sum_{k=-\infty}^{\infty} F(k\Delta_\propto).\frac{sin[\Omega_\alpha csc\propto\{t-k\Delta_\propto\}]}{\Omega_\alpha csc\propto\{t-k\Delta_\propto\}} \qquad (16)$$

where sampling points are , $\Delta_\propto = \frac{\pi sin\alpha}{\Omega_\alpha}$

Using (15) and (16), we have

$$f(t) = \sqrt{\frac{1+icot\propto}{2\pi i}}e^{-\frac{t^2}{2}cot\propto}\sum_{k=-\infty}^{\infty} f(k\Delta_\propto)e^{-(k\Delta_\propto)^2.\frac{cot\propto}{2}}.$$

$$\frac{sin[\Omega_\alpha csc\propto\{t-k\Delta_\propto\}]}{\Omega_\alpha csc\propto\{t-k\Delta_\propto\}} \qquad (17)$$

Now put $k = v + sN$, v and s are integers and $N = 2k+1$ in (17) and using the periodic property of the function, we have $s = 0$, $k = v$ and $N = 2k+1$.

$k = (N-1) + s(2k+1) = (N-1)$ as s=0. $f(t) = e^{-\frac{t^2}{2}cot\propto}\sum_{v=0}^{N-1} f(v\,\Delta_\propto)e^{-\{(v+sN)\Delta_\propto\}^2.\frac{cot\propto}{2}}.$

$$\frac{sin[\Omega_\alpha csc\propto\{t-(v+sN)\Delta_\propto\}]}{\Omega_\alpha csc\propto\{t-(v+sN)\Delta_\propto\}}$$

$$= e^{-\frac{t^2}{2}cot\propto}\sum_{v=0}^{N-1} f(v\,\Delta_\propto).\ G(t,\propto,v,N) \qquad (18)$$

Thus it is clear that any function of $\propto -$ compact support and period-0 can be reconstructed from $(2k+1)$ samples of the function in time domain.

5. CONCLUSION

In this paper, it is observed that fractional Laplace transform of $0-$periodic with $(\propto - \frac{\pi}{2})$ compact support, can be calculated using $(2k+1)$ samples of the function in time domain, where k is the order of positive highest nonzero harmonic component in conventional Laplace domain. It is also shown that only (2k+1) coefficients are sufficient to reconstruct any fractional Laplace transform domain of periodic function with compact support, where k is the order of positive highest nonzero harmonic component in the $\propto^{th}$ domain.

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